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KODAI MATH. J.
32 (2009), 404–419

ON THE GEOMETRY OF CERTAIN IRREDUCIBLE NON-TORUS PLANE SEXTICS

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Abstract

An irreducible non-torus plane sextic with simple singularities is said to be *special* if its fundamental group factors to a dihedral group. There exist (exactly) ten configurations of simple singularities that are realizable by such curves. Among them, six are realizable by *non-special* sextics as well. We conjecture that for each of these six configurations there always exists a non-special curve whose fundamental group is *abelian*, and we prove this conjecture for three configurations (another one has already been treated in one of our previous papers). As a corollary, we obtain new explicit examples of Alexander-equivalent Zariski pairs of irreducible sextics.

1. Introduction

1.1. Motivations. An old conjecture by the second author says that the fundamental group of (the complement of)[1] an irreducible plane sextic with simple singularities and which is not of torus type is abelian. (We recall that a sextic is said to be of *torus type* if its defining polynomial can be written as $F_2^3 + F_3^2$, where F_2 and F_3 are homogeneous polynomials of degree 2 and 3 respectively.) Although this statement has been checked for hundreds of configurations of singularities[2]—with a considerable contribution by A. Degtyarev [5, 6, 7], see also [14] and [11]—it turns out to be false in general. This was also observed by A. Degtyarev who proved in [3] that there exist (exactly) ten equisingular deformation families of irreducible non-torus sextics with simple singularities and whose fundamental groups factor to the dihedral group $\mathbf{D}_k$, where $k = 10$ or 14 (so called *special* sextics, or $\mathbf{D}_k$-sextics when we need to make

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2000 *Mathematics Subject Classification*. 14H30 (14H20, 14H45, 14H50).

Key words and phrases. Irreducible non-torus plane sextic; Fundamental group; Alexander-equivalent Zariski pair.

Received December 9, 2008; revised January 14, 2009.

[1] The word 'complement', always understood, will be systematically omitted.

[2] To be precise, what is proved is that for each of the configurations in question there is a curve with that configuration and whose fundamental group is abelian.

mention of the dihedral group). The $\mathbf{D}_{10}$-sextics form eight equisingular deformation families, one family for each of the following sets of singularities:

$$\begin{gathered} 4\mathbf{A}_4, \quad 4\mathbf{A}_4 \oplus \mathbf{A}_1, \quad 4\mathbf{A}_4 \oplus 2\mathbf{A}_1, \quad 4\mathbf{A}_4 \oplus \mathbf{A}_2, \\ \mathbf{A}_9 \oplus 2\mathbf{A}_4, \quad \mathbf{A}_9 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_1, \quad \mathbf{A}_9 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_2, \quad 2\mathbf{A}_9. \end{gathered} \tag{1.1}$$

The $\mathbf{D}_{14}$-sextics divide into two families, one for each of the following configurations:

$$3\mathbf{A}_6 \quad \text{and} \quad 3\mathbf{A}_6 \oplus \mathbf{A}_1. \tag{1.2}$$

First explicit examples and fundamental groups of $\mathbf{D}_{10}$-sextics were given in [4] (see also [12] for the sets of singularities $4\mathbf{A}_4$ and $4\mathbf{A}_4 \oplus \mathbf{A}_1$).[3] First examples of $\mathbf{D}_{14}$-sextics appeared in [10]. The fundamental group of a $\mathbf{D}_{14}$-sextic with the set of singularities $3\mathbf{A}_6$ was also given in [10]. The fundamental group of a $\mathbf{D}_{14}$-sextic with the configuration $3\mathbf{A}_6 \oplus \mathbf{A}_1$ is still unknown.

A. Degtyarev also observed in [3] that the following six configurations

$$4\mathbf{A}_4, \quad 4\mathbf{A}_4 \oplus \mathbf{A}_1, \quad \mathbf{A}_9 \oplus 2\mathbf{A}_4, \quad \mathbf{A}_9 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_1, \quad 2\mathbf{A}_9, \quad 3\mathbf{A}_6 \tag{1.3}$$

(from the lists (1.1) and (1.2)) can be realized not only by special sextics but also by *non-special* ones (i.e., irreducible non-torus sextics the fundamental groups of which do not admit any dihedral quotient).[4] However he did not give any explicit equation for these non-special curves neither did he calculate their fundamental groups. The first concrete example, together with the calculation of its fundamental group, was given is [13] for the configuration of singularities $\mathbf{A}_9 \oplus 2\mathbf{A}_4$. In fact, we showed in [13] that the non-special sextic in question has an *abelian* fundamental group. It is then natural to ask whether the other five configurations in the list (1.3) can be also realized by non-special sextics having an abelian fundamental group. In this paper, we answer positively this question for the sets of singularities $2\mathbf{A}_9$, $4\mathbf{A}_4$ and $3\mathbf{A}_6$. (In general, it seems (from known examples) that for 'most of' the configurations of singularities realizable by irreducible non-torus sextics, one may find a curve with that configuration and whose fundamental group is abelian. Up to now, the only known exceptions are the four sets of singularities $4\mathbf{A}_4 \oplus 2\mathbf{A}_1$, $4\mathbf{A}_4 \oplus \mathbf{A}_2$, $\mathbf{A}_9 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_2$, $3\mathbf{A}_6 \oplus \mathbf{A}_1$, already mentioned above (cf. footnote 4), and the following two configurations $\mathbf{E}_7 \oplus 2\mathbf{A}_4 \oplus 2\mathbf{A}_2$ and $\mathbf{E}_8 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_2$ that have been discovered recently by A. Degtyarev [9]. Note that for these six configurations the corresponding equisingular moduli space is connected.)

1.2. Statement of the main result. Consider the curves C_1, C_2 and C_3 defined in sections 2, 3 and 4 below. Their configurations of singularities are given by

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[3] In [3] only the existence of special sextics was proved.

[4] Note that all the other configurations appearing in the lists (1.1) and (1.2) (i.e., $4\mathbf{A}_4 \oplus 2\mathbf{A}_1$, $4\mathbf{A}_4 \oplus \mathbf{A}_2$, $\mathbf{A}_9 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_2$ and $3\mathbf{A}_6 \oplus \mathbf{A}_1$) cannot be realized by non-special curves (cf. [3]).

$$
\Sigma_1 := 2\mathbf{A}_9 \\
\Sigma_2 := 4\mathbf{A}_4 \qquad (1.4) \\
\Sigma_3 := 3\mathbf{A}_6
$$

respectively, all of them in the list (1.3).

