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KODAI MATH. J.
32 (2009), 77–90

WEIERSTRASS PRODUCT REPRESENTATIONS OF MULTIPLE GAMMA AND SINE FUNCTIONS

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Abstract

It is well known that the Weierstrass product representation of the Barnes multiple gamma function $\Gamma_r(z)$ can be calculated concretely. However, there has been no study on its explicit formulation. In this paper, its simple formulation is achieved. It is applicable to the Weierstrass product representation of the Vignéras multiple gamma function also. Moreover, the Weierstrass product representation of the Kurokawa multiple sine function $S_r(z)$ is also formulated explicitly.

1. Introduction

Let r be a positive integer. Put $\boldsymbol{\omega} = (\omega_1, \dots, \omega_r) \in (\mathbf{R}_{>0})^r$ and $|\boldsymbol{\omega}| = \omega_1 + \cdots + \omega_r$. The multiple gamma and sine functions are defined by

$$\Gamma_r(z, \boldsymbol{\omega}) = \exp\left(\frac{\partial}{\partial s}\zeta_r(s, z, \boldsymbol{\omega})\Big|_{s=0}\right),$$

$$S_r(z, \boldsymbol{\omega}) = \Gamma_r(z, \boldsymbol{\omega})^{-1}\Gamma_r(|\boldsymbol{\omega}| - z, \boldsymbol{\omega})^{(-1)^r}$$

where $\zeta_r(s, z, \boldsymbol{\omega})$ is the multiple Hurwitz zeta function. These functions are meromorphic in the whole complex z-plane. The multiple gamma function was studied by Barnes [3]. One of his results is a generalization of the Weierstrass product representation of the usual gamma function. To be precise, he obtained

$$\Gamma_r(z, \boldsymbol{\omega})^{-1} = e^{A_r(z, \boldsymbol{\omega})} z \prod_{\substack{\boldsymbol{m} \in (\mathbf{Z}_{\geq 0})^r \\ \boldsymbol{m} \neq \boldsymbol{0}}} P_r\left(-\frac{z}{\boldsymbol{m} \cdot \boldsymbol{\omega}}\right)$$

with $P_r(u) = (1-u)\exp\left(u + \frac{u^2}{2} + \cdots + \frac{u^r}{r}\right)$. Here $A_r(z, \boldsymbol{\omega}) = \sum_{l=0}^r \frac{\gamma_{r,l}(\boldsymbol{\omega})}{l!} z^l$ with

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2000 *Mathematics Subject Classification*. 11M35, 11B73, 33E20.

Key words and phrases. multiple Hurwitz zeta function, multiple gamma function, multiple sine function, Weierstrass product.

Received November 26, 2007; revised June 12, 2008.

$$\gamma_{r,l}(\boldsymbol{\omega}) = -\frac{\partial^l}{\partial z^l} \log(z\Gamma_r(z,\boldsymbol{\omega}))\Big|_{z=0}.$$

Barnes called $\rho_r(\boldsymbol{\omega}) = e^{\gamma_{r,0}(\boldsymbol{\omega})}$ the r-ple Stirling modular form and $\gamma_{r,1}(\boldsymbol{\omega}), \dots, \gamma_{r,r}(\boldsymbol{\omega})$ the first r r-ple gamma modular forms. Regarding the multiple sine function, Kurokawa and Koyama [4] proved its infinite product representation:

$$S_r(z,\boldsymbol{\omega}) = e^{B_r(z,\boldsymbol{\omega})} z \prod_{\substack{\boldsymbol{m} \in (\mathbf{Z}_{\geq 0})^r \\ \boldsymbol{m} \neq \mathbf{0}}} P_r\Big(-\frac{z}{\boldsymbol{m}\cdot\boldsymbol{\omega}}\Big) \prod_{\boldsymbol{m} \in (\mathbf{Z}_{\geq 0})^r} P_r\left(\frac{z}{(\boldsymbol{m}+\mathbf{1})\cdot\boldsymbol{\omega}}\right)^{(-1)^{r-1}}$$

with $\mathbf{1} = (1,\dots,1) \in \mathbf{R}^r$. Here $B_r(z,\boldsymbol{\omega}) = \sum_{l=0}^{r} \frac{\delta_{r,l}(\boldsymbol{\omega})}{l!} z^l$ with

$$\delta_{r,l}(\boldsymbol{\omega}) = \frac{\partial^l}{\partial z^l} \log\big(z^{-1} S_r(z,\boldsymbol{\omega})\big)\Big|_{z=0}.$$

We note that $A_1(z,\omega) = \frac{1}{2}\log\left(\frac{2\pi}{\omega}\right) + \frac{1}{\omega}(\gamma - \log\omega)z$ and $B_1(z,\omega) = \log\left(\frac{2\pi}{\omega}\right)$ where γ is the Euler constant, since $\Gamma_1(z,\omega) = \frac{\Gamma(z/\omega)}{\sqrt{2\pi}}\omega^{z/\omega-1/2}$ and $S_1(z,\omega) = 2\sin\left(\frac{\pi z}{\omega}\right)$. When $r \geq 2$, the explicit expressions of $A_r(z,\boldsymbol{\omega})$ and $B_r(z,\boldsymbol{\omega})$ are unknown.

