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ON THE CANONICAL HERMITIAN CONNECTION IN NEARLY KÄHLER MANIFOLDS

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Abstract

In the present paper we prove that the Hermitian curvature tensor $\tilde{R}$ associated to a nearly Kähler metric g always satisfies the second Bianchi identity $\mathfrak{S}(\tilde{\nabla}_X\tilde{R})(Y,Z,\cdot,\cdot)=0$ and that it satisfies the first Bianchi identity $\mathfrak{S}\tilde{R}(X,Y,Z,\cdot)=0$ if and only if g is a Kähler metric. Furthermore we characterize condition for $\tilde{R}$ to be parallel with respect to the canonical Hermitian connection $\tilde{\nabla}$ in terms of the Riemann curvature tensor and in the last part of the paper we study the curvature of some generalizations of the nearly Kähler structure.

1. Introduction

An almost Hermitian manifold (M,g,J) is called *nearly Kähler* if the covariant derivative of J with respect to the Levi-Civita connection of g defines a skew-symmetric tensor on M (these manifolds were originally also named *K-spaces* or *almost Tachibana spaces*). Nearly Kähler manifolds were intensively studied by Gray in seventies. The first motivations of Gray for taking this kind of manifolds into account were the following:

- the curvature identities of Kähler manifolds naturally generalize to nearly Kähler manifolds; consequently some topological theorems can be proved (see [6, 8, 9]);
- any 3-symmetric space admits a nearly Kähler structure (see [8]);
- nearly Kähler structures are related to the weak holonomy groups theory, since nearly Kähler structures correspond to metrics having the unitary group $\mathrm{U}(n)$ as weak holonomy (see [7]).

Moreover, nearly Kähler structures natural arise as one of the sixteen classes of the Gray-Harvella classification of almost Hermitian structures stated in [10]. Typical examples of nearly Kähler manifolds are 3-symmetric spaces, some

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homogeneous spaces and twistor spaces over quaternionic Kähler manifolds with positive scalar curvature. Today such manifolds are considered interesting for string theory since they are naturally endowed with a Hermitian connection $\tilde{\nabla}$ with totally skew-symmetric torsion (see [4]).

In this paper we study the curvature of $\tilde{\nabla}$. We will prove the following

Theorem 1.1. *Let (M, g, J) be a nearly Kähler manifold and let $\tilde{R}$ be the curvature tensor associated to $\tilde{\nabla}$. Then $\tilde{R}$ satisfies the second Bianchi identity*

$$\mathfrak{S}(\tilde{\nabla}_X \tilde{R})(Y, Z, \cdot, \cdot) = 0, \tag{1}$$

where the symbol $\mathfrak{S}$ denotes the cyclic sum. Furthermore $\tilde{R}$ satisfies the first Bianchi identity

$$\mathfrak{S}\tilde{R}(X, Y, Z, \cdot) = 0, \tag{2}$$

if and only if (M, g, J) is a Kähler manifold.

Theorem 1.1 will be proved in section 3 where we also characterize condition $\tilde{\nabla}\tilde{R} = 0$ in terms of the curvature tensor of a nearly Kähler metric:

Theorem 1.2. *Let (M, g, J) be a nearly Kähler manifold. The following facts are equivalent*

1. $\tilde{\nabla}\tilde{R} = 0$;
2. *∇R has no component in $(\Lambda^{1,1} \odot \Lambda^{1,1} \oplus \Lambda^{2,0} \odot \Lambda^{1,1}) \otimes \Lambda^1$,*

where ∇ and R denote the Levi-Civita connection and the curvature of g, respectively.

Condition $\tilde{\nabla}\tilde{R} = 0$ is very important for nearly Kähler manifolds since, in the complete simply connected case, forces the ambient space to be homogeneous (see [13, 12]) and homogenous simply connected nearly Kähler manifolds are classified by Nagy in [13]. Indeed, since in nearly Kähler manifolds $\tilde{\nabla}$ has always parallel torsion, condition $\tilde{\nabla}\tilde{R} = 0$ forces $\tilde{\nabla}$ to be an Ambrose-Singer connection and complete simply connected manifolds admitting an Ambrose-Singer connection are necessary homogeneous spaces (see [15]).