THEOREM 1.1. *Each curve* C_i $(1 \le i \le 3)$ *is an irreducible non-torus non-special sextic and the fundamental group* $\pi_1(\mathbf{CP}^2 \backslash C_i)$ *is abelian.*

Remark 1.2. In fact, $\pi_1(\mathbf{CP}^2 \backslash C_i) \simeq \mathbf{Z}/6\mathbf{Z}$. Indeed, by Hurewicz's theorem, if $\pi_1(\mathbf{CP}^2 \backslash C_i)$ is abelian, then it is isomorphic to the first integral homology group $H_1(\mathbf{CP}^2 \backslash C_i)$, and it is well known that $H_1(\mathbf{CP}^2 \backslash C_i) \simeq \mathbf{Z}/6\mathbf{Z}$.

Remark 1.3. It is also well known that $2\mathbf{A}_4$ can degenerate into one $\mathbf{A}_9$ and thus $4\mathbf{A}_4$ into $2\mathbf{A}_9$ if no degree condition is given. If this degeneration C_t, $t \in U \subset \mathbf{C}$, can be done in the moduli space of sextics with $4\mathbf{A}_4$ so that the sextic C_0 with $2\mathbf{A}_9$ has an abelian fundamental group, then the commutativity of the fundamental group for a generic sextic with $4\mathbf{A}_4$ follows by the degeneration principle. Unfortunately the practical calculation to find such an explicit family of sextics involves a heavy calculation and we do not have any explicit example.

We expect that the remaining two configurations $4\mathbf{A}_4 \oplus \mathbf{A}_1$ and $\mathbf{A}_9 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_1$ may be also realized by non-special sextics with abelian fundamental groups. (However, for these two configurations, the computations using `Maple` are also very heavy, so it is extremely difficult to produce explicit equations.)

Theorem 1.1 is proved in sections 2, 3 and 4. Note that we only have to show that the fundamental group $\pi_1(\mathbf{CP}^2 \backslash C_i)$ is abelian. Indeed, by [3, 8, 18], an irreducible sextic is of torus type if and only if its fundamental group factors to the dihedral group $\mathbf{D}_6$. So, if $\pi_1(\mathbf{CP}^2 \backslash C_i)$ is abelian, then C_i cannot be of torus type. To show that $\pi_1(\mathbf{CP}^2 \backslash C_i)$ is abelian, we use Zariski-van Kampen's theorem (cf. [20] and [19]). (The calculations being similar for the three curves, we will give full details only for the curve C_1 and merely sketch the proof for C_2 and C_3.)

1.3. Alexander-equivalent Zariski pairs. As mentioned above, each configuration of singularities Σ_i in the list (1.4) can be also realized by a special sextic C_i'. Explicit equations for these special curves can be found in [4] (configurations $2\mathbf{A}_9$ and $4\mathbf{A}_4$), in [12] (configuration $4\mathbf{A}_4$) and in [10] (configuration $3\mathbf{A}_6$). The generic Alexander polynomial of an irreducible non-torus sextic being always trivial (cf. [3]), each pair (C_i, C_i')—where C_i is the non-special sextic given by Theorem 1.1—is a new explicit example of so called *Alexander-equivalent Zariski pair*. (We recall that a pair of *irreducible* curves C, C' with the same degree and the same configuration of singularities is said to be a *Zariski pair* if the pairs of spaces $(\mathbf{CP}^2, C)$ and $(\mathbf{CP}^2, C')$ are not homeomorphic (cf. [1]). A Zariski pair

(C, C') is said to be *Alexander-equivalent* if the generic Alexander polynomials of the curves C and C' are the same. The first example of Zariski pair goes back to Zariski [20, 21, 22] (see also [1] and [14]). It deals with curves of degree 6, which is the lowest degree where Zariski pairs appear. The first examples of Alexander-equivalent pairs are due to the second author [16] (irreducible curves of degree 12) and [17] (irreducible curves of degree 8) and to E. Artal Bartolo and J. Carmona Ruber [2] (*reducible* curves of degree 7)[5]. The *existence* of Alexander-equivalent Zariski pairs on irreducible curves of degree 6 was proved by A. Degtyarev in [3], while the first explicit example was given in our paper [13].)

2. An example of a non-special sextic with the set of singularities $2\mathbf{A}_9$ and whose fundamental group is abelian

Let $(X : Y : Z)$ be homogeneous coordinates on $\mathbf{CP}^2$ and (x, y) the affine coordinates defined by $x := X/Z$ and $y := Y/Z$ on $\mathbf{CP}^2 \setminus \{Z = 0\}$. We consider the projective curve C_1 defined by the affine equation $f_1(x, y) = 0$, where

$$\begin{aligned} f_1(x, y) := {} & -166 + 148x + 12y - 8y^5x\sqrt{5} + 92y^2x^4\sqrt{5} + 186y^4x^2\sqrt{5} - 24y^3 \\ & + 502y^2 - 506y^4 + 12y^5 - 296x^4 - 308x^3 + 450x^2 + 248x^5 + 16y^3x^2 \\ & - 884y^2x^2 - 16yx^2 + 8y^3x + 434y^4x^2 + 148y^4x - 248yx^5 + 244yx^3 \\ & - 4y^5x - 4yx - 244y^3x^3 + 16yx^4 - 296y^2x + 280y^2x^4 + 308y^2x^3 \\ & - 77\sqrt{5} + 170y^6 + 64x^6 - 8yx^2\sqrt{5} - 8yx\sqrt{5} - 380y^2x^2\sqrt{5} \\ & - 112y^2x\sqrt{5} - 116y^3x^3\sqrt{5} + 16y^3x\sqrt{5} + 116yx^3\sqrt{5} - 104yx^5\sqrt{5} \\ & + 8y^3x^2\sqrt{5} + 8yx^4\sqrt{5} + 116y^2x^3\sqrt{5} + 56y^4x\sqrt{5} + 2y\sqrt{5} + 2y^5\sqrt{5} \\ & - 227y^4\sqrt{5} - 4y^3\sqrt{5} + 75y^6\sqrt{5} + 229y^2\sqrt{5} - 100x^4\sqrt{5} + 56x\sqrt{5} \\ & + 194x^2\sqrt{5} - 116x^3\sqrt{5} + 104x^5\sqrt{5}. \end{aligned}$$