In this paper, we treat only the special case $\boldsymbol{\omega} = \mathbf{1} = (1,\dots,1)$. For simplicity, when $\boldsymbol{\omega} = \mathbf{1}$, we omit the parameters, e.g. $\Gamma_r(z) = \Gamma_r(z,\mathbf{1})$, $A_r(z) = A_r(z,\mathbf{1})$ and $\gamma_{r,l} = \gamma_{r,l}(\mathbf{1})$. Then, the infinite product representations are

$$\Gamma_r(z)^{-1} = e^{A_r(z)} z \prod_{m=1}^{\infty} P_r\left(-\frac{z}{m}\right)^{\binom{m+r-1}{r-1}},$$

$$S_r(z) = e^{B_r(z)} z \prod_{m=1}^{\infty} P_r\left(-\frac{z}{m}\right)^{\binom{m+r-1}{r-1}} \prod_{m=r}^{\infty} P_r\left(\frac{z}{m}\right)^{(-1)^{r-1}\binom{m-1}{r-1}}$$

with $A_r(z) = \sum_{l=0}^{r} \frac{\gamma_{r,l}}{l!} z^l$ and $B_r(z) = \sum_{l=0}^{r} \frac{\delta_{r,l}}{l!} z^l$. The values $\gamma_{r,l}$ and $\delta_{r,l}$ are computable. Especially, $\gamma_{r,0}$ has the simple expression (1.1) which was shown by Adamchik [1, Lemma 2]. However their explicit formulations generally remain unanswered. The aim of this paper is to achieve them. Moreover, in the process, a new proof of (1.1) will be given.

Denote by $s(n,k)$ the signed Stirling number of the first kind defined by

$$n!\binom{x}{n} = \sum_{k=0}^{\infty} s(n,k) x^k$$

for $n \in \mathbf{Z}_{\geq 0}$. Let $H_0 = 0$ and H_n be the n-th harmonic number for $n \in \mathbf{Z}_{\geq 1}$: $H_n = \sum_{k=1}^{n} \frac{1}{k}$.

THEOREM 1. *The constants $\gamma_{r,l}$ for the multiple gamma function are given by*

$$\log \rho_r = \gamma_{r,0} = -\frac{1}{(r-1)!}\sum_{m=0}^{r-1} |s(r, m+1)|\zeta'(-m) \tag{1.1}$$

and

$$\gamma_{r,l} = \frac{(-1)^{l-1}(l-1)!}{(r-1)!}\sum_{m=-l}^{r-l-1} |s(r, l+m+1)| f_l(m) \tag{1.2}$$

for $1 \leq l \leq r$, where $\zeta(s)$ is the Riemann zeta function and

$$f_l(m) = \begin{cases} H_{l-1} + \gamma & \text{if } m = -1, \\ \zeta(-m) & \text{if } m \neq -1. \end{cases}$$

Hence $\deg A_r(z) = r$.

Example 1.

(i) $\gamma_{1,0} = \dfrac{\log(2\pi)}{2}$, $\gamma_{1,1} = \gamma$.

(ii) $\gamma_{2,0} = \dfrac{\log(2\pi)}{2} - \zeta'(-1)$, $\gamma_{2,1} = -\dfrac{1}{2} + \gamma$, $\gamma_{2,2} = -1 - \gamma - \dfrac{\pi^2}{6}$.

(iii) $\gamma_{3,0} = \dfrac{\log(2\pi)}{2} + \dfrac{\zeta(3)}{8\pi^2} - \dfrac{3\zeta'(-1)}{2}$, $\gamma_{3,1} = -\dfrac{19}{24} + \gamma$,

$\gamma_{3,2} = -\dfrac{5}{4} - \dfrac{3\gamma}{2} - \dfrac{\pi^2}{6}$, $\gamma_{3,3} = \dfrac{3}{2} + \gamma + \dfrac{\pi^2}{2} + 2\zeta(3)$.

(iv) $\gamma_{4,0} = \dfrac{\log(2\pi)}{2} + \dfrac{\zeta(3)}{4\pi^2} - \dfrac{11\zeta'(-1)}{6} - \dfrac{\zeta'(-3)}{6}$, $\gamma_{4,1} = -1 + \gamma$,

$\gamma_{4,2} = -\dfrac{95}{72} - \dfrac{11\gamma}{6} - \dfrac{\pi^2}{6}$, $\gamma_{4,3} = \dfrac{17}{6} + 2\gamma + \dfrac{11\pi^2}{18} + 2\zeta(3)$,

$\gamma_{4,4} = -\dfrac{11}{6} - \gamma - \pi^2 - \dfrac{\pi^4}{15} - 11\zeta(3)$.