As a direct consequence of Theorem 1.2 we have the following

Corollary 1.3. *Assume that the curvature tensor of a complete and simply connected nearly Kähler manifold has no component in $(\Lambda^{1,1} \odot \Lambda^{1,1} \oplus \Lambda^{2,0} \odot \Lambda^{1,1}) \otimes \Lambda^1$, then M is a homogeneous space.*

In the last section we take into account almost Hermitian manifolds admitting a connection with totally skew-symmetric torsion.

Note. In this paper we adopt the convention that in the indicial expressions the symbol of sum over repeated indices is omitted.

2. Preliminaries

Let (M, g, J) be an almost Hermitian manifold. Then the almost complex structure J induces a splitting of the complexified tangent bundle in $TM \otimes \mathbf{C} = T^{1,0}M \oplus T^{0,1}M$. Consequently the vector bundle $\wedge^p$ of complex p-forms on M splits in $\wedge^p = \bigoplus_{r+s=p} \wedge^{r,s}$ and the de Rham operator d decomposes in $\mathrm{d} = A + \partial + \bar{\partial} + \bar{A}$. The pair (g, J) induces the almost symplectic form $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$. The triple (g, J, ω) is called a *quasi-Kähler structure* if $\bar{\partial}\omega = (\mathrm{d}\omega)^{1,2} = 0$. Quasi-Kähler structures are important since they include both almost Kähler and nearly Kähler structures.

2.1. The canonical Hermitian connection. Let (M, g, J) be an almost Hermitian manifold, then there exists a unique connection $\tilde{\nabla}$ on M, called the *canonical Hermitian connection*, satisfying the following properties

$$\tilde{\nabla} g = 0, \quad \tilde{\nabla} J = 0, \quad \mathrm{Tor}(\tilde{\nabla})^{1,1} = 0,$$

where $\mathrm{Tor}(\tilde{\nabla})^{1,1}$ denotes the $(1,1)$-part of the torsion of $\tilde{\nabla}$ (see for instance [5] where $\tilde{\nabla}$ is called the Chern connection). If (M, g, J, ω) is a *quasi-Kähler manifold*, then $\tilde{\nabla}$ can be described by the following suitable formula

$$\tilde{\nabla} = \nabla - \frac{1}{2} J \nabla J,$$

where ∇ is the Levi-Civita connection of g. In this case the torsion of $\tilde{\nabla}$ is a multiple of the Nijenhuis tensor

$$N(X, Y) = [JX, JY] - J[X, JY] - J[JX, Y] - [X, Y].$$

The connection $\tilde{\nabla}$ induces the *Hermitian curvature tensor*

$$\tilde{R}(X, Y, Z, W) = g([\tilde{\nabla}_X, \tilde{\nabla}_Y]Z - \tilde{\nabla}_{[X,Y]}Z, W).$$

Since $\tilde{\nabla}$ preserves g, one has that

$$\tilde{R}(X, Y, Z, W) = -\tilde{R}(Y, X, Z, W) = -\tilde{R}(X, Y, W, Z).$$

2.1.1. *The canonical Hermitian connection in nearly Kähler manifolds.* Assume now that (M, g, J) is a nearly Kähler manifold. Then the Nijenhuis tensor of J is $\tilde{\nabla}$-parallel (see [11, 1]) and the tensor

$$B(X, Y, Z) = g(N(X, Y), Z)$$

is a skew-symmetric 3-form of complex type $(3,0) + (0,3)$. Hence in the nearly Kähler case the torsion of $\tilde{\nabla}$ is totally skew-symmetric and parallel.