This curve is irreducible, of degree 6, and has two singular points of type $\mathbf{A}_9$ located at $(0, \pm 1)$. Its real plane section $\{(x, y) \in \mathbf{R}^2; f_1(x, y) = 0\}$ is shown

[5] Two *irreducible* curves C, C' with the same degree and the same configuration of singularities always have the same *combinatoric* (i.e., there exist regular neighbourhoods $T(C)$ and $T(C')$ of C and C' respectively such that the pairs $(T(C), C)$ and $(T(C'), C')$ are homeomorphic). However this is not always the case for *reducible* curves. The definition of Zariski pairs for reducible curves should then be adjusted as follows (cf. [1]): a pair of *reducible* curves C, C' with the same degree and the same configuration of singularities is said to be a *Zariski pair* if C and C' have the same combinatoric and if the pairs $(\mathbf{CP}^2, C)$ and $(\mathbf{CP}^2, C')$ are not homeomorphic.

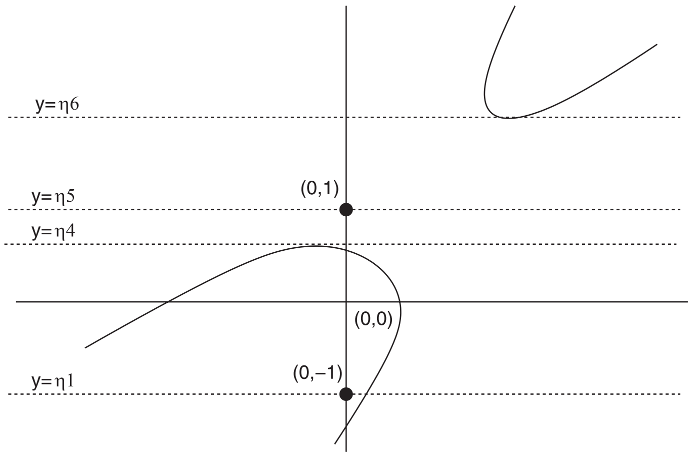

FIGURE 1. $\{(x,y)\in\mathbf{R}^2; f_1(x,y)=0\}$

in Figure 1. (In the figures we do not respect the numerical scale.) Note that, after appropriate changes of coordinates, the Newton principal parts of f_1 at $(0,\pm 1)$ have no real factorization, so these points are isolated in the set $\{(x,y)\in\mathbf{R}^2;\ f_1(x,y)=0\}$.

To show that $\pi_1(\mathbf{CP}^2\backslash C_1)$ is abelian, we use Zariski-van Kampen's theorem with the pencil given by the horizontal lines $L_\eta : y=\eta,\ \eta\in\mathbf{C}$ (cf. [20] and [19]). We take the point $(1:0:0)$ as base point for the fundamental groups. This point is nothing but the axis of the pencil, which is also the point at infinity of the lines L_η. Note that it does not belong to the curve. This pencil has 6 singular lines $L_{\eta_1},\dots,L_{\eta_6}$ with respect to C_1. (A line of the pencil is said to be singular with respect to C_1 if it is tangent to the regular part of C_1 or passes through singular points of C_1.) They correspond to the 6 complex roots

$$\eta_1=-1,$$
$$\eta_2\approx -0.9980-i0.0059,\quad \eta_3=\bar{\eta}_2\approx -0.9980+i0.0059,$$
$$\eta_4\approx 0.9964,\quad \eta_5=1,\quad \eta_6\approx 3.3097$$

of the discriminant of f_1 as a polynomial in x. Note that the lines L_{η_1} and L_{η_5} pass through a singular point of C_1, while all the other singular lines are tangent to the curve. See Figure 1.

We consider the generic line $L_{\eta_4+\varepsilon}$ and we choose generators $\xi_1,\dots,\xi_6$ of the fundamental group $\pi_1(L_{\eta_4+\varepsilon}\backslash C_1)$ as in Figure 2, where $\varepsilon>0$ is small enough. The ξ_k's $(1\le k\le 6)$ are *lassos oriented counter-clockwise* around the six inter-

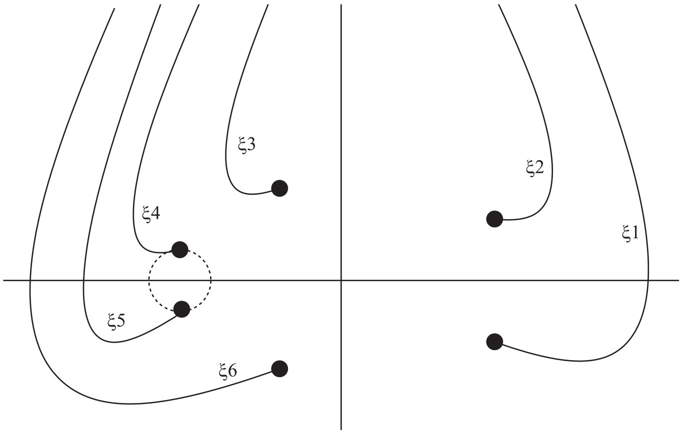

FIGURE 2. Generators at $y = \eta_4 + \varepsilon$

section points of the line $L_{\eta_4+\varepsilon}$ with the curve. (In the figures, a lasso is represented by a path ending with a bullet.) Note that

$$\omega := \xi_6 \cdot \dots \cdot \xi_1 = e, \tag{2.1}$$

where e is the unit element (vanishing relation at infinity). The Zariski-van Kampen theorem says that

$$\pi_1(\mathbf{CP}^2 \backslash C_1) \simeq \pi_1(L_{\eta_4+\varepsilon} \backslash C_1)/G_1,$$

where G_1 is the normal subgroup of $\pi_1(L_{\eta_4+\varepsilon} \backslash C_1)$ generated by the monodromy relations associated with the singular lines of the pencil. As usual, to find these relations we fix a 'standard' system of generators $\sigma_1, \dots, \sigma_6$ for the fundamental group $\pi_1(\mathbf{C} \backslash \{\eta_1, \dots, \eta_6\})$ with base point $\eta_4 + \varepsilon$ (for details we refer to our previous papers [12, 13]). The monodromy relations around the singular line L_{η_j} $(1 \leq j \leq 6)$ are obtained by moving the generic fibre $F \simeq L_{\eta_4+\varepsilon} \backslash C_1$ isotopically above σ_j (the loop surrounding η_j), and by identifying each ξ_k $(1 \leq k \leq 6)$ with its image by the terminal homeomorphism of this isotopy.