(v) $\gamma_{5,0} = \dfrac{\log(2\pi)}{2} + \dfrac{35\zeta(3)}{96\pi^2} - \dfrac{\zeta(5)}{32\pi^4} - \dfrac{25\zeta'(-1)}{12} - \dfrac{5\zeta'(-3)}{12}$, $\gamma_{5,1} = -\dfrac{3349}{2880} + \gamma$,

$\gamma_{5,2} = -\dfrac{95}{72} - \dfrac{25\gamma}{12} - \dfrac{\pi^2}{6}$, $\gamma_{5,3} = \dfrac{569}{144} + \dfrac{35\gamma}{12} + \dfrac{25\pi^2}{36} + 2\zeta(3)$,

$\gamma_{5,4} = -\dfrac{107}{24} - \dfrac{5\gamma}{2} - \dfrac{35\pi^2}{24} - \dfrac{\pi^4}{15} - \dfrac{25\zeta(3)}{2}$,

$\gamma_{5,5} = \dfrac{25}{12} + \gamma + \dfrac{5\pi^2}{3} + \dfrac{5\pi^4}{9} + 35\zeta(3) + 24\zeta(5)$.

Theorem 2. *The constants* $\delta_{r,l}$ *for the multiple sine function are given by*

$$\delta_{r,0} = \log(2\pi) + \frac{1}{(r-1)!} \sum_{m=1}^{[(r-1)/2]} \frac{(-1)^{m-1}(2m)!}{(2\pi)^{2m}} |s(r, 2m+1)| \zeta(2m+1) \tag{1.3}$$

and

$$\delta_{r,l} = \frac{(-1)^{l-1} 2(l-1)!}{(r-1)!} \sum_{m=0}^{[l/2]} |s(r, l-2m+1)| \zeta(2m) \tag{1.4}$$

for $1 \le l \le r$. *Hence* $\deg B_1(z) = 0$ *and* $\deg B_r(z) = r$ *for* $r \ge 2$.

Example 2.
(i) $\delta_{1,0} = \log(2\pi)$, $\delta_{1,1} = 0$.
(ii) $\delta_{2,0} = \log(2\pi)$, $\delta_{2,1} = -1$, $\delta_{2,2} = -\dfrac{\pi^2}{3}$.
(iii) $\delta_{3,0} = \log(2\pi) + \dfrac{\zeta(3)}{4\pi^2}$, $\delta_{3,1} = -\dfrac{3}{2}$, $\delta_{3,2} = \dfrac{1}{2} - \dfrac{\pi^2}{3}$, $\delta_{3,3} = \pi^2$.
(iv) $\delta_{4,0} = \log(2\pi) + \dfrac{\zeta(3)}{2\pi^2}$, $\delta_{4,1} = -\dfrac{11}{6}$, $\delta_{4,2} = 1 - \dfrac{\pi^2}{3}$,
$\delta_{4,3} = -\dfrac{1}{3} + \dfrac{11\pi^2}{9}$, $\delta_{4,4} = -2\pi^2 - \dfrac{2\pi^4}{15}$.
(v) $\delta_{5,0} = \log(2\pi) + \dfrac{35\zeta(3)}{48\pi^2} - \dfrac{\zeta(5)}{16\pi^4}$, $\delta_{5,1} = -\dfrac{25}{12}$, $\delta_{5,2} = \dfrac{35}{24} - \dfrac{\pi^2}{3}$,
$\delta_{5,3} = -\dfrac{5}{6} + \dfrac{25\pi^2}{18}$, $\delta_{5,4} = \dfrac{1}{4} - \dfrac{35\pi^2}{12} - \dfrac{2\pi^4}{15}$, $\delta_{5,5} = \dfrac{10\pi^2}{3} + \dfrac{10\pi^4}{9}$.

Finally, we apply Theorem 1 to the Vignéras multiple gamma function $G_r(z)$ defined by the next conditions:
(1) $G_r(z+1) = G_{r-1}(z) G_r(z)$ for $z \in \mathbf{C}$,
(2) $G_r(1) = 1$,
(3) $\dfrac{d^{r+1}}{dz^{r+1}} \log G_r(z) \ge 0$ for $z \ge 1$,
(4) $G_0(z) = z$.

In 1977 Vignéras [9, p. 239] proved the uniqueness of the functions satisfying the above conditions. According to the Bohr-Mollerup theorem, $G_1(z)$ coincides with $\Gamma(z)$. So Vignéras called $G_r(z)$'s multiple gamma functions. Especially, the function $G_2(z)$ was originated by Barnes [2] and so is called the Barnes G-function.

The Vignéras multiple gamma function $G_r(z)$ has the following infinite product representation.

Theorem 3. *For* $r \ge 1$

$$G_r(z+1) = e^{C_r(z)} \prod_{n=1}^{\infty} P_r\left(-\frac{z}{n}\right)^{(-1)^r \binom{n+r-2}{r-1}} \tag{1.5}$$

with $C_r(z) = \sum_{l=1}^{r} \frac{\varepsilon_{r,l}}{l!} z^l$. *Here*

(1.6)
$$\varepsilon_{r,l} = \frac{(-1)^{r-l} l!}{(r-1)!} \times \left\{ \sum_{m=0}^{r-l-1} \binom{l+m}{l} |s(r-1, l+m)| \zeta'(-m) - \frac{1}{l} \sum_{m=-l}^{r-l-1} |s(r-1, l+m)| f_l(m) \right\}$$

for $1 \le l \le r$ *where* $f_l(m)$ *is as in Theorem* 1. *Hence* $\deg C_r(z) = r$.

