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ON SUBMANIFOLDS WITH PARALLEL MEAN CURVATURE VECTOR

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Abstract

We consider M^n, $n \geq 3$, a complete, connected submanifold of a space form $\tilde{M}^{n+p}(\tilde{c})$, whose non vanishing mean curvature vector H is parallel in the normal bundle. Assuming the second fundamental form h of M satisfies the inequality $\langle h\rangle^2 \leq n^2|H|^2/(n-1)$, we show that for $\tilde{c} \geq 0$ the codimension reduces to 1. When M is a submanifold of the unit sphere, then M^n is totally umbilic. For the case $\tilde{c} < 0$, one imposes an additional condition that is trivially satisfied when $\tilde{c} \geq 0$. When M is compact and has non-negative Ricci curvature then it is a geodesic hypersphere in the hyperbolic space. An alternative additional condition, when $\tilde{c} < 0$, reduces the codimension to 3.

1. Introduction

Submanifolds of space forms with parallel mean curvature vector have been investigated, in recent years, by several authors such as Alencar-do Carmo [AdC], Bérard-Santos [BS], Cheng-Nonaka [CN], Cheung-Leung-Leung [CLL], de Barros-Brasil-de Souza [BdBdS], do Carmo-Cheung-Santos [dCCS], Li [L], Mo [M], Santos [Sa], Sun [Su], Wang-Li [WL].

The main results of this paper extend to submanifolds of the sphere and of the hyperbolic space, a result proved by Cheng-Nonaka in [CN], for submanifolds of the Euclidean space.

We consider M^n, $n \geq 3$, a complete, connected submanifold of a space form $\tilde{M}^{n+p}(\tilde{c})$, whose mean curvature vector H does not vanish and it is parallel in the normal bundle. Assuming the second fundamental form h of M satisfies the inequality $\langle h\rangle^2 \leq n^2|H|^2/(n-1)$, we show (Theorem 3.2) that whenever $\tilde{c} \geq 0$ then the codimension reduces to 1. As a consequence, we show that when M is a submanifold of the unit sphere, then M^n is totally umbilic (Corollary 3.3). We remark that, when $\tilde{c} = 0$, Theorem 3.2 was proved by Cheng and Nonaka [CN].

A result analogous to Theorem 3.2, when $\tilde{c} < 0$, is proved by imposing an additional condition (see Theorem 3.4) that is trivially satisfied when $\tilde{c} \geq 0$.

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Under these conditions we prove (Theorem 3.5) that when the submanifold M is compact and has non-negative Ricci curvature then it is a geodesic hypersphere in the hyperbolic space $\mathbf{H}^{n+1}(-1)$ and therefore it is totally umbilic.

We also consider an alternative additional condition (see (Theorem 3.6)), for submanifolds M^n, $n \geq 4$, of the hyperbolic space, which implies that in this case the codimension reduces to 3.

One should mention that Santos [Sa], Sun [Su] and Wang [W] considered submanifolds of the sphere, with parallel mean cuvature, assuming different inequalities.

2. Preliminaries

This section contains preliminary results that will be necessary for the proofs of our main results. Let $\Phi : M^n \to \tilde{M}^{n+p}(\tilde{c})$ be an isometric immersion of an n-dimensional differential manifold M in an $(n+p)$-dimensional space form $\tilde{M}$ with constant sectional curvature $\tilde{c}$. Locally we can consider Φ as being an embedding and we identify $x \in M$ with $\Phi(x) \in \tilde{M}$. In this context, the tangent space T_xM is identified with a subspace of $T_x\tilde{M}$. The normal space $T_x^{\perp}M$ is the subspace of $T_x\tilde{M}$ of all $\xi \in T_x\tilde{M}$ that are orthogonal to T_xM with respect to the metric $\tilde{g}$ of $\tilde{M}$. We denote by $\chi(M)$ and $\chi(M)^{\perp}$, the sets of the C^{∞} vector fields, tangent and normal to M, respectively. Let ∇ and $\tilde{\nabla}$ be the Riemannian connections of M and $\tilde{M}$, respectively. We denote by $D^{\perp}$ the connection of the normal bundle. For each $\xi \in T_x^{\perp}M$ we have a linear transformation A_{ξ} on T_xM defined by