The monodromy relation around the singular line L_{η_4} is a multiplicity 2 tangent relation given by

$$\xi_5 = \xi_4. \tag{2.2}$$

To find the monodromy relations around the line L_{η_5}, we first need to get to know how the ξ_k's are deformed when y moves on the real axis from $\eta_4 + \varepsilon$ to $\eta_5 - \varepsilon$. We proceed as follows. At $(0, 1)$, the curve has two branches K and K' given by

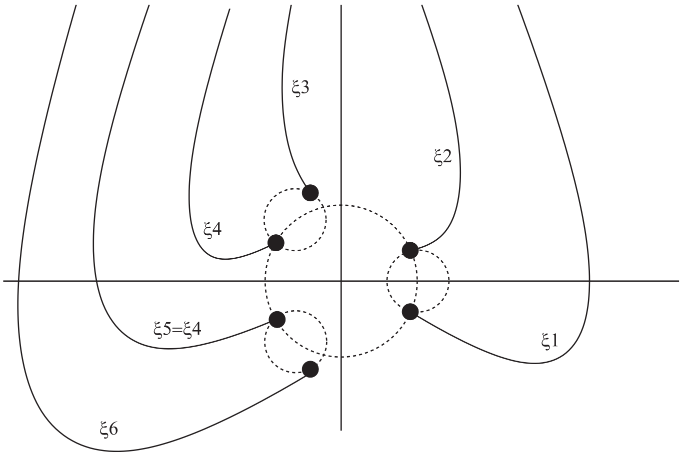

FIGURE 3. Generators at $y = \eta_5 - \varepsilon$

$$K: \quad y = 1 - x^3 + a_4 x^4 + a_5 x^5 + \text{higher terms},$$

$$K': \quad y = 1 - x^3 + a_4 x^4 + \bar{a}_5 x^5 + \text{higher terms},$$

where $a_4 = -6/(\sqrt{5} - 3)$, $a_5 \approx -65.4412 + i18.0732$, and $\bar{a}_5$ is the complex conjugate of a_5. These two branches are smooth at $(0, 1)$ and intersect at this point with intersection multiplicity 5. As we use the pencil $\{y = \eta \mid \eta \in \mathbf{C}\}$, it is more convenient to have a parametrization taking y as a parameter. An easy computation gives Puiseux parametrizations of K and K' at $(0, 1)$:

$$K: \quad y - 1 = t^3, \quad x = -t + \frac{a_4}{3} t^2 - \frac{a_4^2 + a_5}{3} t^3 + \text{higher terms}, \tag{2.3}$$

$$K': \quad y - 1 = t^3, \quad x = -t + \frac{a_4}{3} t^2 - \frac{a_4^2 + \bar{a}_5}{3} t^3 + \text{higher terms}. \tag{2.4}$$

From these equations we can easily find the position of the 6 complex roots of the polynomial $f_1(x, \eta_5 - \varepsilon)$. In fact, for a fixed $y = 1 + \eta$ with $|\eta|$ sufficiently small, there are three choices of t (the cubic roots of η) which give three corresponding points on K (those associated with ξ_1, ξ_4 and ξ_6) and three points on K' (those associated with ξ_2, ξ_3 and ξ_5). Then it follows from the next lemma that, when y moves on the real axis from $\eta_4 + \varepsilon$ to $\eta_5 - \varepsilon$, the ξ_k's are deformed as shown in Figure 3.

LEMMA 2.1. *Let y_0 be any point in the interval $[\eta_4 + \varepsilon, \eta_5 - \varepsilon]$. It is not possible to have four complex solutions of the equation $f_1(x, y_0) = 0$ aligned on a vertical line $u = u_0$ in the complex plane $(\mathbf{C}, x = u + iv)$.*

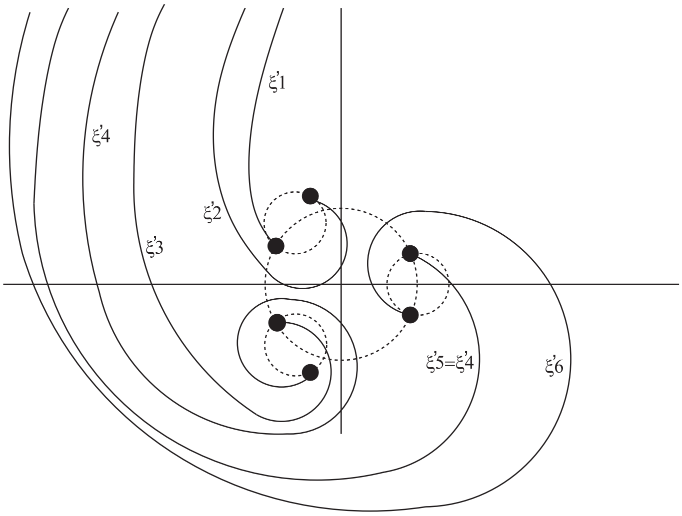

FIGURE 4. Images of the generators at $y = \eta_5 - \varepsilon$ after one y-turn

We postpone the proof of this lemma till the end of the section and first complete the calculation of $\pi_1(\mathbf{CP}^2 \backslash C_1)$.