Results of this type have been proved by Vignéras [9, p. 241] and by Ueno and Nishizawa [7, Theorem 6.1], but are more complicated than our result.

Example 3.

(i) $\varepsilon_{1,1} = -\gamma$.

(ii) $\varepsilon_{2,1} = -\frac{1}{2} + \frac{\log(2\pi)}{2}$, $\varepsilon_{2,2} = -1 - \gamma$.

(iii) $\varepsilon_{3,1} = \frac{7}{24} - \frac{\log(2\pi)}{4} + \zeta'(-1)$, $\varepsilon_{3,2} = \frac{1}{4} + \frac{\gamma}{2} + \frac{\log(2\pi)}{2}$, $\varepsilon_{3,3} = -\frac{3}{2} - \gamma - \frac{\pi^2}{6}$.

(iv) $\varepsilon_{4,1} = -\frac{5}{24} + \frac{\log(2\pi)}{6} + \frac{\zeta(3)}{8\pi^2} - \zeta'(-1)$, $\varepsilon_{4,2} = -\frac{5}{72} - \frac{\gamma}{3} - \frac{\log(2\pi)}{2} + \zeta'(-1)$,

$\varepsilon_{4,3} = \frac{4}{3} + \gamma + \frac{\pi^2}{9} + \frac{\log(2\pi)}{2}$, $\varepsilon_{4,4} = -\frac{11}{6} - \gamma - \frac{\pi^2}{2} - 2\zeta(3)$.

(v) $\varepsilon_{5,1} = \frac{469}{2880} - \frac{\log(2\pi)}{8} - \frac{3\zeta(3)}{16\pi^2} + \frac{11\zeta'(-1)}{12} + \frac{\zeta'(-3)}{6}$,

$\varepsilon_{5,2} = \frac{\gamma}{4} + \frac{11\log(2\pi)}{24} + \frac{\zeta(3)}{8\pi^2} - \frac{3\zeta'(-1)}{2}$,

$\varepsilon_{5,3} = -\frac{161}{144} - \frac{11\gamma}{12} - \frac{\pi^2}{12} - \frac{3\log(2\pi)}{4} + \zeta'(-1)$,

$\varepsilon_{5,4} = \frac{21}{8} + \frac{3\gamma}{2} + \frac{11\pi^2}{24} + \frac{\log(2\pi)}{2} + \frac{3\zeta(3)}{2}$,

$\varepsilon_{5,5} = -\frac{25}{12} - \gamma - \pi^2 - \frac{\pi^4}{15} - 11\zeta(3)$.

2. Lemmas

Lemma 2.1 (cf. Adamchik [1, Proposition 2]). (i) *Put* $\zeta'(s,z) = \frac{\partial}{\partial s}\zeta(s,z)$. *For* $z > 0$,

$$\log \Gamma_r(z) = \sum_{m=0}^{r-1} \frac{(-1)^{r-m-1} B_{r-m-1}^{(r)}(z)}{m!(r-m-1)!} \zeta'(-m,z) \tag{2.1}$$

where $B_m^{(r)}(z)$ *is the generalized Nörlund polynomial defined by*

$$\left(\frac{t}{e^t-1}\right)^r e^{zt} = \sum_{m=0}^{\infty} \frac{B_m^{(r)}(z)}{m!} t^m.$$

(ii) *For* $0 < z < 1$

$$\log S_r(z) = \sum_{m=0}^{r-1} \frac{(-1)^{r-m-1} B_{r-m-1}^{(r)}(z)}{m!(r-m-1)!} \log S_{1,m}(z), \tag{2.2}$$

where $S_{1,m}(z) = \exp\{-\zeta'(-m,z) + (-1)^{m+1}\zeta'(-m,1-z)\}$ *is called Milnor's multiple sine function* (*refer to Kurokawa, Ochiai and Wakayama* [5]).

Remark 1. Basic properties of $B_m^{(r)}(z)$ are found in the book of Nörlund [6]. The following formulas are needed in this paper:

$$B_m^{(r)}(r-z) = (-1)^m B_m^{(r)}(z) \qquad (r \geq 1, m \geq 0), \tag{2.3}$$

$$B_{r-1}^{(r)}(z) = (z-1)\cdots(z-r+1) \qquad (r \geq 2), \tag{2.4}$$

$$\frac{d}{dz} B_m^{(r)}(z) = m B_{m-1}^{(r)}(z) \qquad (r \geq 1, m \geq 1), \tag{2.5}$$

$$B_m^{(r)}(z+a) = \sum_{l=0}^{m} \binom{m}{l} B_{m-l}^{(r)}(a) z^l \quad (r \geq 1, m \geq 0). \tag{2.6}$$