2.2. The curvature of a Nearly Kähler metric. Let (M, g, J) be a nearly Kähler manifold and let $R(X, Y, Z, W) = g([\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z, W)$ be the curvature tensor of g. It is known that R satisfies the second Gray identity

$$R(X,Y,Z,W) - R(JX,JY,Z,W) - R(JX,Y,JZ,W) - R(JX,Y,Z,JW) = 0 \tag{3}$$

and that, consequently, R is a J-invariant tensor, i.e.

$$R(JX,JY,JZ,JW) = R(X,Y,Z,W).$$

Condition (3) can be written in complex coordinates as

$$R_{ijkr} = R_{\bar{i}jkr} = 0.$$

Note that condition $R_{\bar{i}jkr} = 0$ simply says that R is J-invariant. Moreover Gray in [6] proved the following identity

$$R(X,Y,JZ,JW) - R(X,Y,Z,W) = g((\nabla_X J)Y, (\nabla_Z J)W).$$

Hence in particular one has that

$$R(X,Y,JX,JY) - R(X,Y,X,Y) = \|(\nabla_X J)Y\|^2.$$

Let $\tilde{R}$ be the Hermitian curvature tensor associated to (g,J). $\tilde{R}$ is related to the standard curvature tensor R by the following formula

$$4\tilde{R}(X,Y,Z,W) = 3R(X,Y,Z,W) + 2R(X,Y,JZ,JW) + R(X,Z,JW,JY) + R(X,W,JY,JZ) \tag{4}$$

(see [6], Corollary 9.2 and [9], formula (3.1)). Formula (4) can be expressed in terms of complex frames with the following three equations

$$\tilde{R}_{kj\bar{r}s} = R_{kj\bar{r}s}; \tag{5}$$

$$\tilde{R}_{\bar{i}\bar{j}kl} = \tilde{R}_{ijkl} = \tilde{R}_{i\bar{j}kl} = 0; \tag{6}$$

$$\tilde{R}_{k\bar{j}\bar{r}s} = R_{k\bar{j}\bar{r}s} - 1/2R_{ks\bar{j}\bar{r}}. \tag{7}$$

Formulae (5) and (6) hold also in the quasi-Kähler case (see for instance [3]), while equation (7) is characteristic of nearly Kähler structures.

2.3. Special frames on nearly Kähler manifolds. Let (M,g,J,ω) be a $2n$-dimensional quasi-Kähler manifold. We recall the following

Lemma 2.1 ([16], Lemma 1). *It is possible to find around any point o of M a local complex $(1,0)$-frame $\{Z_1,\dots,Z_n\}$ satisfying the following properties*:

a. $\nabla_i Z_{\bar{j}}(o) = 0$;
b. $\nabla_i Z_j(o)$ *is of type* $(0,1)$;
c. $g_{i\bar{j}}(o) = \delta_{ij}$, $dg_{i\bar{j}}(o) = 0$;
d. $\nabla_i \nabla_{\bar{j}} Z_k(o) = 0$;

for every $i,j,k = 1,\dots,n$. *Such a frame is called a* generalized normal holomorphic frame *around* o (*shortly a* g.n.h.f.).

Remark 2.2. Although Lemma 2.1 as stated in [16] refers to symplectic structures, its proof make use only of the fact that ω is a quasi-Kähler form. Hence the existence of generalized normal holomorphic frames is equivalent to require that the form ω is quasi-Kähler.

Furthermore all the results of [16] stated for symplectic structures hold in the quasi-Kähler case.

It is easy to prove that the curvature tensor in terms of a g.n.h.f. $\{Z_1,\dots,Z_n\}$ writes as

$$\begin{aligned}
R_{i\bar{j}k\bar{l}}(o) &= -g(\nabla_{\bar{j}}\nabla_i Z_k, Z_{\bar{l}})(o);\\
R_{\bar{i}jkl}(o) &= g(\nabla_{\bar{i}}\nabla_j Z_k, Z_l)(o);\\
R_{\bar{i}\bar{j}kl}(o) &= -g(\nabla_{[Z_{\bar{i}},Z_{\bar{j}}]} Z_k, Z_l)(o);\\
R_{ijkl}(o) &= g(\nabla_i\nabla_j Z_k, Z_l)(o) - g(\nabla_j\nabla_i Z_k, Z_l)(o)
\end{aligned}$$