The Puiseux parametrizations (2.3) and (2.4) also show that, when y runs once counter-clockwise on the circle $\mathbf{S}_\varepsilon(\eta_5)$ with centre η_5 and radius ε, the three small dotted circles in Figure 3 (which correspond to the term $-t$ in equations (2.3) and (2.4)) make (1/3)-turn in the counter-clockwise direction along the big dotted circle, while each of the six bullets (corresponding to the terms in t^3) runs once counter-clockwise on the corresponding small dotted circle. Figure 4 shows the images $\xi'_1, \ldots, \xi'_6$ of $\xi_1, \ldots, \xi_6$ after this movement. The monodromy relations around L_{η_5}, obtained by identifying ξ'_k with ξ_k $(1 \le k \le 6)$, are then given by

$$\xi_1 = \xi_4 \tag{2.5}$$

$$\xi_2 = \xi_4 \xi_3 \xi_4^{-1} \tag{2.6}$$

$$\xi_3 = \xi_6 \xi_4 \xi_6^{-1} \tag{2.7}$$

$$\xi_4 = (\xi_6 \xi_4) \cdot \xi_6 \cdot (\xi_6 \xi_4)^{-1} \tag{2.8}$$

$$\xi_5 \ (= \xi_4) = \omega \xi_2 \omega^{-1} = \xi_2 \tag{2.9}$$

$$\xi_6 = \omega \cdot \xi_2 \xi_1 \xi_2^{-1} \cdot \omega^{-1} = \xi_2 \xi_1 \xi_2^{-1}. \tag{2.10}$$

It follows immediately from (2.2) and (2.5)–(2.10) that $\pi_1(\mathbf{CP}^2\backslash C_1)$ is abelian. (We do not need to look for the monodromy relations associated with the singular lines L_{η_j} for $j = 1, 2, 3, 6$.) To complete the calculation it remains to prove Lemma 2.1.

Proof of Lemma 2.1. We consider the polynomial

$$f(u, v, y) := f_1(u + iv, y)$$

for u, v and y real (we recall that f_1 is the defining polynomial of C_1). We denote by $f_{\Re}(u, v, y)$ and $f_{\Im}(u, v, y)$ the real and the imaginary parts of $f(u, v, y)$ respectively. They have degree 6 and 5 respectively in v. Suppose the equation $f_1(x, y_0) = 0$ has 4 complex solutions aligned on a vertical line $u = u_0$ in the complex plane $(\mathbf{C}, x = u + iv)$. This implies that the equations

$$f_{\Re}(u_0, v, y_0) = f_{\Im}(u_0, v, y_0) = 0$$

have 4 common real solutions v_1, v_2, v_3 and v_4. The v_i's are non-zero since the discriminant of f_1 as a polynomial in x does not have any solution in $[\eta_4 + \varepsilon, \eta_5 - \varepsilon]$. Then, since v divides $f_{\Im}(u, v, y)$, the equations

$$f_{\Re}(u_0, v, y_0) = f_{\Im}(u_0, v, y_0)/v = 0,$$

also have the v_i's ($1 \leq i \leq 4$) as common solutions. As $f_{\Im}(u, v, y)/v$ has degree 4 in v, it follows that $f_{\Im}(u_0, v, y_0)/v$ divides $f_{\Re}(u_0, v, y_0)$. The remainder $R(u, v, y)$ of $f_{\Re}(u, v, y)$ by $f_{\Im}(u, v, y)/v$, as polynomials in v, is then zero for $u = u_0$ and $y = y_0$. One checks easily using `Maple` that $R(u, v, y)$ has the form

$$R(u, v, y) = \frac{R_2'(u, y)}{R_2''(u, y)} v^2 + \frac{R_0'(u, y)}{R_0''(u, y)},$$

where R_2', R_2'', R_0' and R_0'' are polynomials in u and y. Thus (u_0, y_0) is a common real solution of the equations

$$R_2'(u, y) = R_0'(u, y) = 0. \tag{2.11}$$

This implies that y_0 is a root of the resultant

$$\mathrm{Res}_u(R_2', R_0')$$

of R_2' and R_0' as polynomials in u. There is only one real solution $y_0 \approx 0.9965$ of the equation $\mathrm{Res}_u(R_2', R_0')(y) = 0$ in the interval $[\eta_4 + \varepsilon, \eta_5 - \varepsilon]$. This solution gives a real number $u_0 \approx 0.0459$ such that the pair (u_0, y_0) is a solution of (2.11). The condition $(u_0, y_0) \approx (0.0459, 0.9965)$ is then a necessary condition to have 4 complex solutions of the equation $f_1(x, y_0) = 0$ aligned on a vertical line $u = u_0$. However it is not sufficient. In fact, one checks easily using `Maple` that for $(u_0, y_0) \approx (0.0459, 0.9965)$ the polynomial (in v) $f_{\Im}(u_0, v, y_0)/v$ does not have any real roots, and the discussion above then shows that it is not possible to find 4 complex roots of $f_1(x, y_0)$ aligned on the vertical line $u = u_0$. □

3. An example of a non-special sextic with the set of singularities $4A_4$ and whose fundamental group is abelian

In this section, we consider the curve C_2 defined by $f_2(x,y)=0$, where

$$\begin{aligned} f_2(x,y) := {} & -5995519872x^4y\sqrt{2}+390096y^2x\sqrt{2}-14372748x^2y \\ & -551664y^2x-10015508672x^5y+8478937128x^4y-872359488x^3y \\ & +1344433152y^2x^3+11107764y^2x^2+551664xy^3-551664xy^5 \\ & -31808052y^4x^2+551664y^4x-472073664x^3y^3 \\ & +35073036x^2y^3-3160180972y^2x^4-7371y^2+14742y^4 \\ & -7025413356x^4+19412557632x^5-7371y^6-13411000576x^6 \\ & -24800544x^2y^3\sqrt{2}+333798336x^3y^3\sqrt{2}+2234567488y^2x^4\sqrt{2} \\ & +390096xy^5\sqrt{2}-950658048y^2x^3\sqrt{2}-390096y^4x\sqrt{2} \\ & +616859712x^3y\sqrt{2}+22490208y^4x^2\sqrt{2}-390096xy^3\sqrt{2} \\ & +10163232x^2y\sqrt{2}-7852896y^2x^2\sqrt{2}+7082017088x^5y\sqrt{2} \\ & -10368y^4\sqrt{2}+5184y^6\sqrt{2}+4967717184x^4\sqrt{2}+5184y^2\sqrt{2} \\ & +9482996224x^6\sqrt{2}-13726742208x^5\sqrt{2}. \end{aligned}$$

This curve is an irreducible sextic with four $\mathbf{A}_4$-singularities located at $(0,\pm 1)$, $(0,0)$ and $(1,-1)$ respectively. Its real plane section is shown in Figure 5.

