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PSEUDO-JACOBI OPERATORS AND OSSERMAN LIGHTLIKE HYPERSURFACES

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Abstract

We study pseudo-Jacobi operators associated to algebraic curvature maps on lightlike hypersurfaces M and investigate conditions for an induced Riemann curvature tensor to be an algebraic curvature map on M. Two examples are provided with explicit determination of their pseudo-Jacobi operators. Finally, we introduce the notion of lightlike Osserman hypersurfaces and prove some characterization results.

1. Introduction

One of the most important and central concepts in differential geometry is the notion of curvature. As it is well known [11], Jacobi and Szabó operators have been extremely useful in their study. Let (M, g) be a semi-Riemannian manifold, $p \in M$. $F \in \otimes^4 T_p^\star M$ is said to be an *algebraic curvature map* (*tensor*) on T_pM if it satisfies the following symmetries:

$$
\begin{aligned}
&F(x, y, z, w) = -F(y, x, z, w) = F(z, w, x, y), \\
&F(x, y, z, w) + F(y, z, x, w) + F(z, x, y, w) = 0.
\end{aligned} \tag{1}
$$

The Riemann curvature tensor R is an algebraic curvature tensor on the tangent space T_pM, for every $p \in M$. For an algebraic curvature map F on T_pM, the associated Jacobi operator J is the linear map on T_pM characterized by the identity

$$
g(J(x)y, z) = F(y, x, x, z). \tag{2}
$$

$J(x)$ is a self-adjoint map and F is spacelike (resp. timelike) Osserman tensor if Spec$\{J\}$ is constant on the pseudo-sphere of unit spacelike (resp. unit timelike) vectors in T_pM. These are equivalent notions and such a tensor is called an Osserman tensor. The basic problem is to what extent general sectional curvatures can provide information on the curvature and metric tensors. Osserman

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condition has been under much scrutiny in recent years and we refer to [11] for an extensive bibliography.

Since any semi-Riemannian manifold has lightlike subspaces, we reasonably expect a role of Jacobi and Szabó type operators in the study of lightlike manifolds. But, on the latter, the degenerate metric tensor has a non-trivial kernel so (2) is not well defined in the usual way. Also, in general, induced Riemann curvature tensors on lightlike manifolds are not algebraic curvature tensors, i.e (1) does not hold. Therefore, it is our first objective to find conditions on a lightlike hypersurface to have an induced algebraic Riemann curvature tensor (Theorems 3.1 and 3.2) so that (1) holds.

Secondly, we introduce and study a class of lightlike Osserman hypersurfaces. We first observe that, as in semi-Riemannian case, being spacelike or timelike Osserman are equivalent notions (Theorem 4.1) and under some embedding conditions, being Osserman at a point $p \in M$ sometimes reduces to being Osserman for the semi-Riemannian screen leaf through this point (Theorem 4.2). Also, we show that a totally umbilical lightlike hypersurface is locally Einstein and pointwise Osserman (Theorem 4.3).

2. Preliminaries

Let (M, g) be a hypersurface of an $(n+2)$-dimensional semi-Riemannian manifold $(\overline{M}, \bar{g})$ of constant index $0 < v < n+2$. In the classical theory of nondegenerate hypersurfaces, the normal bundle has trivial intersection $\{0\}$ with the tangent one and plays an important role in the introduction of main induced geometric objects of M. In case of lightlike hypersurfaces, the situation is totally different. The normal bundle $TM^{\perp}$ is a rank-one distribution over M: $TM^{\perp} \subset TM$ and then coincides with the radical distribution $\operatorname{Rad} TM = TM \cap TM^{\perp}$. Hence, the induced metric tensor g is degenerate with constant rank n.

A complementary bundle of $\operatorname{Rad} TM$ in TM is a rank n nondegenerate distribution over M, called a *screen distribution* of M, denoted by $S(TM)$. Existence of $S(TM)$ is secured provided M be paracompact. A lightlike hypersurface with a specific screen distribution is denoted by $(M, g, S(TM))$.

It is well-known [9] that for such a triplet, there exists a unique vector sub bundle $\operatorname{tr}(TM)$ of rank 1 over M, such that for any non-zero section ξ of $TM^{\perp}$ on a coordinate neighborhood $\mathscr{U} \subset M$, there exists a unique section N of $\operatorname{tr}(TM)$ on $\mathscr{U}$ satisfying

$$(3) \qquad \bar{g}(N, \xi) = 1, \quad \bar{g}(N, N) = \bar{g}(N, W) = 0, \quad \forall W \in \Gamma(ST(M)|_{\mathscr{U}}).$$

TM and $T\overline{M}$ are decomposed as follows:

$$(4) \qquad TM = S(TM) \overset{\perp}{\oplus} TM^{\perp},$$

$$(5) \qquad T\overline{M}|_M = TM \oplus \operatorname{tr}(TM).$$

We denote by $\Gamma(E)$ the $\mathscr{F}(M)$-module of smooth sections of a vector bundle E over M, $\mathscr{F}(M)$ being the algebra of smooth functions on M. Also, all manifolds are supposed to be smooth, paracompact and connected.

The induced connection, say ∇, on M is defined by

$$\nabla_X Y = Q(\overline{\nabla}_X Y), \quad \forall X, Y \in \Gamma(TM),$$

where $\overline{\nabla}$ denotes the Levi-Civita connection on $(\overline{M}, \bar{g})$ and Q the projection morphism on TM with respect to the decomposition (4). Notice that ∇ depends on both g and a screen distribution $S(TM)$ of M.

Let P be the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ with respect to the decomposition (5). The local Gauss and Weingarten type formulas are given by

$$\begin{aligned} \overline{\nabla}_X Y &= \nabla_X Y + B(X, Y)N, \\ \overline{\nabla}_X N &= -A_N X + \tau(X)N, \\ \nabla_X PY &= \overset{\star}{\nabla}_X PY + C(X, PY)\xi, \\ \nabla_X \xi &= -\overset{\star}{A}_\xi X - \tau(X)\xi, \quad \forall X, Y \in \Gamma(TM|_{\mathscr{U}}), \end{aligned} \tag{6}$$

where B and C are the local second fundamental forms on $\Gamma(TM)$ and $\Gamma(S(TM))$, respectively, $\overset{\star}{\nabla}$ is a metric connection on $\Gamma(S(TM))$, $\overset{\star}{A}_\xi$ the local shape operator on $S(TM)$ and τ a 1-form on TM. Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $TM/\operatorname{Rad} TM$. As per [9, page 83], although the second fundamental form B of M is independent of the choice of a screen distribution, but, it depends on the choice of N. Also, B satisfies for all $X, Y \in \Gamma(TM)$,

$$B(X, \xi) = 0 \quad \text{and} \quad B(X, Y) = g(\overset{\star}{A}_\xi X, Y).$$

It is important to mention that there are a large classes of lightlike hypersurfaces with canonical screen distribution [1, 3, 8, 9, 10].