Proof of Lemma 2.1. (i) When $r = 1$ the formula (2.1) is trivial. We now consider the case $r \geq 2$. Since

$$\zeta_r(s,z) = -\frac{z-r+1}{r-1}\zeta_{r-1}(s,z) + \frac{1}{r-1}\zeta_{r-1}(s-1,z),$$

we obtain

$$\zeta_r(s,z) = \frac{1}{(r-1)!}\sum_{m=0}^{r-1} c_{r,r-m-1}(z)\zeta(s-m,z)$$

where $c_{r,0}(z) = 1$ and

$$c_{r,m}(z) = (-1)^m \sum_{1 \leq i_1 < \cdots < i_m \leq r-1} \cdots \sum (z-i_1)\cdots(z-i_m)$$

for $1 \leq m \leq r-1$. The equations (2.4) and (2.5) imply

$$\begin{aligned} c_{r,m}(z) &= \frac{(-1)^m}{(r-m-1)!} \frac{d^{r-m-1}}{dz^{r-m-1}} (z-1)\cdots(z-r+1) \\ &= \frac{(-1)^m}{(r-m-1)!} \frac{d^{r-m-1}}{dz^{r-m-1}} B_{r-1}^{(r)}(z) \\ &= (-1)^m \binom{r-1}{m} B_m^{(r)}(z). \end{aligned}$$

Thus

$$\zeta_r(s,z) = \sum_{m=0}^{r-1} \frac{(-1)^{r-m-1} B_{r-m-1}^{(r)}(z)}{m!(r-m-1)!} \zeta(s-m,z).$$

This shows (2.1).

(ii) Using the relation

$$\zeta_r(s,z) = -\frac{z-1}{r-1}\zeta_{r-1}(s,z-1) + \frac{1}{r-1}\zeta_{r-1}(s-1,z-1),$$

we have, for $z > r-1$

$$\zeta_r(s,z) = \sum_{m=0}^{r-1} \frac{(-1)^{r-m-1} B_{r-m-1}^{(r)}(z)}{m!(r-m-1)!} \zeta(s-m,z-r+1).$$

Hence, by (2.3),

$$\log \Gamma_r(r-z) = \sum_{m=0}^{r-1} \frac{B_{r-m-1}^{(r)}(z)}{m!(r-m-1)!} \zeta'(-m,1-z).$$

Thus we deduce (2.2) from the definition of $S_r(z)$. □

LEMMA 2.2. *Let l and m be integers.*

(i) *For $l \geq 1$ and $0 \leq m \leq l$*

$$\sum_{k=0}^{m} \binom{l}{k} (-1)^k = \begin{cases} 0 & \text{if } m = l, \\ (-1)^m \binom{l-1}{m} & \text{if } 0 \leq m \leq l-1. \end{cases} \tag{2.7}$$

(ii) *For $l \geq 2$*

$$\sum_{k=2}^{l} \binom{l}{k} (-1)^k H_{k-1} = H_{l-1}. \tag{2.8}$$

(iii) *For $l \geq 1$ and $m \geq 0$*

$$\sum_{k=0}^{l} \binom{l}{k} (-1)^k H_{m+k} = -\frac{(l-1)!m!}{(l+m)!}. \tag{2.9}$$

Proof. (i): The formula is well known.
(ii),(iii): The results follow from the next expression of the harmonic number:

$$H_n = \int_0^1 \frac{1 - x^n}{1 - x}\, dx$$

for $n \geq 0$. □

3. Proofs of Theorem 1 and 2

Proof of Theorem 1. Using (2.1) and the formula $\zeta(s, z+1) = \zeta(s, z) - z^{-s}$, we see

$$\log \Gamma_r(z) = \sum_{m=0}^{r-1} \frac{(-1)^{r-m-1} B_{r-m-1}^{(r)}(z)}{m!(r-m-1)!} \zeta'(-m, z+1) - \log z \sum_{m=0}^{r-1} \frac{(-1)^{r-m-1} B_{r-m-1}^{(r)}(z)}{m!(r-m-1)!} z^m.$$

By (2.4) and (2.6) we have

$$\log(z\Gamma_r(z)) = \sum_{m=0}^{r-1} \frac{(-1)^{r-m-1} B_{r-m-1}^{(r)}(z)}{m!(r-m-1)!} \zeta'(-m, z+1).$$

Hence

$$\gamma_{r,l} = -\sum_{m=0}^{r-1} \sum_{k=0}^{l} \binom{l}{k} \frac{d^{l-k}}{dz^{l-k}} \frac{(-1)^{r-m-1} B_{r-m-1}^{(r)}(z)}{(r-m-1)!} \frac{d^k}{dz^k} \frac{\zeta'(-m, z+1)}{m!} \bigg|_{z=0}.$$

We see that

$$\frac{d^{l-k}}{dz^{l-k}} \frac{(-1)^{r-m-1} B_{r-m-1}^{(r)}(z)}{(r-m-1)!} \bigg|_{z=0} = \frac{(-1)^{l-k}(m+l-k)!}{(r-1)!} |s(r, m+l-k+1)|,$$

since

$$\binom{r-1}{m} B_m^{(r)}(z) = \sum_{k=0}^{m} \binom{r-m+k-1}{k} s(r, r-m+k) z^k$$

for $0 \leq m \leq r-1$. We can compute

$$\begin{aligned} \frac{d^k}{dz^k} \frac{\zeta'(-m, z+1)}{m!} \bigg|_{z=0} &= \frac{1}{m!} \frac{\partial^{k+1}}{\partial z^k \partial s} \zeta(s, z+1) \bigg|_{s=-m,\, z=0} \\ &= \frac{(-1)^k}{m!} \frac{\partial}{\partial s} s(s+1) \cdots (s+k-1) \zeta(s+k, z+1) \bigg|_{s=-m,\, z=0} \\ &= f(m, k), \end{aligned}$$

where

(3.1)
$$f(m,k) = \begin{cases} \dfrac{1}{(m-k)!}(\zeta'(-m+k) - (H_m - H_{m-k})\zeta(-m+k)) & \text{if } 0 \le k \le m, \\ H_m - \gamma & \text{if } k = m+1, \\ (-1)^{m-k}(k-m-1)!\zeta(-m+k) & \text{if } k \ge m+2. \end{cases}$$