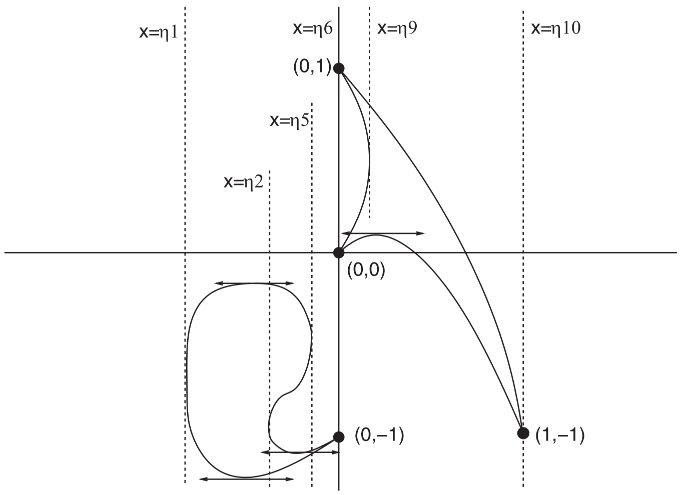

FIGURE 5. $\{(x,y)\in\mathbf{R}^2; f_2(x,y)=0\}$

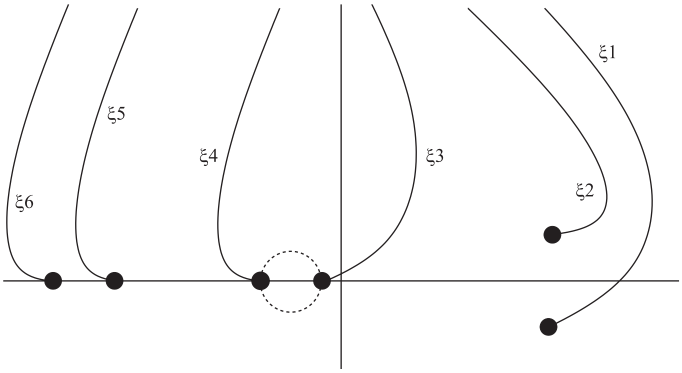

FIGURE 6. Generators at $x = \eta_5 - \varepsilon$

To show that $\pi_1(\mathbf{CP}^2 \backslash C_2)$ is abelian, we use here Zariski-van Kampen's theorem with the pencil given by the vertical lines $L_\eta : x = \eta$, $\eta \in \mathbf{C}$. We take the axis of the pencil (i.e., the point $(0:1:0)$) as base point for the fundamental groups. Note that it does not belong to the curve. This pencil has 12 singular lines $L_{\eta_1}, \dots, L_{\eta_{12}}$ with respect to C_2. They correspond to the 12 complex roots

$$\eta_1 \approx -0.3756, \quad \eta_2 \approx -0.0932,$$

$$\eta_3 \approx -0.0833 - i0.0692, \quad \eta_4 = \bar{\eta}_3 \approx -0.0833 + i0.0692,$$

$$\eta_5 \approx -0.0500, \quad \eta_6 = 0,$$

$$\eta_7 \approx 0.0759 - i0.0786, \quad \eta_8 = \bar{\eta}_7 \approx 0.0759 + i0.0786,$$

$$\eta_9 \approx 0.0840, \quad \eta_{10} = 1,$$

$$\eta_{11} \approx 1.0465 - i0.0299, \quad \eta_{12} = \bar{\eta}_{11} \approx 1.0465 + i0.0299$$

of the discriminant of f_2 as a polynomial in y. The lines L_{η_6} and $L_{\eta_{10}}$ pass through singular points of the curve. All the other singular lines are tangent to it. See Figure 5.

We consider the generic line $L_{\eta_5-\varepsilon}$ and take generators $\xi_1, \dots, \xi_6$ of $\pi_1(L_{\eta_5-\varepsilon} \backslash C_2)$ as in Figure 6, where $\varepsilon > 0$ is small enough. As above, to find the monodromy relations associated with the L_{η_j}'s we move the generic fibre $F \simeq L_{\eta_5-\varepsilon} \backslash C_2$ above a 'standard' system of generators of $\pi_1(\mathbf{C} \backslash \{\eta_1, \dots, \eta_{12}\})$ with base point $\eta_5 - \varepsilon$.

The monodromy relations around the singular lines L_{η_j}, where $j = 5, 2, 1$, can be found easily. They are all multiplicity 2 tangent relations:

$$\xi_4 = \xi_3 \quad \text{(monodromy relation around } L_{\eta_5}\text{)},$$

$$\xi_5 = \xi_4 \quad \text{(monodromy relation around } L_{\eta_2}\text{)},$$

$$\xi_3 = \xi_4^{-1} \xi_6 \xi_4 \quad \text{(monodromy relation around } L_{\eta_1}\text{)}.$$

Altogether, they give

$$\xi_6 = \xi_5 = \xi_4 = \xi_3. \tag{3.1}$$

The monodromy relation around L_{η_9} is also a multiplicity 2 tangent relation given by

$$\xi_2 = \xi_1 \xi_3 \xi_1^{-1}. \tag{3.2}$$

(To see how the ξ_k's are deformed, when x moves on the real axis from $\eta_5 + \varepsilon$ to $\eta_6 - \varepsilon$, one may proceed as in Lemma 2.1. To see how the ξ_k's are deformed when x makes half-turn around the singular line L_{η_6}, use the Puiseux parametrizations of C_2 at $(0,0)$, $(0,1)$ and $(0,-1)$ given by

$$x = t^2, \quad y = a_4 t^4 + a_5 t^5 + \text{higher terms},$$
$$x = t^2, \quad y = 1 + a_4' t^4 + a_5' t^5 + \text{higher terms},$$
$$x = t^2, \quad y = -1 + a_2'' t^2 + a_4'' t^4 + a_5'' t^5 + \text{higher terms},$$

respectively, where a_j, a_j', a_j'' are non-zero complex numbers.)