and that the canonical Hermitian connection satisfies

$$\tilde{\nabla}_i Z_j(o) = \tilde{\nabla}_{\bar{i}} Z_j(o) = 0. \tag{8}$$

We consider now the case of a nearly Kähler manifold (M,g,J). First of all we notice that if $\{W_1,\dots,W_n\}$ is an arbitrary unitary frame, then one has

$$g((\nabla_i J)W_j, W_k) = g(\mathrm{i}\nabla_i W_j - J\nabla_i W_j, W_k) = 2\mathrm{i}g(\nabla_i W_j, W_k)$$

and consequently the nearly Kähler defining equation

$$(\nabla_X J)Y = -(\nabla_Y J)X \tag{9}$$

implies

$$g(\nabla_i W_j, W_k) = -g(\nabla_j W_i, W_k). \tag{10}$$

Equation (10) says that

$$g([W_i, W_j], W_k) = 2g(\nabla_i W_j, W_k),$$

i.e. $[W_i, W_j] - 2\nabla_i W_j = -\nabla_i W_j - \nabla_j W_i$ is a vector field of type $(1,0)$. Since nearly Kähler manifolds are in particular quasi-Kähler, any point o of M admits a nearby g.n.h.f. $\{Z_1,\dots,Z_n\}$. The nearly Kähler equation (9) in terms of $\{Z_1,\dots,Z_n\}$ can be written as

$$\nabla_i Z_j(o) = -\nabla_j Z_i(o), \quad i,j = 1,\dots,n. \tag{11}$$

In particular in the nearly Kähler case one has

$$[Z_i, Z_j](o) = 2\nabla_i Z_j(o) = 2\Gamma_{ij}^{\bar{k}}(o)Z_{\bar{k}}, \quad N_{ij}(o) = -8\nabla_i Z_j(o) = -8\Gamma_{ij}^{\bar{k}}(o)Z_{\bar{k}}, \tag{12}$$

where $\Gamma_{ij}^{\bar{k}}(o) = g(\nabla_i Z_j, Z_k)(o)$. Note that the Christoffel symbols $\Gamma_{ij}^{\bar{k}}(o)$ have the following symmetries

$$\Gamma^{\bar{k}}_{ij}(o) = -\Gamma^{\bar{k}}_{ji}(o) = -\Gamma^{\bar{j}}_{ik}(o). \tag{13}$$

Moreover, since the curvature is J-invariant, the frame $\{Z_1, \dots, Z_n\}$ satisfies

e. $\nabla_{\bar{i}}\nabla_j Z_k(o) \in T^{1,0}_o M$.

Also the identity $\tilde{\nabla}N = 0$ can be described in terms of g.n.h.f.'s. Indeed, $\tilde{\nabla}N$ vanishes at o if and only if the following equations

$$g((\tilde{\nabla}_{\bar{i}}N)_{jk}, Z_r)(o) = 0, \quad g((\tilde{\nabla}_i N)_{jk}, Z_r)(o) = 0$$

hold for $i, j, k, r = 1, \dots, n$. A straightforward computation yields that equation $g((\tilde{\nabla}_{\bar{i}}N)_{jk}, Z_r)(o) = 0$ is implied by the fact that the curvature of g is J-invariant, while, taking into account (8), the second equation gives

$$\begin{aligned} 0 = g((\tilde{\nabla}_i N)_{jk}, Z_r)(o) &= g(\tilde{\nabla}_i N_{jk}, Z_r)(o) = Z_i g(N_{jk}, Z_r)(o) \\ &= g(\nabla_i [Z_j, Z_k], Z_r)(o) = 2g(\nabla_i \nabla_j Z_k, Z_r)(o), \end{aligned}$$

i.e.

f. $\nabla_i \nabla_j Z_k(o) \in T^{1,0}_o M$.