Combining the above results we obtain

$$\gamma_{r,l} = \frac{1}{(r-1)!}\sum_{m=0}^{r-1}\sum_{k=0}^{l}\binom{l}{k}(-1)^{l-k+1}(m+l-k)!|s(r,m+l-k+1)|f(m,k)$$

for $0 \le l \le r$, which gives (1.1). So we hereafter consider the case $1 \le l \le r$. We split the summation into three parts S_1, S_2, S_3 according to the conditions $0 \le k \le m$, $k = m+1$ and $k \ge m+2$. We first calculate the sum S_1. Since $|s(n,m)| = 0$ for $m \ge n+1$, we have

$$\begin{aligned} S_1 &= \frac{1}{(r-1)!}\sum_{k=0}^{l}\sum_{m=k}^{r-1}\binom{l}{k}(-1)^{l-k+1}(m+l-k)!|s(r,m+l-k+1)|f(m,k) \\ &= \frac{1}{(r-1)!}\sum_{k=0}^{l}\sum_{m=0}^{r-l-1}\binom{l}{k}(-1)^{l-k+1}(l+m)!|s(r,l+m+1)|f(m+k,k) \\ &= \frac{(-1)^{l-1}}{(r-1)!}\sum_{m=0}^{r-l-1}\frac{(l+m)!}{m!}|s(r,l+m+1)| \\ &\quad \times \sum_{k=0}^{l}\binom{l}{k}(-1)^k\{\zeta'(-m) - (H_{m+k} - H_m)\zeta(-m)\}. \end{aligned}$$

By (2.7) and (2.9),

$$S_1 = \frac{(-1)^{l-1}(l-1)!}{(r-1)!}\sum_{m=0}^{r-l-1}|s(r,l+m+1)|\zeta(-m).$$

Next we use (2.7) and (2.8) to obtain

$$\begin{aligned} S_2 &= \frac{(-1)^{l-1}(l-1)!}{(r-1)!}|s(r,l)|\sum_{k=1}^{l}\binom{l}{k}(-1)^k(H_{k-1} - \gamma) \\ &= \frac{(-1)^{l-1}(l-1)!}{(r-1)!}|s(r,l)|(H_{l-1} + \gamma). \end{aligned}$$

Finally the formula (2.7) shows

$$\begin{aligned} S_3 &= \frac{1}{(r-1)!}\sum_{k=0}^{l}\sum_{m=0}^{k-2}\binom{l}{k}(-1)^{l-k+1}(m+l-k)!|s(r,m+l-k+1)|f(m,k) \\ &= \frac{1}{(r-1)!}\sum_{k=0}^{l}\sum_{m=2}^{k}\binom{l}{k}(-1)^{l-k+1}(l-m)!|s(r,l-m+1)|f(k-m,k) \\ &= \frac{(-1)^{l-1}}{(r-1)!}\sum_{m=2}^{l}(-1)^m(l-m)!(m-1)!|s(r,l-m+1)|\zeta(m)\sum_{k=m}^{l}\binom{l}{k}(-1)^k \\ &= \frac{(-1)^{l-1}(l-1)!}{(r-1)!}\sum_{m=2}^{l}|s(r,l-m+1)|\zeta(m). \end{aligned}$$

Combining the above results we deduce (1.2). Especially, the equation

$$\gamma_{r,r} = (-1)^{r-1}(H_{r-1}+\gamma) + (-1)^{r-1}\sum_{m=2}^{r}|s(r,r-m+1)|\zeta(m) \tag{3.2}$$

implies $\deg A_r(z) = r$. Thus the proof of Theorem 1 is complete. □

We next prove Theorem 2 by the similar procedure as in the proof of Theorem 1.