Definition 2.1. A lightlike hypersurface $(M, g, S(TM))$ of a semi-Riemannian manifold $(\overline{M}, \bar{g})$ is called null transversally closed if its transversal lightlike bundle $\operatorname{tr}(TM)$ is parallel along the radical direction, that is

$$\overline{\nabla}_U V \in \operatorname{tr}(TM), \quad \forall U \in \operatorname{Rad} TM \quad \text{and} \quad V \in \operatorname{tr}(TM).$$

Definition 2.2. $(M, g, S(TM))$ is called screen conformal [3] if on any coordinate neighborhood $\mathscr{U} \subseteq M$ and for any normalizing pair $\{\xi, N\}$ there exists a non-vanishing smooth function φ on $\mathscr{U}$ such that $A_N = \varphi \overset{\star}{A}_\xi$.

Denote by $\overline{R}$ and R the Riemann curvature tensors of $\overline{\nabla}$ and ∇, respectively. Recall the following Gauss-Codazzi equations [9, p. 93]

$$\langle \bar{R}(X,Y)Z,\xi\rangle = (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z), \tag{7}$$

$$\langle \bar{R}(X,Y)Z,PW\rangle = \langle R(X,Y)Z,PW\rangle + B(X,Z)C(Y,PW) - B(Y,Z)C(X,PW), \tag{8}$$

$$\langle \bar{R}(X,Y)\xi,N\rangle = \langle R(X,Y)\xi,N\rangle = C(Y,\overset{\star}{A}_\xi X) - C(X,\overset{\star}{A}_\xi Y) - 2\,d\tau(X,Y), \quad \forall X,Y,Z,W \in \Gamma(TM|_{\mathscr{U}}). \tag{9}$$

Finally, we recall from [2] the following results. Consider on M a normalizing pair $\{\xi, N\}$ satisfying (3) and define the one-form

$$\eta(\bullet) = \bar{g}(N,\bullet).$$

For all $X \in \Gamma(TM)$, $X = PX + \eta(X)\xi$ and $\eta(X) = 0$ if and only if $X \in \Gamma(S(TM))$. Now, we define $\flat$ by

$$\begin{aligned} \flat : \Gamma(TM) &\to \Gamma(T^*M) \\ X &\mapsto X^\flat = g(X,\bullet) + \eta(X)\eta(\bullet). \end{aligned} \tag{10}$$

Clearly, such a $\flat$ is an isomorphism of $\Gamma(TM)$ onto $\Gamma(T^*M)$, and generalize the usual nondegenerate theory. In the latter case, $\Gamma(S(TM))$ coincides with $\Gamma(TM)$, and as a consequence the 1-form η vanishes identically and the projection morphism P becomes the identity map on $\Gamma(TM)$. We let $\sharp$ denote the inverse of the isomorphism $\flat$ given by (10). For $X \in \Gamma(TM)$ (resp. $\omega \in T^*M$), $X^\flat$ (resp. $\omega^\sharp$) is called the dual 1-form of X (resp. the dual vector field of ω) with respect to the degenerate metric g. It follows from (10) that if ω is a 1-form on M, we have for $X \in \Gamma(TM)$,

$$\omega(X) = g(\omega^\sharp, X) + \omega(\xi)\eta(X).$$

Define a $(0,2)$-tensor $\tilde{g}$ by $\tilde{g}(X,Y) = X^\flat(Y)$, $\forall X,Y \in \Gamma(TM)$. Clearly, $\tilde{g}$ defines a non-degenerate metric on M which plays an important role in defining the usual differential operators *gradient*, *divergence*, *Laplacian* with respect to degenerate metric g on lightlike hypersurfaces (details be seen in [2]). Also, observe that $\tilde{g}$ coincides with g if the latter is non-degenerate. The $(0,2)$-tensor $g^{[\cdot,\cdot]}$, inverse of $\tilde{g}$ is called *the pseudo-inverse of* g. With respect to the quasi orthonormal local frame field $\{\partial_0 := \xi, \partial_1, \dots, \partial_n, N\}$ adapted to the decompositions (4) and (5) we have

$$\begin{aligned} \tilde{g}(\xi,\xi) = 1, \quad \tilde{g}(\xi,X) = \eta(X), \\ \tilde{g}(X,Y) = g(X,Y) \quad \forall X,Y \in \Gamma(S(TM)). \end{aligned} \tag{11}$$

Let (M,g) be a lightlike hypersurface and $p \in M$. We denote

$$\mathscr{S}_p^-(M) = \{x \in T_pM \mid g(x,x) = -1\}$$

$$\mathscr{S}_p^+(M) = \{x \in T_pM \mid g(x,x) = 1\}$$

$$\mathscr{S}_p(M) = \{x \in T_pM \mid |g(x,x)| = 1\} = \mathscr{S}_p^-(M) \cup \mathscr{S}_p^+(M)$$

3. Pseudo-Jacobi operators

Let us start by an intrinsic interpretation of relation (2) which in semi-Riemannian setting characterizes the Jacobi operator J associated to an algebraic curvature map $R \in \otimes^4 T_p^\star M$, $(p \in M)$. Indeed, we have equivalently for x in the unit bundle, y, w in T_pM,

$$(J_R(x)y)^{\flat_g}(w) = R(y,x,x,w), \quad \text{that is,} \tag{12}$$

$$J_R(x)y = R(y,x,x,\bullet)^{\sharp_g}, \tag{13}$$

where $\flat_g$ and $\sharp_g$ are the usual isomorphisms between T_pM and its dual $T_p^\star M$, for a non-degenerate g. As stated above, the metric g and its associated metric $\tilde{g}$ coincide if the former is nondegenerate, and equivalently, relation (13) can be written in the form

$$\tilde{g}(J_R(x)y,w) = R(y,x,x,w),$$

in which $J_R(x)y$ is well defined. This leads to the following definition.

Definition 3.1. Let $(M,g,S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\overline{M},\bar{g})$, $p \in M$, $x \in \mathscr{S}_p(M)$ and $R \in \otimes^4 T_p^\star M$ an algebraic curvature map. By pseudo-Jacobi operator of R with respect to x, we call the self-adjoint linear map $J_R(x)$ of $x^\perp$ defined by

$$J_R(x)y = R(y,x,x,\bullet)^{\sharp_g},$$

where $\sharp_g$ denotes the dual isomorphism on the triplet $(M,g,S(TM))$.

Contrary to non-null hypersurfaces, the induced Riemann curvature tensor of a lightlike hypersurface $(M,g,S(TM))$ may not be an algebraic curvature tensor. For this, we prove the following.