Summarizing we have the following

Proposition 2.3. *Let (M, g, J) be a nearly Kähler manifold. Then any point o of M admits a local complex frame $\{Z_1, \dots, Z_n\}$ of type $(1,0)$ satisfying*

a. $\nabla_i Z_{\bar{j}}(o) = 0$;

b. $\nabla_i Z_j(o) \in T^{0,1}_o M$;

c. $g_{i\bar{j}}(o) = \delta_{ij}$, $\mathrm{d}g_{i\bar{j}}(o) = 0$;

d. $\nabla_i \nabla_{\bar{j}} Z_k(o) = 0$;

e. $\nabla_{\bar{i}} \nabla_j Z_k(o) \in T^{1,0}_o M$;

f. $\nabla_i \nabla_j Z_k(o) \in T^{1,0}_o M$;

h. $[Z_i, Z_j] - 2\nabla_i Z_j$ *is a local vector field of type* $(1,0)$;

for $i, j, k = 1, \dots, n$.

3. Proof of the results

In this section we will prove the results stated in the introduction. We will proceed in this order: first of all we will give the proof of Theorem 1.2, then we will show that the Hermitian curvature tensor in the nearly Kähler case always satisfies the second Bianchi identity (1) and finally we will prove that the Hermitian curvature tensor associated to a nearly Kähler metric g satisfies the first Bianchi identity (2) if and only if g is a Kähler metric. The last part will be obtained as a corollary of a theorem of [3], while Theorem 1.2 and formula (1) will be proved using the existence of a generalized normal holomorphic frame.

In order to prove Theorem 1.2 and formula (1), we need two preliminary lemmas:

LEMMA 3.1. *Let (M, g, J, ω) be a $2n$-dimensional quasi-Kähler manifold. Fix a point o in M and consider a g.n.h.f $\{Z_1, \dots, Z_n\}$ around o. Then the following equations*

$$(\tilde{\nabla}_r \tilde{R})_{i\bar{j}k\bar{l}}(o) = Z_r(\tilde{R}_{i\bar{j}k\bar{l}})(o), \tag{14}$$

$$(\tilde{\nabla}_r \tilde{R})_{ijk\bar{l}}(o) = Z_r(R_{ijk\bar{l}})(o), \tag{15}$$

$$(\tilde{\nabla}_{\bar{r}} \tilde{R})_{ijk\bar{l}}(o) = Z_{\bar{r}}(R_{ijk\bar{l}})(o) \tag{16}$$

hold for $r, i, j, k, l = 1, \dots, n$.

Proof. First of all we observe that if $\{Z_1, \dots, Z_n\}$ is a g.n.h.f. around a point o, then in view of (8) one has

$$(\tilde{\nabla}_\alpha \tilde{R})_{\beta\gamma\delta\sigma}(o) = Z_\alpha(\tilde{R}_{\beta\gamma\delta\sigma})(o)$$

for every $\alpha, \beta, \gamma, \delta, \sigma = 1, \dots n, \bar{1}, \dots, \bar{n}$ (note that with the Greek letters we also include the bar indexes). Hence identity (14) follows immediately. For identities (15) and (16) it is enough to keep in mind that locally one has

$$\tilde{R}_{ijk\bar{l}} = R_{ijk\bar{l}}, \quad i, j, k, l = 1, \dots, n. \qquad \square$$

Using equation (5) and (6), one can deduce that if M is a quasi-Kähler manifold, then the equation $\tilde{\nabla}\tilde{R} = 0$ in terms of complex frames writes as

$$(\tilde{\nabla}_r \tilde{R})_{i\bar{j}k\bar{l}} = (\tilde{\nabla}_r \tilde{R})_{ijk\bar{l}} = (\tilde{\nabla}_{\bar{r}} \tilde{R})_{ijk\bar{l}} = 0.$$

Using the previous lemma, we can rewrite these equations at a point o in terms of a g.n.h.f. as

$$Z_r(\tilde{R}_{i\bar{j}k\bar{l}})(o) = Z_r(R_{ijk\bar{l}})(o) = Z_{\bar{r}}(R_{ijk\bar{l}})(o) = 0.$$

Note that if R is J-invariant, then condition $\tilde{\nabla}\tilde{R} = 0$ reduces to

$$Z_r(\tilde{R}_{i\bar{j}k\bar{l}})(o) = 0.$$

Now we have

LEMMA 3.2. *Let (M, g, J, ω) be a quasi-Kähler manifold such that the curvature tensor R satisfies the second Gray identity* (3). *Fix a point o in M and let $\{Z_1, \dots, Z_n\}$ be a g.n.h.f. around o. Then*

$$(\nabla_{\bar{r}} R)_{i\bar{j}kl}(o) = 0; \tag{17}$$

$$Z_r(R_{\bar{i}\bar{j}kl})(o) = 0, \tag{18}$$

for every $r, i, j, k, l = 1, \dots, n$.