$$\tilde{\nabla}_X\xi = -A_{\xi}(X) + D_X^{\perp}\xi. \tag{1}$$

Given orthonormal vector fields $\xi_1, \dots, \xi_p$ normal to M, we denote $A_{\alpha} = A_{\xi_{\alpha}}$, $\alpha = 1, \dots, A_p$ and we say that A_{α}, is the second fundamental forms associated to ξ_{α}. We define the normal connection forms $S_{\alpha\beta}$ by

$$D_X^{\perp}\xi_{\alpha} = \sum_{\beta=1}^{p} S_{\alpha\beta}(X)\xi_{\beta}, \tag{2}$$

where $X \in \chi(M)$. For all α and β, $S_{\alpha\beta} + S_{\beta\alpha} = 0$. A vector field ξ normal to M is *parallel on the normal bundle*, or simply parallel, if $D_X^{\perp}\xi = 0$, $\forall X \in \chi(M)$.

The second fundamental form h of M is defined by

$$h(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y.$$

Therefore,

$$\tilde{g}(h(X, Y), \xi) = g(A_{\xi}(X), Y)$$

where $X, Y \in \chi(M)$ and $\xi \in \chi(M)^{\perp}$.

Let $E_1, \dots, E_n$ be orthonormal vectors tangent to M at $x \in M$ and $\xi_1, \dots, \xi_p$ be orthonormal vectors normal to M at x. Then,

$$H = \frac{1}{n}\sum_{\alpha=1}^{p}(\mathrm{tr}\ A_\alpha)\xi_\alpha$$

is called the *mean curvature vector* of the immersion Φ. We observe that if H is parallel then $|H|$ is constant.

The *length of the second fundamental form* is defined to be

$$\langle h\rangle^2 = \sum_{\alpha=1}^{p}\mathrm{tr}\ A_\alpha^2. \tag{3}$$

Erbacher [E1] proved that if the mean curvature vector H is parallel in the normal bundle, then the laplacian Δ of $\langle h\rangle^2$ is given by

$$\begin{aligned}\frac{1}{2}\Delta\langle h\rangle^2 &= \tilde{c}n\langle h\rangle^2 - \tilde{c}\sum_{\alpha=1}^{p}(\mathrm{tr}\ A_\alpha)^2 + \sum_{\alpha,\beta=1}^{p}\mathrm{tr}[A_\alpha, A_\beta]^2 \\ &\quad + \sum_{\alpha,\beta=1}^{p}(\mathrm{tr}\ A_\alpha)(\mathrm{tr}\ A_\alpha A_\beta^2) - \sum_{\alpha,\beta=1}^{p}(\mathrm{tr}\ A_\alpha A_\beta)^2 + \sum_{\alpha=1}^{p}\|\nabla^* A_\alpha\|^2,\end{aligned} \tag{4}$$

where ∇^* denotes the sum of the normal and tangent connections,

$$\nabla_X^* A_\alpha = \nabla_X A_\alpha - \sum_{\beta=1}^{p} S_{\alpha\beta}(X)A_\beta. \tag{5}$$

We now choose orthonormal vector fields, normal to M, in such a way that the first one is in the direction of H. Suppose, the mean curvature vector does not vanish at any point of M, i.e. $|H| \neq 0$ in M. Then, we can choose orthonormal vector fields $\xi_1, \dots, \xi_p$ normal to M such that

$$H = |H|\xi_1.$$

We then have the following relations:

$$\mathrm{tr}\ A_1 = n|H| \tag{6}$$

$$\mathrm{tr}\ A_\alpha = 0, \quad \alpha = 2, 3, \dots, p. \tag{7}$$

Considering the normal connection forms $S_{\alpha\beta}$, as defined in (2), we have

$$S_{1\beta} = 0 \quad \beta = 1, \dots, p. \tag{8}$$

We observe that if the mean curvature vector is parallel, then

$$A_1 A_\alpha = A_\alpha A_1, \quad \text{for all } \alpha. \tag{9}$$

In fact, this follows from the Ricci equation

$$\langle R^\perp(X, Y)\xi_1, \xi_\alpha\rangle = \langle [A_1, A_\alpha]X, Y\rangle.$$

Since the left hand side vanishes, we conclude that $[A_1, A_\alpha] = 0$ for all α.