By (3.1) and (3.2), the vanishing relation at infinity is written as

$$\xi_1 = \xi_3^{-5}. \tag{3.3}$$

Altogether, the relations (3.1)–(3.3) show that the group $\pi_1(\mathbf{CP}^2 \backslash C_2)$ is abelian. (We do not need to find the monodromy relations around the singular lines L_{η_j} for $j = 3, 4, 6, 7, 8, 10, 11, 12$.)

4. An example of a non-special sextic with the set of singularities $3\mathbf{A}_6$ and whose fundamental group is abelian

In this section, we consider the curve C_3 defined by $f_3(x, y) = 0$, where

$$\begin{aligned}
f_3(x,y) := {}& -168iy^5\sqrt{7} - 852iy^4\sqrt{7} - 186iy^2\sqrt{7} - 434iy^4\sqrt{7}x^2 \\
& - 48iy^2x^4\sqrt{7} - 102ix^3y\sqrt{7} - 1392iy^3x\sqrt{7} - 54iy^2x^2\sqrt{7} \\
& - 80iy^2x^3\sqrt{7} - 656iy^5x\sqrt{7} + 872iy^3\sqrt{7} + 334iy^6\sqrt{7} - 898y^2 \\
& - 1912y^3 - 2090y^6 + 472y^5 + 4428y^4 + 7x^6 + 7x^4 - 14x^5 \\
& + 98x^3y + 103y^2x^4 + 618y^2x^2 + 1072y^2x - 42x^2y - 888y^2x^3 \\
& - 5808y^4x - 56x^4y - 174y^3x^2 + 790y^3x^3 + 3440y^5x - 402y^4x^2 \\
& + 368iy^2x\sqrt{7} + 426iy^3x^2\sqrt{7} + 62ix^2y\sqrt{7} + 18ix^4y\sqrt{7} \\
& + 182iy^3x^3\sqrt{7} + 1680iy^4x\sqrt{7} + 22ix^5y\sqrt{7} + 1296xy^3.
\end{aligned}$$

(Note that some of the coefficients of f_3 are non-real.) This curve is irreducible, of degree 6, and has three $\mathbf{A}_6$-singularities located at $(0,0)$, $(0,1)$ and $(1,0)$

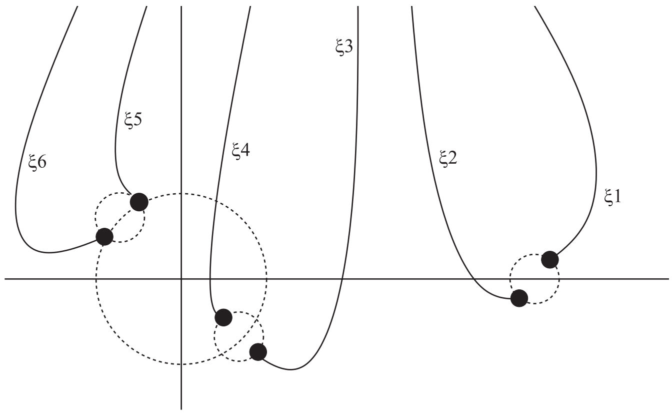

FIGURE 7. Generators at $y = \eta_4 + \varepsilon$

respectively. To prove that $\pi_1(\mathbf{CP}^2 \backslash C_3)$ is abelian, we apply here Zariski-van Kampen's theorem with the pencil given by the horizontal lines $L_\eta : y = \eta$, $\eta \in \mathbf{C}$. (We take $(1:0:0)$ as base point for the fundamental groups.) This pencil has 7 singular lines $L_{\eta_1}, \dots, L_{\eta_7}$ with respect to C_3, corresponding to the 7 complex roots

$$\eta_1 \approx -0.1288 - i0.2140, \quad \eta_2 \approx -0.0328 - i0.4507,$$
$$\eta_3 \approx -0.0158 + i0.0368, \quad \eta_4 = 0, \quad \eta_5 \approx 0.0139 + i0.0359,$$
$$\eta_6 = 1, \quad \eta_7 \approx 1.0778 - i0.0105$$

of the discriminant of f_3 as a polynomial in x. The lines L_{η_4} and L_{η_6} pass through singular points of C_3. All the other singular lines are tangent to the curve.

We consider the generic line $L_{\eta_4+\varepsilon}$ and take generators $\xi_1, \dots, \xi_6$ of $\pi_1(L_{\eta_4+\varepsilon} \backslash C_3)$ as in Figure 7, where $\varepsilon > 0$ is small enough. As above, to find the monodromy relations, we fix a 'standard' system of generators of $\pi_1(\mathbf{C} \backslash \{\eta_1, \dots, \eta_7\})$ with base point $\eta_4 + \varepsilon$. It turns out that we only need to determine the monodromy relations around the singular lines L_{η_4} and L_{η_6}.

The monodromy relations around L_{η_4} are given by

(4.1) $$\xi_3 = \xi_6 \xi_5 \xi_6^{-1}$$

(4.2) $$\xi_4 = (\xi_6 \xi_5) \cdot \xi_6 \cdot (\xi_6 \xi_5)^{-1}$$

(4.3) $$\xi_5 = (\xi_6 \xi_5 \xi_4 \xi_3) \cdot \xi_4 \cdot (\xi_6 \xi_5 \xi_4 \xi_3)^{-1}$$

(4.4) $$\xi_6 = (\xi_6 \xi_5 \xi_4 \xi_3) \cdot \xi_4 \xi_3 \xi_4^{-1} \cdot (\xi_6 \xi_5 \xi_4 \xi_3)^{-1}$$

(4.5) $$\xi_1 = (\xi_2 \xi_1)^3 \cdot \xi_2 \cdot (\xi_2 \xi_1)^{-3}.$$

(To find these relations and determine the exact position of the roots of $f_3(x, \eta_4 + \varepsilon)$, use the Puiseux parametrizations of the curve at $(0,0)$ and $(1,0)$ given by

$$y = t^4, \quad x = a_2 t^2 + a_4 t^4 + a_5 t^5 + \text{higher terms},$$
$$y = t^2, \quad x = 1 + a_2' t^2 + a_4' t^4 + a_6' t^6 + a_7' t^7 + \text{higher terms},$$

respectively, where

$$a_2 \approx 2.7472 - i2.1324,$$
$$a_5 \approx 1.9739 - i1.8813,$$
$$a_7' \approx 313.6754 + i208.7007.$$

(The values of the other coefficients are of no use.))