Proof. We have

$$(\nabla_{\bar r} R)_{i\bar j kl}(o) = Z_{\bar r}(R_{i\bar j kl})(o) - R(\nabla_{\bar r} Z_i, Z_{\bar j}, Z_k, Z_l)(o) - R(Z_i, \nabla_{\bar r} Z_{\bar j}, Z_k, Z_l)(o)$$
$$- R(Z_i, Z_{\bar j}, \nabla_{\bar r} Z_k, Z_l)(o) - R(Z_i, Z_{\bar j}, Z_k, \nabla_{\bar r} Z_l)(o)$$

Since the mixed terms $\nabla_i Z_{\bar j}(o)$ vanish, we obtain

$$(\nabla_{\bar r} R)_{i\bar j kl}(o) = Z_{\bar r}(R_{i\bar j kl})(o) - R(Z_i, \nabla_{\bar r} Z_{\bar j}, Z_k, Z_l)(o).$$

Since $\nabla_r Z_j(o) \in T_o^{0,1} M$ and R satisfies the second Gray identity (3), we get

$$(\nabla_{\bar r} R)_{i\bar j kl}(o) = 0.$$

A similar computation gives

$$Z_r(R_{\overline{i\bar j} kl})(o) = (\nabla_r R)_{\overline{i\bar j} kl}(o).$$

Hence, taking into account formula (17), we get

$$Z_r(R_{\overline{i\bar j} kl})(o) = (\nabla_r R)_{\overline{i\bar j} kl}(o) = -(\nabla_{\bar j} R)_{r\bar i kl}(o) - (\nabla_{\bar i} R)_{\bar j r kl}(o) = 0$$

which proves formula (18). □

Now we can prove Theorem 1.2:

Proof of Theorem 1.2. Let $o \in M$ and let $\{Z_1, \dots, Z_n\}$ be a g.n.h.f. around o. Since R satisfies the second Gray identity, then condition $\tilde\nabla \tilde R = 0$ reduces at o to $Z_r(\tilde R_{i\bar j k\bar l})(o) = 0$ for $r, j, k, l = 1, \dots, n$. Taking into account formula (7), Lemma 3.2 and that R satisfies the second Gray identity (3), we get

$$(19) \qquad Z_r(\tilde R_{i\bar j k\bar l})(o) = Z_r\left(R_{i\bar j k\bar l} + \frac{1}{2} R_{ik\bar j\bar l}\right)(o) = Z_r(R_{i\bar j k\bar l})(o) = (\nabla_r R)_{i\bar j k\bar l}(o)$$

which implies the statement. □

We remark again that condition $\tilde\nabla \tilde R = 0$ forces a simply connected complete nearly Kähler manifold to be a homogeneous space, since in such a case $\tilde\nabla$ is an Ambrose-Singer connection. Hence Corollary 1.3 follows immediately.

Now we are ready to prove the Hermitian curvature tensor associated to a nearly metric satisfies the second Bianchi identity (1):

Proof of formula (1). Since the curvature tensor of a nearly Kähler metric is J-invariant, the second Bianchi identity (1) in terms of g.n.h.f.'s reads at o as

$$\underset{r,i,\bar j}{\mathfrak{S}} Z_r(\tilde R_{i\bar j k\bar l})(o) = 0, \quad r, i, j, k, l = 1, \dots, n.$$