Theorem 3.1. *Let $(M,g,S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\overline{M},\bar{g})$. If the induced Riemann curvature tensor of M is an algebraic curvature tensor, then, locally at least one of the following holds*

(a) *M is totally geodesic.*

(b) *M is null transversally closed.*

Proof. Let $\overline{\nabla}$ and ∇ be the Levi-Civita connection on $\overline{M}$ and the induced connection on M, respectively. Denote by $\overline{R}$ and R the curvature tensors of $\overline{\nabla}$ and ∇, respectively. Let indices $\alpha,\beta,\gamma,\ldots$ range from 0 to n; $i,j,k,\ldots$ from 1 to n and $A,B,C,\ldots$ from 0 to $n+1$, where $\dim(\overline{M}) = n+2$. Consider on $\overline{M}$ a local frame $\left\{\xi = \frac{\partial}{\partial u^0}, \frac{\partial}{\partial u^i}, N\right\}$ such that $\left\{\xi = \frac{\partial}{\partial u^0}, \frac{\partial}{\partial u^i}\right\}$ be a frame on M. Denote ∂_A for $\frac{\partial}{\partial u^A}$. Using the local expressions of $\overline{R}$ and R (see [9, page 96]) and (6) leads to the following relations.

$$R_{ijkh} = \bar{R}_{ijkh} + C_{ih}B_{jk} - C_{ik}B_{jh}, \tag{14}$$

$$R_{ij0h} = \bar{R}_{ij0h} - C_i B_{jh}. \tag{15}$$

where by definition $R_{ijkh} = g(R(\partial_h, \partial_k)\partial_j, \partial_i)$ and $B_{ij} = B(\partial_i, \partial_j)$, $C_{ij} = C(\partial_i, \partial_j)$, $C_i = C(\xi, \partial_i)$ are the components of second fundamental forms of M and $S(TM)$. Thus, the 4-tensor $R \in \otimes^4 T_p^*M$ does not have the usual curvature tensor symmetries as in the semi-Riemannian setting. Assume that the induced curvature tensor R defines an algebraic curvature map and consider relation (15). We have

$$R_{ij0h} = \bar{R}_{ij0h} - C_i B_{jh} = -\bar{R}_{ji0h} - C_i B_{jh} = -(\bar{R}_{ji0h} + C_i B_{jh}).$$

Thus,

$$R_{ij0h} = -R_{ji0h} \Leftrightarrow C_i B_{jh} = -C_j B_{ih}, \quad 1 \le i, j, h \le n.$$

Using the symmetry of B and (3) leads to

$$\begin{aligned} C_i B_{jh} &= -C_j B_{ih} = -C_j B_{hi} = C_h B_{ji} \quad \text{(From (3))} \\ &= C_h B_{ij} = -C_i B_{hj} = -C_i B_{jh} \quad \forall 1 \le i, j, h \le n. \end{aligned}$$

Hence,

$$C_i B_{jh} = 0, \quad \forall\ \forall i, j, h, \tag{16}$$

that is, $C_i = 0\ \forall i$ or $B_{jh} = 0\ \forall j, h$. Since $B(\xi, \bullet) = 0$, $B_{jh} = 0\ \forall j, h$ leads to M totally geodesic. Now, assume that in (16) there exist h_0 and j_0 such that $B_{j_0 h_0} \neq 0$. Then $C_i = 0\ \forall i$. This leads to the following:

$$\begin{aligned} C_i = C(\xi, \partial_i) &= \bar{g}(\nabla_\xi \partial_i, N) = \bar{g}(\bar{\nabla}_\xi \partial_i, N) \\ &= -\bar{g}(\partial_i, \bar{\nabla}_\xi N), \quad \forall 1 \le i \le n. \end{aligned}$$

Hence, null transversally closed condition (see Definition 2.1) is equivalent to $C_i = 0\ \forall i$ and the proof is complete. □

Note. As a trivial case if both C_i and B_{ij} in (16) vanish, then, both (a) and (b) may hold simultaneously.

Observe that a large number of lightlike hypersurfaces of Lorentzian manifolds do have integrable screen distributions [3, 8, 9, 10]. So it seems reasonable to prove the following characterization result.

THEOREM 3.2. *Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$, with non totally geodesic integrable screen distribution $S(TM)$. Then, the induced Riemann curvature tensor of M defines an algebraic curvature map if and only if M is either totally geodesic or locally screen conformal, with ambient holonomy condition*

$$\bar{R}(X, PY)(\text{Rad}\ TM) \subset \text{Rad}\ TM \quad \forall X, Y \in \Gamma(TM).$$

Proof. Assume the induced Riemann curvature tensor defines an algebraic curvature map and consider relation (14). We have

$$\begin{aligned} R_{ijkh} &= \bar{R}_{ijkh} + C_{ih}B_{jk} - C_{ik}B_{jh} \\ &= -\bar{R}_{jikh} + C_{ih}B_{jk} - C_{ik}B_{jh} \\ &= -(\bar{R}_{jikh} + C_{ik}B_{jh} - C_{ih}B_{jk}). \end{aligned}$$

Thus,

$$R_{ijkh} = -R_{jikh} \Leftrightarrow C_{ik}B_{jh} - C_{ih}B_{jk} = C_{jh}B_{ik} - C_{jk}B_{ih}.$$

Also,

$$\begin{aligned} R_{ijkh} &= \bar{R}_{ijkh} + C_{ih}B_{jk} - C_{ik}B_{jh} \\ &= \bar{R}_{khij} + C_{ih}B_{jk} - C_{ik}B_{jh}. \end{aligned}$$

So,

$$R_{ijkh} = R_{khij} \Leftrightarrow C_{ih}B_{jk} - C_{ik}B_{jh} = C_{kj}B_{hi} - C_{ki}B_{hj}.$$

Then, since $S(TM)$ is integrable, C is symmetric and we have

$$C_{jk}B_{ih} = C_{ih}B_{kj}, \quad \forall i, j, k, h.$$

Now, we distinguish two cases: M is totally geodesic or not. If M is totally geodesic then M is not screen conformal since $C \neq 0$. If M is not totally geodesic, there exist j_0 and k_0 such that $B_{j_0k_0} \neq 0$. Then, we have

$$C_{ih} = \frac{C_{k_0j_0}}{B_{k_0j_0}} B_{ih}, \quad \forall i, h.$$

Observe that $C_{k_0j_0} \neq 0$ otherwise C would vanish identically at some $p \in M$. Also, by continuity $B_{k_0j_0}$ is nonzero in a neighborhood $\mathscr{U}$ of p in M. Define locally the function φ on $\mathscr{U}$ by $\varphi(x) = \dfrac{C_{k_0j_0}}{B_{k_0j_0}}(x)$. Then $C(X, Y) = \varphi B(X, Y)$ for all X, Y in $\Gamma(S(TM|_{\mathscr{U}}))$, which is equivalent to $A_N X = \varphi \overset{\star}{A}_\xi X$, for all X, Y in $\Gamma(S(TM|_{\mathscr{U}}))$. Finally note that $\overset{\star}{A}_\xi \xi = 0$. Also, since M is non totally geodesic, it is null transversally closed, that is $A_N \xi = 0$. Thus $A_N X = \varphi \overset{\star}{A}_\xi X$ for all X, Y in $\Gamma(TM|_{\mathscr{U}})$, that is, M is screen locally conformal (see Definition 2.2). In addition,

$$\begin{aligned} \langle \bar{R}(X, PY)\xi, Z \rangle &= -\langle \bar{R}(Z, \xi)X, PY \rangle \overset{(8)}{=} -\langle R(Z, \xi)X, PY \rangle \\ &\quad - B(Z, X)C(\xi, PY) + B(\xi, X)C(Z, PY) \\ &= -\langle R(Z, \xi)X, PY \rangle = -\langle R(X, PY)Z, \xi \rangle = 0. \end{aligned}$$

Thus, $\bar{R}(X, PY)$ Rad $TM \subset$ Rad TM.