Proof of Theorem 2. By (2.2) we obtain

$$\log(z^{-1}S_r(z)) = -\sum_{m=0}^{r-1}\frac{(-1)^{r-m-1}B_{r-m-1}^{(r)}(z)}{m!(r-m-1)!}(\zeta'(-m,z+1)+(-1)^m\zeta'(-m,1-z))$$

for $0 < z < 1$. Hence

$$\begin{aligned} \delta_{r,l} = \frac{1}{(r-1)!}\sum_{m=0}^{r-1}\sum_{k=0}^{l}\binom{l}{k}(-1)^{l-k+1}\left(1+(-1)^{k+m}\right) \\ \times (m+l-k)!|s(r,m+l-k+1)|f(m,k) \end{aligned}$$

for $0 \le l \le r$, where $f(m,k)$ is defined by (3.1). Thus

$$\begin{aligned} \delta_{r,0} &= -\frac{2}{(r-1)!}\sum_{m=0}^{[(r-1)/2]}|s(r,2m+1)|\zeta'(-2m) \\ &= \log(2\pi) + \frac{1}{(r-1)!}\sum_{m=1}^{[(r-1)/2]}\frac{(-1)^{m-1}(2m)!}{(2\pi)^{2m}}|s(r,2m+1)|\zeta(2m+1) \end{aligned}$$

which shows (1.3). We next consider the case $1 \le l \le r$. Split the sum into the subparts T_1, T_2, T_3 according to $0 \le k \le m$, $k = m+1$ and $k \ge m+2$. Then

$$T_1 = \frac{(-1)^{l-1}(l-1)!}{(r-1)!} \sum_{m=0}^{r-l-1} (1+(-1)^m)|s(r, l+m+1)|\zeta(-m)$$
$$= \frac{(-1)^l (l-1)!}{(r-1)!} |s(r, l+1)|$$

and $T_2 = 0$. Finally we see

$$T_3 = \frac{(-1)^{l-1} 2(l-1)!}{(r-1)!} \sum_{m=1}^{[l/2]} |s(r, l-2m+1)|\zeta(2m).$$

Combining the above results we deduce (1.4). □

4. Application to the Vignéras multiple gamma function

In this section, we apply Theorem 1 to obtain Theorem 3 via the next proposition.

PROPOSITION 4.1 (Vardi [8, Proposition 2.3]). *For* $r \geq 1$

$$G_r(z) = e^{T_r(z)} \Gamma_r(z)^{(-1)^{r-1}} \tag{4.1}$$

where

$$T_r(z) = \sum_{k=1}^{r} (-1)^{r-k} \binom{z}{k-1} \log \rho_{r-k+1}.$$

Proof of Theorem 3. When $r = 1$, the formula (1.5) is the Weierstrass product representation of $\Gamma(z+1)$. So we now consider the case $r \geq 2$. Using the relation $G_r(z+1) = G_{r-1}(z) G_r(z)$ and the formula (4.1), we have

$$\begin{aligned} G_r(z+1) &= e^{T_r(z)+T_{r-1}(z)} \Gamma_{r-1}(z)^{(-1)^{r-2}} \Gamma_r(z)^{(-1)^{r-1}} \\ &= e^{U_r(z)+(-1)^r V_r(z)} W_r(z), \end{aligned} \tag{4.2}$$

where $U_r(z) = T_r(z) + T_{r-1}(z)$, $V_r(z) = A_r(z) - A_{r-1}(z)$ and

$$W_r(z) = \prod_{n=1}^{\infty} P_r\left(-\frac{z}{n}\right)^{(-1)^r \binom{n+r-1}{r-1}} P_{r-1}\left(-\frac{z}{n}\right)^{(-1)^{r-1}\binom{n+r-2}{r-2}}.$$

We first consider $U_r(z)$. Since (1.1) shows

$$T_r(z) = \sum_{k=1}^{r} \sum_{m=0}^{r-k} \frac{(-1)^{m+1}}{(r-k)!} \binom{z}{k-1} s(r-k+1, m+1)\zeta'(-m),$$

we have

$$\begin{aligned}U_r(z) = &\sum_{k=1}^{r-1}\sum_{m=0}^{r-k-1}\frac{(-1)^{m+1}}{(r-k)!}\binom{z}{k-1}\\ &\times\{s(r-k+1,m+1)+(r-k)s(r-k,m+1)\}\zeta'(-m)\\ &+\sum_{k=1}^{r}\frac{(-1)^{r-k+1}}{(r-k)!}\binom{z}{k-1}\zeta'(-r+k).\end{aligned}$$

By the recurrence relation

$$s(n,m) = s(n-1,m-1)-(n-1)s(n-1,m) \tag{4.3}$$

for $n \geq 1$ and $1 \leq m \leq n$, we obtain

$$\begin{aligned}U_r(z) &= \sum_{k=1}^{r-1}\sum_{m=0}^{r-k-1}\frac{(-1)^{m+1}}{(r-k)!}\binom{z}{k-1}s(r-k,m)\zeta'(-m)\\ &\quad+\sum_{k=1}^{r}\frac{(-1)^{r-k+1}}{(r-k)!}\binom{z}{k-1}\zeta'(-r+k)\\ &= \sum_{k=1}^{r}\sum_{m=0}^{r-k}\frac{(-1)^{m+1}}{(r-k)!}\binom{z}{k-1}s(r-k,m)\zeta'(-m).\end{aligned}$$

Moreover

$$\begin{aligned}U_r(z) &= \sum_{k=1}^{r}\sum_{m=0}^{r-k}\sum_{l=0}^{k-1}\frac{(-1)^{m+1}}{(k-1)!(r-k)!}s(k-1,l)s(r-k,m)\zeta'(-m)z^l\\ &= \frac{1}{(r-1)!}\sum_{l=0}^{r-1}z^l\sum_{m=0}^{r-l-1}(-1)^{m+1}\zeta'(-m)\sum_{k=l+1}^{r-m}\binom{r-1}{k-1}s(k-1,l)s(r-k,m).\end{aligned}$$