The monodromy relations around L_{η_6} are given by

(4.6) $$(\xi_2\xi_1\xi_2^{-1})^{-1} \cdot \xi_4 \cdot (\xi_2\xi_1\xi_2^{-1}) = \xi_6\xi_5\xi_6^{-1}$$

(4.7) $$\xi_2\xi_1\xi_2^{-1} = (\xi_6\xi_5) \cdot \xi_6 \cdot (\xi_6\xi_5)^{-1}$$

(4.8) $$\xi_5 = (\xi_6\xi_5\xi_4) \cdot \xi_2\xi_1\xi_2^{-1} \cdot (\xi_6\xi_5\xi_4)^{-1}$$

(4.9) $$\xi_6 = (\xi_6\xi_5\xi_4) \cdot (\xi_2\xi_1\xi_2^{-1}) \cdot \xi_4 \cdot (\xi_2\xi_1\xi_2^{-1})^{-1} \cdot (\xi_6\xi_5\xi_4)^{-1}.$$

(To see how the ξ_k's are deformed, when y moves on the real axis from $\eta_4 + \varepsilon$ to $\eta_6 - \varepsilon$, one may look at the position of the roots of the equation (in x)

$$f_3\left(x, (\eta_4 + \varepsilon) + \frac{n}{40}((\eta_6 - \varepsilon) - (\eta_4 + \varepsilon))\right) = 0$$

for $n = 0, 1, \ldots, 40$. To determine the exact position of the roots of $f_3(x, \eta_6 - \varepsilon)$, use the Puiseux parametrization of C_3 at $(0,1)$ given by

$$y = 1 + t^4, \quad x = a_2'' t^2 + a_4'' t^4 + a_5'' t^5 + \text{higher terms},$$

where $a_2'' \approx 1.6560 + i2.3963$ and $a_5'' \approx 0.8700 + i0.0641$.)

These relations are enough to conclude. (We do not need to calculate the monodromy relations around L_{η_j} for $j = 1, 2, 3, 5, 7$.) Indeed, by (4.2) and (4.7), we have

(4.10) $$\xi_4 = \xi_2\xi_1\xi_2^{-1}.$$

Putting into (4.6) gives $\xi_2\xi_1\xi_2^{-1} = \xi_6\xi_5\xi_6^{-1}$. Combined with (4.7), this new relation shows that $\xi_6 = \xi_5$. The latter, combined with (4.1) (respectively (4.2)), shows that $\xi_3 = \xi_6$ (respectively $\xi_4 = \xi_6$). Altogether, $\xi_6 = \xi_5 = \xi_4 = \xi_3$. Then the vanishing relation at infinity can be written as $\xi_2\xi_1 = \xi_6^{-4}$, while (4.10) turns into $\xi_6\xi_2 = \xi_2\xi_1$. This shows that $\xi_2 = \xi_6^{-5}$, and then $\xi_1 = \xi_6$. Finally, the group $\pi_1(\mathbf{CP}^2 \backslash C_3)$ is abelian.

References

[1] E. Artal Bartolo, Sur les couples de Zariski, J. Algebraic Geom. **3** (1994), 223–247.

[2] E. Artal Bartolo and J. Carmona Ruber, Zariski pairs, fundamental groups and Alexander polynomials, J. Math. Soc. Japan **50** (1998), 521–543.

[3] A. Degtyarev, Oka's conjecture on irreducible plane sextics, J. London Math. Soc. **78** (2008), 329–351.

[4] A. Degtyarev, On irreducible sextics with non-abelian fundamental group, Proceedings of Niigata-Toyama Conferences 2007, to appear in Adv. Stud. Pure Math.

[5] A. Degtyarev, Irreducible plane sextics with large fundamental groups, to appear in J. Math. Soc. Japan.

[6] A. Degtyarev, Fundamental groups of symmetric sextics, J. Math. Kyoto Univ. **48** (2008), 765–792.

[7] A. Degtyarev, Fundamental groups of symmetric sextics, II, to appear in Proc. London Math. Soc.

[8] A. Degtyarev, Oka's conjecture on irreducible plane sextics, II, to appear in J. Knot Theory Ramifications.

[9] A. Degtyarev, Plane sextics via dessins d'enfants, arXiv:0812.3258v1 [math.AG] 17 Dec. 2008.

[10] A. Degtyarev and M. Oka, A plane sextic with finite fundamental group, Proceedings of Niigata-Toyama Conferences 2007, to appear in Adv. Stud. Pure Math.

[11] C. Eyral and M. Oka, On the fundamental groups of the complements of plane singular sextics, J. Math. Soc. Japan **57** (2005), 37–54.

[12] C. Eyral and M. Oka, A proof of a conjecture of Degtyarev on non-torus plane sextics, Proceedings of Niigata-Toyama Conferences 2007, to appear in Adv. Stud. Pure Math.

[13] C. Eyral and M. Oka, Alexander-equivalent Zariski pairs of irreducible sextics, to appear in J. Topol.

[14] M. Oka, Symmetric plane curves with nodes and cups, J. Math. Soc. Japan **44** (1992), 375–414.

[15] M. Oka, Two transforms of plane curves and their fundamental groups, J. Math. Sci. Univ. Tokyo **3** (1996), 399–443.

[16] M. Oka, Flex curves and their applications, Geometriae Dedicata **75** (1999), 67–100.

[17] M. Oka, A new Alexander-equivalent Zariski pair, Acta Math. Vietnam. **27** (2002), 349–357.

[18] H. Tokunaga, A note on triple covers of $\mathbf{P}^2$, to appear.

[19] E. R. van Kampen, On the fundamental group of an algebraic curve, Amer. J. Math. **55** (1933), 255–260.

[20] O. Zariski, On the problem of existence of algebraic functions of two variables possessing a given branch curve, Amer. J. Math. **51** (1929), 305–328.

[21] O. Zariski, On the irregularity of cyclic multiple planes, Annals of Math. **32** (1931), 445–489.

[22] O. Zariski, The topological discriminant group of a Riemann surface of genus p, Amer. J. Math. **59** (1937), 335–358.

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