Conversely, assume that M is either totally geodesic or screen locally conformal with required ambient holonomy condition. Observe that the first

Bianchi identity is straightforward. Also, if M is totally geodesic there is nothing more to prove since $\bar{R}|_{TM} = R$. Now we consider M to be screen conformal with $B \neq 0$ and show that R defines an algebraic curvature map. From (8) we have for $X, Y, Z \in \Gamma(TM)$ and $W \in S(TM)$,

$$\langle \bar{R}(X,Y)Z, W\rangle = \langle R(X,Y)Z, W\rangle + B(X,Z)C(Y,W) - B(Y,Z)C(X,W).$$

Thus, since $C(X,W) = \varphi B(X,W)$, above equation becomes

$$\langle \bar{R}(X,Y)Z, W\rangle = \langle R(X,Y)Z, W\rangle + \varphi[B(X,Z)B(Y,W) - B(Y,Z)B(X,W)].$$

Put

$$\mathscr{B}(X,Y,Z,W) = B(X,Z)B(Y,W) - B(Y,Z)B(X,W).$$

We have

$$\langle R(X,Y)Z, W\rangle = \langle \bar{R}(X,Y)Z, W\rangle - \varphi\mathscr{B}(X,Y,Z,W), \tag{17}$$

and it is straightforward that $\mathscr{B}$ has the required symmetries. So the left hand side of (17) has the required symmetries. For the right hand side, first we have $\langle R(X,Y)Z, \xi\rangle = -\langle R(Y,X)Z, \xi\rangle = 0$. Now,

$$\begin{aligned}\langle R(Z,\xi)X, Y\rangle &= \langle R(Z,\xi)X, PY\rangle \\ &= \langle \bar{R}(Z,\xi)X, PY\rangle - B(Z,X)C(\xi,PY) + B(\xi,X)C(Z,PY), \\ &= \langle \bar{R}(Z,\xi)X, PY\rangle = -\langle \bar{R}(X,PY)\xi, Z\rangle = 0,\end{aligned}$$

by ambient holonomy condition and the proof is complete. □

Example 1. A simple but basic example is that of the lightlike cone $\bigwedge_0^{n+1}$ at the origin of $\mathbf{R}_1^{n+2}$ for which the null transversal normalization

$$N = \frac{1}{2(x_0)^2}\left[-x^0\frac{\partial}{\partial x_0} + \sum_{a=1}^{n+1} x^a \frac{\partial}{\partial x^a}\right]$$

induces the algebraic curvature tensor

$$R(X,Y)Z = \frac{1}{2(x_0)^2}[g(Y,Z)PX - g(X,Z)PY],$$

where P is the projection morphism on the screen associated to N. Its associated pseudo-Jacobi operator is then given, for $x \in \mathscr{S}_p(\bigwedge_0^{n+1})$, by

$$J_R(x) = \frac{1}{2(x_0)^2}\langle x, x\rangle \circ P.$$

Example 2. Consider (M, g) a real lightlike hypersurface of an indefinite almost Hermitian manifold $(\bar{M}, \bar{g}, \bar{J})$, where $\bar{g}$ is a semi-Riemannian metric of constant index [9, Chapter 6, p. 194]. It is easy to check that $\{\xi, N\}$ being a normalizing pair verifying (3), $\bar{J}(TM^{\perp} \oplus \bar{J}\,\mathrm{tr}(TM))$ is a vector sub bundle of $S(TM)$ of rank 2 with hyperbolic fibers. Then,

$$S(TM) = \{\bar{J}(TM^{\perp}) \oplus \bar{J}(\mathrm{tr}\; TM)\} \overset{\perp}{\oplus} D_0,$$

with D_0 a non-null almost complex distribution with respect to $\bar{J}$. Thus,

$$TM = \{\bar{J}(TM^{\perp}) \oplus \bar{J}(\mathrm{tr}\; TM)\} \overset{\perp}{\oplus} D_0 \overset{\perp}{\oplus} TM^{\perp}. \tag{18}$$

Now, consider the almost complex distribution

$$D = \{TM^{\perp} \overset{\perp}{\oplus} \bar{J}(TM^{\perp})\} \overset{\perp}{\oplus} D_0,$$

and let $\mathscr{S}$ denote the projection morphism of TM on D. Put $U = -\bar{J}N$ and $V = -\bar{J}\xi$. Then, for all $X \in TM$,

$$X = \mathscr{S}X + u(X)U, \tag{19}$$

with $u = g(\cdot, V)$ a local 1-form on M. It follows that

$$\bar{J}X = FX + u(X)N,$$

with $F = \bar{J} \circ \mathscr{S}$. Clearly, we have

$$F^2X = -X + u(X)U, \quad u(U) = 1.$$

Thus, provided ξ and N be globally defined on M, (F, u, U) defines an almost contact structure on M [9, p. 195].

We construct an algebraic curvature map R^F on M using F as follows:

$$\begin{aligned} R^F(x, y, z, w) = {} & (\langle Fy, z\rangle + \langle y, Fz\rangle)(\langle Fx, w\rangle + \langle x, Fw\rangle) \\ & - (\langle Fx, z\rangle + \langle x, Fz\rangle)(\langle Fy, w\rangle + \langle y, Fw\rangle), \end{aligned} \tag{20}$$

for all $x, y, z, w \in T_pM$, $p \in M$. It is easy to check that such a R^F is an algebraic curvature map on M. Put $v = \langle \bullet, U\rangle$ and get

$$\langle Fx, y\rangle + \langle x, Fy\rangle = u(x)v(Fy) + u(y)v(Fx). \tag{21}$$

Now, we compute the pseudo-Jacobi operator $J_{R^F}(x)$ for $x \in \mathscr{S}_p(M)$, $p \in M$. We have for all y in $x^{\perp}$,

$$\begin{aligned} J_{R^F}(x)y &= R^F(y, x, x, \bullet)^{\sharp_g} \\ &= [2\langle Fx, x\rangle(\langle Fy, \bullet\rangle + \langle y, F(\bullet)\rangle) \\ &\quad - (\langle Fy, x\rangle + \langle y, Fx\rangle)(\langle Fx, \bullet\rangle + \langle x, F(\bullet)\rangle)]^{\sharp_g}. \end{aligned}$$