Since

$$\binom{m}{l}s(n,m) = \sum_{k=m-l}^{n-l}\binom{n}{k}s(k,m-l)s(n-k,l)$$

for non-negative integers l, m, n with $l \leq m$, we have

$$\begin{aligned}U_r(z) &= \frac{1}{(r-1)!}\sum_{l=0}^{r-1}z^l\sum_{m=0}^{r-l-1}(-1)^{m+1}\binom{l+m}{l}s(r-1,l+m)\zeta'(-m)\\ &= \sum_{l=0}^{r-1}\frac{z^l}{l!}\sum_{m=0}^{r-l-1}\frac{(-1)^{r-l}(l+m)!}{(r-1)!m!}|s(r-1,l+m)|\zeta'(-m).\end{aligned} \tag{4.4}$$

Secondly, we deal with

$$V_r(z) = \sum_{l=0}^{r-1} (\gamma_{r,l} - \gamma_{r-1,l}) \frac{z^l}{l!} + \gamma_{r,r} \frac{z^r}{r!}.$$

By (1.1) and (4.3) we can calculate that

(4.5)

$$\begin{aligned}
\gamma_{r,0} - \gamma_{r-1,0} &= \sum_{m=0}^{r-2} \frac{(-1)^{r+m}}{(r-1)!} (s(r, m+1) + (r-1)s(r-1, m+1))\zeta'(-m) - \frac{\zeta'(-r+1)}{(r-1)!} \\
&= \sum_{m=0}^{r-2} \frac{(-1)^{r+m}}{(r-1)!} s(r-1, m)\zeta'(-m) - \frac{\zeta'(-r+1)}{(r-1)!} \\
&= -\frac{1}{(r-1)!} \sum_{m=1}^{r-1} |s(r-1, m)| \zeta'(-m).
\end{aligned}$$

Also, by (1.2) we can show that for $1 \le l \le r-1$

$$\gamma_{r,l} - \gamma_{r-1,l} = \frac{(-1)^{l-1}(l-1)!}{(r-1)!} \sum_{m=-l+1}^{r-l-1} |s(r-1, l+m)| f_l(m) \tag{4.6}$$

The value $\gamma_{r,r}$ has been calculated as (3.2).

Thirdly, we treat $W_r(z)$. We see

$$W_r(z) = \prod_{n=1}^{\infty} P_r\left(-\frac{z}{n}\right)^{(-1)^r \binom{n+r-2}{r-1}} \exp\left\{ \binom{n+r-2}{r-2} \frac{z^r}{rn^r} \right\}.$$

Since

$$\binom{n+r-2}{r-2} = (-1)^{r-1} \frac{r-1}{n} \binom{-n}{r-1} = \frac{(-1)^{r-1}}{(r-2)!} \sum_{l=1}^{r-1} (-1)^l s(r-1, l) n^{l-1},$$

we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \binom{n+r-2}{r-2} \frac{z^r}{rn^r} &= \frac{(-1)^{r-1} z^r}{r \cdot (r-2)!} \sum_{l=1}^{r-1} (-1)^l s(r-1, l) \zeta(r+1-l) \\
&= \frac{z^r}{r!} \sum_{m=2}^{r} (r-1) |s(r-1, r-m+1)| \zeta(m).
\end{aligned}$$

Hence

$$W_r(z) = \exp\left(\frac{z^r}{r!} \sum_{m=2}^{r} (r-1) |s(r-1, r-m+1)| \zeta(m) \right) \prod_{n=1}^{\infty} P_r\left(-\frac{z}{n}\right)^{(-1)^r \binom{n+r-2}{r-1}}. \tag{4.7}$$

Combining the above results we obtain

$$G_r(z+1) = \exp\left(\sum_{l=0}^{r} \frac{\alpha_{r,l}}{l!} z^l\right) \prod_{n=1}^{\infty} P_r\left(-\frac{z}{n}\right)^{(-1)^r \binom{n+r-2}{r-1}}$$

with certain $\alpha_{r,l} \in \mathbf{R}$. We now determine $\alpha_{r,l}$ for $0 \le l \le r$. From (4.2), (4.4) and (4.5), we can check $\alpha_{r,0} = 0$. When $1 \le l \le r-1$, we use (4.4) and (4.6) to obtain $\alpha_{r,l} = \varepsilon_{r,l}$ where $\varepsilon_{r,l}$ is defined by (1.6). Finally (3.2) and (4.7) show that

$$\begin{aligned} \alpha_{r,r} &= -H_{r-1} - \gamma - \sum_{m=2}^{r} (-1)^{m+1} \{s(r, r-m+1) + (r-1)s(r-1, r-m+1)\}\zeta(m) \\ &= -H_{r-1} - \gamma - \sum_{m=2}^{r} (-1)^{m+1} s(r-1, r-m)\zeta(m) \\ &= \varepsilon_{r,r}. \end{aligned}$$

Here we used (4.3) for the second equality. Therefore we deduce Theorem 3. □

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