Using formulae (18) and (19) we obtain

$$\begin{aligned}\mathfrak{S}_{r,i,\bar{j}} Z_r(\tilde{R}_{i\bar{j}k\bar{l}})(o) &= Z_r(\tilde{R}_{i\bar{j}k\bar{l}})(o) + Z_{\bar{j}}(\tilde{R}_{rik\bar{l}})(o) + Z_i(\tilde{R}_{\bar{j}rk\bar{l}})(o)\\ &= Z_r(\tilde{R}_{i\bar{j}k\bar{l}})(o) + Z_i(\tilde{R}_{\bar{j}rk\bar{l}})(o)\\ &= (\nabla_r R)_{i\bar{j}k\bar{l}}(o) + (\nabla_i R)_{\bar{j}rk\bar{l}}(o)\\ &= \mathfrak{S}_{r,i,\bar{j}}(\nabla_r R)_{i\bar{j}k\bar{l}}(o) - (\nabla_{\bar{j}} R)_{rik\bar{l}}(o)\\ &= -(\nabla_{\bar{j}} R)_{rik\bar{l}}(o) = 0,\end{aligned}$$

which implies the statement. □

In this last part of this section we prove that the Hermitian curvature tensor associated to a nearly Kähler metric satisfies the first Bianchi identity if and only if the metric is Kähler. We recall the following

THEOREM 3.3 ([3], Theorem 1.1). *Let (M, g, J, ω) be a quasi-Kähler manifold. The Hermitian curvature tensor $\tilde{R}$ satisfies the first Bianchi identity*

$$\mathfrak{S}\tilde{R}(X, Y, Z, \cdot) = 0, \quad \textit{for every } X, Y, Z \in \Gamma(TM)$$

if and only if the following conditions hold:

1. *the curvature tensor R is J-invariant*;
2. $4R_{ij\bar{k}\bar{l}} = g((\nabla_{\bar{k}} N)_{ij}, Z_{\bar{l}})$, *where ∇ is the Levi-Civita connection of g and N denotes the Nijenhuis tensor associated to J.*

Since nearly Kähler manifolds are quasi-Kähler, Theorem 3.3 can be applied in the nearly Kähler case:

Proof of Theorem 1.1. Let (M, g, J) be a nearly Kähler manifold and let R be the curvature tensor associated to g. Fix a point $o \in M$ and let $\{Z_1, \dots, Z_n\}$ be a generalized normal holomorphic frame around o. Then, applying formulae (12) and (13), we have

$$\begin{aligned}4R_{ij\bar{k}\bar{l}}(o) &= -4g(\nabla_{[Z_i, Z_j]} Z_{\bar{k}}, Z_{\bar{l}})(o) = -8\Gamma_{ij}^{\bar{r}}(o) g(\nabla_{\bar{r}} Z_{\bar{k}}, Z_{\bar{l}})(o)\\ &= -8\Gamma_{ij}^{\bar{r}}(o) g(\nabla_{\bar{k}} Z_{\bar{l}}, Z_{\bar{r}})(o) = -4g([Z_i, Z_j], \nabla_{\bar{k}} Z_{\bar{l}})(o)\\ &= -g((\nabla_{\bar{k}} N)_{ij}, Z_{\bar{l}})(o),\end{aligned}$$

i.e.

$$4R_{ij\bar{k}\bar{l}}(o) = -g((\nabla_{\bar{k}} N)_{ij}, Z_{\bar{l}})(o),$$

where the last equality comes from [3], Lemma 2.7. Since R is a J-invariant tensor, then in view of Theorem 3.3, $\tilde{R}$ satisfies the first Bianchi identity if and only if

$$g((\nabla_{\bar{k}}N)_{ij}, Z_{\bar{l}})(o) = 0.$$

Furthermore a direct computation gives

$$0 = g((\nabla_{\bar{i}}N)_{ij}, Z_{\bar{j}})(o) = 8|\nabla_i Z_j|^2(o),$$

which implies the statement. □

4. Generalizations

In this section we take into account almost Hermitian manifolds admitting a Hermitian connection with totally skew-symmetric torsion and almost Hermitian manifolds whose almost complex structures satisfies $\tilde{\nabla}(\nabla J) = 0$. The first ones arise a class of the Gray-Hervella classification of almost complex structures and are usually called $\mathcal{G}_1$-manifolds. Such manifolds were recently studied by Nagy in [14].