Then, using (21) leads to

$$\begin{aligned} J_{R^F}(x)y = {} & 2u(x)v(Fx)[u(y)(v \circ F)^{\sharp_g} + v(Fy)u^{\sharp_g}] \\ & - (u(x)v(Fy) + u(y)v(Fx))[u(x)(v \circ F)^{\sharp_g} + v(Fx)u^{\sharp_g}]. \end{aligned}$$

Using (19) and the Hermitian structure of $(\overline{M}, \bar{g}, \bar{J})$ we obtain $v \circ F = -\eta$. Then, $(v \circ F)^{\sharp_g} = -\xi$. Also, since $u(\xi) = 0$, we have $u^{\sharp_g} = -\bar{J}\xi$. Thus,

$$J_{R^F}(x)y = u(x)[\eta(x)u(y) - \eta(y)u(x)]\xi + \eta(x)[\eta(x)u(y) - \eta(y)u(x)]\bar{J}\xi.$$

This implies

$$J_{R^F}(x) = [\eta(x)u(\bullet) - \eta(\bullet)u(x)](u(x)\xi + \eta(x)\bar{J}\xi)$$

Observe that the pseudo-Jacobi operator $J_{R^F}(x)$, $x \in \mathscr{S}_pM$, has values in the holomorphic plane $TM^\perp \overset{\perp}{\oplus} \bar{J}(TM^\perp)$.

Remarks. (a) If the screen distribution $S(TM)$ is integrable and for any $x \in \mathscr{S}_p(M)$, $(p \in M)$, $J_R(x)$ preserves the radical distribution, then, since g and $\tilde{g}$ coincide on $S(TM)$, relation (12) (or equivalently (13)) shows that the pseudo-Jacobi operator J_R induces a Jacobi operator $J_{R'}$ on $(M', g' = g|_{M'})$, where M' is a leaf of $S(TM)$ and R' the restriction on $S(TM)$ of R.

(b) Let R be the induced (algebraic) Riemann curvature tensor of M, $(p \in M)$ and $\xi \in \mathrm{Rad}\, T_pM$. Then, we have

$$J_R(x)\xi = 0. \tag{22}$$

Indeed, for all $x \in \mathscr{S}_p(M)$, $z \in T_pM$,

$$\tilde{g}(J_R(x)\xi, z) = R(\xi, x, x, z) = g(R(z, x)x, \xi) = 0,$$

and since $\tilde{g}$ is non-degenerate on TM, we have $J_R(x)\xi = 0$.

Note also that the screen subspace is preserved by J_R. For this, it suffices to show that for all $x \in \mathscr{S}_p(M)$, $z \in S(T_pM)$, $\eta(J_R(x)z) = 0$ which is equivalent to $\tilde{g}(J(x)z, \xi) = 0$ using (11). But

$$\tilde{g}(J_R(x)z, \xi) = R(z, x, x, \xi) \overset{def.}{=} g(R(\xi, x)x, z) \overset{(1)}{=} -g(R(x, z)x, \xi) = 0.$$

4. Lightlike Osserman hypersurfaces

By the approach developed in this paper (following [9]), the extrinsic geometry of lightlike hypersurfaces (M, g) depends on a choice of screen distribution, or equivalently, normalization. Since the screen distribution is not uniquely determined, a well defined concept of Osserman condition is not possible for an arbitrary lightlike hypersurface of a semi-Riemannian manifold. Thus, one must look for a class of normalization for which the induced Riemann curvature and associated Jacobi operator has the desired symmetries and properties. In short, we precise the following.

Definition 4.1. A screen distribution $S(TM)$ is said to be admissible if the associated induced Riemann curvature is an algebraic curvature.

Examples. Based on Theorem 3.2, we observe that any screen conformal lightlike hypersurface [3] in a semi-Euclidean space admits an admissible screen distribution since its induced curvature tensor defines an algebraic curvature map. In particular, the canonical screens on the lightlike cones, Monge hypersurfaces and totally geodesic lightlike hypersurfaces all of them admit admissible screens.

On the other hand, there are large classes of such hypersurfaces (including above examples) with canonical screen as follows:

(a) Duggal and Bejancu constructed in [9, Chapter 4] a canonical screen distribution in semi-Euclidean spaces which provided significant geometrically and physically results on its lightlike hypersurfaces. Moreover, they proved that a canonical screen distribution is integrable (a desirable property) on any lightlike hypersurface of a Minkowski space $\mathbf{R}_1^n$ and on any lightlike cone $\bigwedge_{q-1}^n$ of $\mathbf{R}_q^{n+1}$.

(b) We know from [3, Theorem 2] that any screen locally conformal lightlike hypersurface (see Definition 2.2) admits an integrable screen distribution $S(TM)$. Using this result, Atindogbe and Duggal (same paper [3]) constructed a canonical screen as follows:

Denote by $\mathscr{S}^1$ the first derivative of $S(TM)$ given by

$$\mathscr{S}^1(x) = \operatorname{span}\{[X, Y]|_x, \quad X_x, Y_x \in S(T_xM)\}, \forall x \in M.$$

As $S(TM)$ is integrable, $\mathscr{S}^1$ is a subbundle of $S(TM)$. They proved (see [3, Theorem 5]) that if $\mathscr{S}^1$ coincide with $S(TM)$ then there is a unique screen locally conformal distribution, up to an orthogonal transformation.

(c) Let $(\overline{M}, \bar{g})$ be an $(n+2)$-dimensional globally hyperbolic spacetime manifold, with the metric $\bar{g}$ given by [4]

$$ds^2 = -dt^2 + e^{\mu}(dx^1)^2 \oplus \bar{g}_{ab}\, dx^a dx^b, \quad 2 \leq a, b \leq n+1,$$

where μ is a function of t and x^1 alone. Recently, in [8], it has been shown that $\overline{M}$ admits a lightlike hypersurface with canonical screen up to an orthogonal transformation. Also, see [1] and [10] for the construction of invariant normalization and a unique distinguished structure for large classes of physically significant lightlike hypersurfaces of Lorentzian manifolds.

Based on above, one may ask whether there exists a general class of semi-Riemannian manifolds of an arbitrary signature which admit admissible canonical screen distributions. To answer this in affirmative, we first quote the following recent result.