We start by recalling the following

Theorem 4.1 ([4], Theorem 10.1). *Let (M, g, J) be an almost Hermitian manifold. M admits a Hermitian connection with totally skew-symmetric torsion if and only if the Nijenhuis tensor $N(X, Y, Z) = g(N(X, Y), Z)$ is a 3-form. In this case the connection is unique and it is determined by the formula*

$$T(X, Y, Z) = -\mathrm{d}\omega(JX, JY, JZ) + g(N(X, Y), Z).$$

Let (M, g, J) be an almost Hermitian manifold admitting a Hermitian connection with totally skew-symmetric torsion. Then the Nijenhuis of J satisfies

$$g(N(X, Y), Z) + g(N(X, Z), Y) = 0.$$

This condition reads in terms of complex fields as

$$g(N_{ij}, Z_k) + g(N_{ik}, Z_j) = 0,$$

or equivalently

$$g([Z_i, Z_j], Z_k) + g([Z_i, Z_k], Z_j) = 0.$$

Hence we have

$$\begin{aligned}
0 &= g([Z_i, Z_j], Z_k) + g([Z_i, Z_k], Z_j) \\
&= g(\nabla_i Z_j, Z_k) - g(\nabla_j Z_i, Z_k) + g(\nabla_i Z_k, Z_j) - g(\nabla_k Z_i, Z_j) \\
&= g(\nabla_i Z_j, Z_k) + g(\nabla_j Z_k, Z_i) - g(\nabla_i Z_j, Z_k) + g(\nabla_k Z_j, Z_i) \\
&= g(\nabla_j Z_k, Z_i) + g(\nabla_k Z_j, Z_i)
\end{aligned}$$

i.e.

$$g(\nabla_j Z_k, Z_i) + g(\nabla_k Z_j, Z_i) = 0 \tag{20}$$

which is equivalent to require that $\nabla_j Z_k + \nabla_k Z_j$ is a vector field of type $(1,0)$. This can be used to prove the following

Proposition 4.2. *Let (M, g, J) be an almost Hermitian manifold. Then the following are equivalent*

1. *M is nearly Kähler*;
2. *M is quasi-Kähler and admits a connection with totally skew-symmetric torsion.*

Proof. Clearly 1 implies 2. In order to show the other implication we fix an arbitrary $o \in M$ and a g.n.h.f. $\{Z_1, \dots, Z_n\}$ around o. Since $\nabla_i Z_j(o) \in T_o^{0,1}M$, then in this case equation (20) implies $\nabla_i Z_j(o) + \nabla_j Z_i(o) = 0$ which is equivalent to $(\nabla_i J)Z_j(o) + (\nabla_j J)Z_i(o) = 0$. Since (g, J) is by hypothesis a quasi-Kähler structures, then $(\nabla_j J)Z_{\bar{i}} = 0$. Hence $\nabla J(o)$ is a skew-symmetric tensor and the claim follows. □

Now we consider the case $\tilde{\nabla}(\nabla J) = 0$. We have the following Proposition which can be proved by a direct computation.

Proposition 4.3. *Let (M, g, J, ω) be a $2n$-dimensional quasi-Kähler manifold. Then condition $\tilde{\nabla}(\nabla J) = 0$ reads in terms of g.n.h.f's as*

$$g(\nabla_r \nabla_i Z_k, Z_s)(o) = g(\nabla_{\bar{r}} \nabla_i Z_k, Z_s)(o) = 0, \quad r, i, k, s = 1, \dots, n.$$

Proposition 4.3 has the following direct consequence

Corollary 4.4. *Let (M, g, J, ω) be a quasi-Kähler manifold satisfying $\tilde{\nabla}(\nabla J) = 0$. Then the curvature tensor of g satisfies the second Gray identity* (3).

Remark 4.5. In [3], it is described an explicit example of a quasi-Kähler non-nearly Kähler structure (g_0, J_0, ω_0) on the Iwasawa manifolds having the curvature $\tilde{R}$ equal to zero. It can be noted that (g_0, J_0, ω_0) satisfies condition $\tilde{\nabla}(\nabla J) = 0$.

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