Theorem Duggal [7]. *Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\overline{M}\bar{g})$. Let E be a complementary vector bundle of $TM^{\perp}$ in $S(TM)^{\perp}$ such that E admits a covariant constant timelike vector field. Then, with respect to a section ξ of* Rad TM, *M is screen conformal. Thus, M can admit an integrable canonical screen distribution.*

Consequently, there exist large classes of lightlke hypersurfaces of semi-Riemannian manifolds which admit admissible canonical screen distributions. Using this information, we make the following definition:

Definition 4.2. A lightlike hypersurface (M, g) of a semi-Riemannian manifold $(\overline{M}, \bar{g})$ of constant index is called timelike (resp. spacelike) Osserman at $p \in M$ if, for each admissible screen distribution $S(TM)$ and associate induced algebraic curvature R, the characteristic polynomial of $J_R(x)$ is independent of $x \in \mathscr{S}_p^-(M)$ (resp. $x \in \mathscr{S}_p^+(M)$). Moreover, if this holds at each $p \in M$, then (M, g) is called pointwise Osserman.

Based on discussion so far, it is clear that the above definition of Osserman condition is independent on the choice of admissible screen distribution. This conclusion is noteworthy for the entire study of the geometry of Osserman lightlike hypersurfaces.

Examples. (a) Being totally umbilical is independent on the choice of screen distribution. Now, for a given admissible screen on the lightlike cone $\bigwedge_0^{n+1}$ of $\mathbf{R}_1^{n+2}$, the induced curvature tensor is given by

$$R(X, Y)Z = \frac{1}{2(x^0)^2}[g(Y, Z)PX - g(X, Z)PY]$$

with P the projection morphism of the tangent bundle $T\bigwedge_0^{n+1}$ onto the screen distribution and the pseudo-Jacobi operator is given for $z \in S_p(\bigwedge_0^{n+1})$ by

$$J_R(z) = \frac{1}{2(x^0)^2}\langle z, z\rangle \circ P.$$

It follows that the characteristic polynomial is given by

$$f_z(t) = -t\left[\frac{\varepsilon}{2(x^0)^2} - t\right]^{n-1}, \quad \varepsilon = \operatorname{sign}(z) = \pm 1,$$

which is independent on both admissible screen distributions and $z \in S_p^-(\bigwedge_0^{n+1})$ (resp. $z \in S_p^+(\bigwedge_0^{n+1})$). The lightlike cone is then timelike (resp. spacelike) pointwise Osserman.

(b) Consider a real lightlike hypersurface of indefinite almost Hermitian manifold $(\overline{M}, \bar{g}, \bar{J})$ and suppose an admissible screen distribution induces a Riemann curvature R^F as in Eq. (20). We know that the pseudo-Jacobi operator J_{R^F} is given by

$$J_{R^F}(z) = [\eta(z)u(\bullet) - \eta(\bullet)u(z)](u(z)\xi + \eta(z)\bar{J}\xi),$$

and for $z \in S_p(M) \cap (\bar{J}(TM^\perp) \oplus \bar{J}(\operatorname{tr} TM))$, the characteristic polynomial reduces to

$$f_z(t) = (-1)^n(t + u(z)^2)t^{n-1},$$

which obviously depends on $z \in S_p^-(M)$ and $z \in S_p^+(M)$. Such a hypersurface is neither timelike nor spacelike Osserman at any $p \in M$. Observe that $u = -g(\cdot, \bar{J}\xi)$ is nonzero at every $p \in M$.

A straight adaptation of technique in [11, pp. 4–5] to the lightlike case shows that (M, g) being timelike Osserman at $p \in M$ is equivalent to (M, g) being spacelike Osserman at p. More precisely, we have

THEOREM 4.1. *Let (M, g) be a lightlike hypersurface of a semi-Riemannian manifold $(\overline{M}, \bar{g})$. Then, (M, g) is timelike Osserman at p if and only if it is spacelike Osserman at p.*

From now on we refer Osserman at p to both timelike and spacelike. Recall that the screen distribution $S(TM)$ is totally umbilical [9, page 109] if and only if on any coordinate neighborhood $\mathscr{U} \in M$, there exists a smooth function λ such that $C(X, PY) = \lambda g(X, Y)$, $\forall X, Y \in TM|_{\mathscr{U}}$. Then, since C is symmetric in $S(TM)$, it follows from [9, Theorem 23, page 89] that any totally umbilical $S(TM)$ is integrable. In case $\lambda = 0$ there is totally geodesic screen foliation on M. For this later case, the following holds.

THEOREM 4.2. *Let (M, g) be a lightlike hypersurface of a semi-Riemannian manifold $(\overline{M}, \bar{g})$, all of whose admissible screen distributions are totally geodesic in a neighborhood $\mathscr{U}$ of a $p \in M$. Then, (M, g) is Osserman at p if and only if the semi-Riemannian screen leaves are Osserman at this point. In particular, if $(\overline{M}, \bar{g})$ has constant index $\nu = 2$, then, (M, g) is Osserman at p if and only if semi-Riemannian admissible screen leaves are of constant sectional curvature at p.*

Proof. Consider a generic totally geodesic admissible screen distribution $S(TM)$ on $\mathscr{U} \subset M$. Let R, R' and $\overset{\star}{R}$ denote the algebraic curvature tensors induced on (M, g) by $S(TM)$, the restriction of R on $S(TM)$ and the Riemann curvature tensor given by the Levi-Civita connection $\overset{\star}{\nabla}$ on the screen distribution, respectively. We first show that under the hypothesis, we have $R' = \overset{\star}{R}$ at p. Let $x, y, z \in S(T_pM)$. By straightforward calculation using the last two equations of (6), we have

$$\begin{aligned} R'(x, y)z = R(x, y)z &= \overset{\star}{R}(x, y)z + [C(x, z)\overset{\star}{A}_\xi y - C(y, z)\overset{\star}{A}_\xi x] \\ &\quad + [(\nabla_x C)(y, z) - (\nabla_y C)(x, z) + \tau(y)C(x, z) - \tau(x)C(y, z)]\xi. \end{aligned}$$

Thus, we get $R'(x, y)z = \overset{\star}{R}(x, y)z$ from $C = 0$. Also, $x \in \mathscr{S}_p(M)$ if and only if $\overset{\star}{x} \in \mathscr{S}_p(M^\star)$, with $\overset{\star}{x} = Px$ and $M^\star$ the leaf of $S(TM)$ through p. Moreover, $x^\perp = (Px)^\perp$ and $J_R(x) = J_R(Px)$. We infer that $J_{\overset{\star}{R}}(\overset{\star}{x})$ is the restriction of $J_R(x)$ to $\overset{\star}{x}^{\perp_{S(TM)}}$. On the other hand, observe that

$$x^\perp = \overset{\star}{x}^{\perp_{S(TM)}} \overset{\perp}{\oplus} TM^\perp$$

and from (22) we have $J_R(x)\xi = 0$ for all $\xi \in \operatorname{Rad} TM$. Then, let $f_x(t)$ and $h_{\overset{\star}{x}}$ denote the characteristic polynomials of $J_R(x)$ $(x \in \mathscr{S}_p^-(M))$ and $J_{\overset{\star}{R}}(\overset{\star}{x})$ $(\overset{\star}{x} \in \mathscr{S}_p^-(M^\star))$ with $\overset{\star}{x} = Px$, respectively. We have

$$f_x(t) = th_{\overset{\star}{x}}(t)$$

which shows that the characteristic polynomial of $J_R(x)$ is independent of $x \in \mathscr{S}_p^-(M)$ if and only if the characteristic polynomial of $J_{\overset{\star}{R}}(\overset{\star}{x})$ is independent of $\overset{\star}{x} \in \mathscr{S}_p^-(M^\star)$. Hence, (M, g) is timelike Osserman at p if and only if $M^\star$ (as a semi-Riemannian manifold) is timelike Osserman at p. Similar is the case for $S_p^+(M)$ and $\overset{\star}{x} \in \mathscr{S}_p^+(M^\star)$. Since $S(TM)$ is an arbitrary admissible screen, the first part of the theorem is proved.

Now, assume that $(\overline{M}, \bar{g})$ has constant index $\nu = 2$. Then, T_pM is a degenerate space of signature $(0, -, +, \dots, +)$. It follows that screen leaves through p are Lorentzian manifolds. But it is well-known [11, p. 41] that the latter are Osserman at p if and only if they are constant sectional curvature at this point. This completes our proof. □

Observe that induced Ricci tensor on lightlike $(M, g, S(TM))$ is not necessarily symmetric. In an ambient space form, we prove the following:

Theorem 4.3. *Let (M, g) be a (proper) totally umbilical lightlike hypersurface of a $(n+2)$-dimensional $(n > 1)$ semi-Riemannian manifold of constant sectional curvature $(\overline{M}(c), \bar{g})$. Then, the set of admissible screens reduce to totally umbilical ones. Also, M is pointwise Osserman and for each admissible $S(TM)$,* $\mathrm{Ric}^{S(TM)}$ *is symmetric and M is locally Einstein.*

Proof. Recall that a lightlike hypersurface M is totally umbilical [9, page 107] if and only if on any coordinate neighborhood $\mathscr{U} \in M$, there exists a smooth function ρ such that $B(X, Y) = \rho g(X, Y)$, $\forall X, Y \in TM|_{\mathscr{U}}$. Equivalently, we have $\overset{\star}{A}_\xi X = \rho PX$ $\forall X \in TM|_{\mathscr{U}}$. In case $\rho \neq 0$ then M is proper totally umbilical. Then, it is known [9, page 108] that the induced Riemann curvature takes the form

$$(23)\quad R(X, Y)Z = c\{g(Y, Z)X - g(X, Z)Y\} + \rho\{g(Y, Z)A_NX - g(X, Z)A_NY\}.$$

Now pick an admissible screen $S(TM)$ and let R denote the associate induced curvature tensor. Then, due to $g(R(X, Y)Z, V) = g(R(Z, V)X, Y)$ for all X, Y, Z, V, we have

$$\begin{aligned}\rho\{g(Y, Z)g(A_NX, V) - g(X, Z)g(A_NY, V)\\ - g(V, X)g(A_NZ, Y) + g(Z, X)g(A_NV, Y)\} = 0\end{aligned}$$

$\forall X, Y, Z, V \in \Gamma(TM)$. Since $\rho \neq 0$, choose a $Z \perp X$ and $g(Y, Z) = 1$ to get

$$g(A_NX - g(A_NZ, Y)X, V) = 0$$

for all $X, V \in TM|_{\mathscr{U}}$. Thus, $A_NX = \lambda PX$ with $\lambda = g(A_NZ, Y)$, that is the screen distribution is totally umbilical.

Conversely, suppose $A_NX = \lambda PX$ for some smooth λ in $C^\infty(M)$. Then, (23) becomes

$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\} + \lambda\rho\{g(Y,Z)PX - g(X,Z)PY\}, \tag{24}$$

which defines an algebraic curvature map, that is, $S(TM)$ is admissible.

Now, let $S(TM)$ be an arbitrary admissible screen distribution on M. To compute the induced Ricci curvature with respect to $S(TM)$ using (24) we consider a quasi-orthonormal basis $\{\xi, W_1, \dots, W_n\}$ on $TM|_{\mathscr{U}}$. Then,

$$\begin{aligned}
\mathrm{Ric}(X,Y) &= \sum_{i=1}^{n} g(R(X,W_i)Y,W_i) + \bar{g}(R(X,\xi)Y,N) \\
&= c[g(X,Y) - ng(X,Y)] + \lambda\rho[g(X,Y) - ng(X,Y)] - cg(X,Y) \\
&= [(1-n)\lambda\rho - nc]g(X,Y).
\end{aligned}$$

Hence, the Ricci curvature is symmetric. Moreover M is locally Einstein. Finally, let $x \in \mathscr{S}_p(M)$, $p \in M$, $y \in x^{\perp}$. Then,

$$\begin{aligned}
J_R(x)y &= R(y,x,x,\cdot)^{\sharp_g} \\
&\overset{(24)}{=} [c\{g(x,x)g(\cdot,y) - g(\cdot,x)g(x,y)\}\lambda\rho\{g(x,x)g(\cdot,y) - g(\cdot,x)g(x,y)\}]^{\sharp_g} \\
&= (c+\lambda\rho)g(x,x)g(\cdot,y)^{\sharp_g} \\
&= (c+\lambda\rho)g(x,x)Py
\end{aligned}$$

Hence, in adapted quasi-orthonormal basis and using remark 4.1(c), matrix of $J_R(x)$ has the form

$$\begin{pmatrix}
0 & \cdots & \cdots & 0 \\
\vdots & & & \\
\vdots & (c+\lambda\rho)g(x,x)I_{n-1} & & \\
0 & & &
\end{pmatrix}.$$

Then, the characteristic polynomial f_x of $J_R(x)$ is given by

$$f_x(t) = -t[(c+\lambda\rho)g(x,x) - t]^{n-1},$$

with $g(x,x) = \pm 1$ and for arbitrary given admissible screen distribution. Thus M is pointwise Osserman, which completes the proof. □

COROLLARY 4.1. *A lightlike surface M of a 3-dimensional Lorentz manifold $\overline{M}(c)$ is pointwise Osserman if it is null transversally closed.*

Proof. It is well known [9, page 111] that any lightlike surface of a 3-dimensional Lorentz manifold $\overline{M}$ is either proper totally umbilical or totally geodesic. Hence, it remains only to find necessary and sufficient condition for existence of umbilical screen line bundle $S(TM)$ on M. As such a $S(TM)$ is non-degenerate, let $\lambda = \dfrac{C(W,W)}{g(W,W)}$ with $S(TM) = \mathrm{span}\{W\}$. Then $C(X,PY) =$

$\lambda g(X, Y)$ $\forall X, Y \in TM|_{\mathscr{U}}$ if and only if $C(\xi, W) = 0$, that is M is null transversally closed. $\square$

In semi-Riemannian case, we know [11] that being Osserman at a point simplifies the geometry at that point as the manifold is Einstein at that point. Moreover, if the latter is connected and of at least dimension 3, by Schur lemma 1 [5], it is Einstein. For lightlike hypersurface, this is not always the case as is shown in next theorem using the following lemma.

LEMMA 4.1. *Let $(M, g, S(TM))$ be a lightlike hypersurface of a $(n+2)$-dimensional semi-Riemannian manifold $(\overline{M}, \bar{g})$, with induced algebraic Riemannian curvature map R. For all $x \in S_p M$, $p \in M$ we have*

$$\operatorname{trace} J_R(x) = \operatorname{Ric}(x, x) - \eta(\overline{R}(\xi, x)x).$$

Proof. Let $(e_0 = \xi, e_1 = Px, e_2, \ldots, e_n, N)$ be a $\bar{g}$-quasi orthonormal basis of $T_p\overline{M}$ with $T_pM = \operatorname{span}\{(e_0, e_1, e_2, \ldots, e_n)\}$ and $S(T_pM) = \operatorname{span}\{(e_1, e_2, \ldots, e_n)\}$. We have

$$\begin{aligned}
\operatorname{trace} J_R(x) &= \sum_{\substack{\alpha=0\\ \alpha\neq 1}}^{n} g^{[\alpha\alpha]}\tilde{g}(J_R(x)e_\alpha, e_\alpha)\\
&= \sum_{\alpha=2}^{n} g^{[\alpha\alpha]}\tilde{g}(J_R(x)e_\alpha, e_\alpha) + g^{[00]}\tilde{g}(J_R(x)\xi, \xi)\\
&= \sum_{\alpha=2}^{n} g^{[\alpha\alpha]}R(e_\alpha, x, x, e_\alpha) = \sum_{\alpha=2}^{n} g^{[\alpha\alpha]}g(R(e_\alpha, x)x, e_\alpha)\\
&= g^{[e_1e_1]}g(R(x, x)x, x) + \sum_{\alpha=2}^{n} g^{[\alpha\alpha]}g(R(e_\alpha, x)x, e_\alpha)\\
&\quad + \bar{g}(R(\xi, x)x, N) - \bar{g}(R(\xi, x)x, N)\\
&= \operatorname{Ric}(x, x) - \eta(R(\xi, x)x) \stackrel{(9)}{=} \operatorname{Ric}(x, x) - \eta(\overline{R}(\xi, x)x). \qquad \blacksquare
\end{aligned}$$

THEOREM 4.4. Let (M, g) be a lightlike hypersurface that is Osserman at $p \in M$. If for an admissible screen distribution $S(TM)$, $R^{S(TM)}(\xi, \bullet)\xi$ is zero for a $\xi \in \operatorname{Rad} TM$, and $|\eta(\overline{R}(\xi, x)x)| < \mu \in \mathbf{R}$ for every $x \in S_p^-(M)$ (or every $x \in S_p^+(M)$), then $(M, g, S(TM))$ is Einstein at $p \in M$.

Proof. Let $\overline{M}$ be the ambient semi-Riemannian manifold of M. Denote by R' and g' the restriction on $S(TM)$ of the induced algebraic curvature tensor R and the metric tensor g on M, respectively. The Osserman condition at p implies that the characteristic polynomial of J_R is the same for every $x \in S_p^-(M)$ (or every $x \in S_p^+(M)$). Then $|\operatorname{trace} J_R(x)|$ is bounded on $S_p^-(M)$ (and $S_p^+(M)$).

Now, using Lemma 4.1, we have for every $x \in S_p^-(M)$ (or every $x \in S_p^+(M)$),

$$|\mathrm{Ric}(x,x)| \le |\mathrm{trace}\, J_R(x)| + |\eta(\overline{R}(\xi,x)x)|.$$

It follows that there exist $\alpha \in \mathbf{R}$ such that $|\mathrm{Ric}(x,x)| \le \alpha$ for every $x \in S_p^-(M)$ (or every $x \in S_p^+(M)$). In particular, we have

$$|\mathrm{Ric}'(x,x)| \le \alpha$$

for every $x \in S(T_pM) \cap S_p^-(M)$ (or every $x \in S(T_pM) \cap S_p^+(M)$). Therefore, since $(S(T_pM), g')$ is non-degenerate, it follows from a well known algebraic result (see [6]) that

$$\mathrm{Ric}'(x,y) = \lambda g'(x,y) \quad \forall x,y \in S(T_pM), \text{ with } \lambda \in \mathbf{R}. \tag{25}$$

Consider $(e_0 = \xi, e_1, \dots, e_n, N)$ a $\bar{g}$-quasi orthonormal basis of $T_p\overline{M}$ with $T_pM = \mathrm{span}\{(e_0, e_1, \dots, e_n)\}$ and $S(T_pM) = \mathrm{span}\{(e_1, \dots, e_n)\}$. We show that for all $x \in T_pM$, $\mathrm{Ric}(\xi,x) = \mathrm{Ric}(x,\xi) = 0$. Indeed, we have

$$\begin{aligned}
\mathrm{Ric}(\xi,x) &= g^{[00]}\tilde{g}(R(\xi,\xi)x,\xi) + \sum_{i=1}^{n} g^{[ii]}\tilde{g}(R(e_i,\xi)x,e_i) \\
&= \sum_{i=1}^{n} g'^{ii} g(R(e_i,\xi)x,e_i) = \sum_{i=1}^{n} g'^{ii} g(R(x,e_i)e_i,\xi) = 0.
\end{aligned}$$

Now,

$$\begin{aligned}
\mathrm{Ric}(x,\xi) &= g^{[00]}\tilde{g}(R(\xi,x)\xi,\xi) + \sum_{i=1}^{n} g'^{ii} g(R(x,e_i)\xi,e_i) \\
&= \eta(R(\xi,x)\xi) \overset{(9)}{=} \eta(\overline{R}(\xi,x)\xi) = 0
\end{aligned}$$

by hypothesis. Hence, since $g(\xi,\bullet) = g(\bullet,\xi) = 0$, the latter together with (25) leads to $\mathrm{Ric}(x,y) = \lambda g(x,y)$, for all $x, y \in T_pM$, that is, $(M, g, S(TM))$ is Einstein at $p \in M$. $\square$

Corollary 4.2. *Let (M,g) be an admissible lightlike hypersurface of a flat semi-Riemannian manifold $\overline{M}$. If (M,g) is Osserman at $p \in M$ then it is Einstein at p.*

Proof. This is immediate consequence of Theorem 4.4 since the flat condition implies $\overline{R}(\xi,\cdot)\xi = 0$, $\forall \xi \in \mathrm{Rad}\, TM$ and $\eta(\overline{R}(\xi,x)x) = 0$. $\square$

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