Consider the function

$$|T|^2 = \sum_{\alpha=2}^{p} \operatorname{tr} A_\alpha^2, \tag{10}$$

globally defined on M. Our first lemma describes an expression for the laplacian of $|T|^2$, that will be extremely important to prove our main results in the next section.

For the proof of the lemma, we need the following remark. If B is a tensor of type $1-1$ in M^n, then we have

$$\frac{1}{2}\Delta(\operatorname{tr} B^2) = \operatorname{tr}((\Delta' B)B) + \|\nabla B\|^2 \tag{11}$$

where

$$(\Delta' B)(x) = \sum_{i=1}^{n} \nabla_{E_i}(\nabla_{E_i} B) - \nabla_{\nabla_{E_i} E_i} B$$

and $E_1, \dots, E_n$ are orthonormal tangent vector fields.

Lemma 2.1. *Let $\Phi : M^n \to \tilde{M}^{n+p}(\tilde{c})$ be an isometric immersion. Suppose the mean curvature vector H does not vanish at any point of M, and it is parallel in the normal bundle. Let $\xi_1, \dots, \xi_p$ be an orthonormal frame in $T^\perp M$, such that $H = |H|\xi_1$. Considering $|T|^2$, as defined by* (10), *we have*

$$\begin{aligned} \frac{1}{2}\Delta|T|^2 = {} & \sum_{\alpha=2}^{p}\{(\operatorname{tr} A_1)(\operatorname{tr} A_1 A_\alpha^2) - (\operatorname{tr} A_1 A_\alpha)^2\} \\ & - \sum_{\alpha,\beta=2}^{p}\{\operatorname{tr}[A_\alpha, A_\beta]^t[A_\alpha, A_\beta] + (\operatorname{tr} A_\alpha A_\beta)^2\} + \tilde{c}n|T|^2 + \sum_{\alpha=2}^{p}\|\nabla^* A_\alpha\|^2. \end{aligned} \tag{12}$$

Proof. Initially we note, from (10), that

$$\langle h \rangle^2 = \sum_{\alpha=1}^{p} \operatorname{tr} A_\alpha^2 = \operatorname{tr} A_1^2 + |T|^2. \tag{13}$$

Hence, from (4), we have

$$\begin{aligned} \frac{1}{2}\Delta|T|^2 = {} & \tilde{c}n\langle h\rangle^2 - \tilde{c}\sum_{\alpha=1}^{p}(\operatorname{tr} A_\alpha)^2 + \sum_{\alpha,\beta=1}^{p}\operatorname{tr}[A_\alpha, A_\beta]^2 + \sum_{\alpha,\beta=1}^{p}(\operatorname{tr} A_\alpha)(\operatorname{tr} A_\alpha A_\beta^2) \\ & - \sum_{\alpha,\beta=1}^{p}(\operatorname{tr} A_\alpha A_\beta)^2 + \sum_{\alpha=1}^{p}\|\nabla^* A_\alpha\|^2 - \frac{1}{2}\Delta(\operatorname{tr} A_1^2). \end{aligned}$$

Using (9), (7) and the fact that $[A_\alpha, A_\beta]^t = -[A_\alpha, A_\beta]$, we have

$$(14)\quad \frac{1}{2}\Delta|T|^2 = \tilde{c}n\langle h\rangle^2 - \tilde{c}(\mathrm{tr}\ A_1)^2 + \sum_{\alpha=2}^{p}\{(\mathrm{tr}\ A_1)(\mathrm{tr}\ A_1A_\alpha^2) - 2(\mathrm{tr}\ A_1A_\alpha)^2\}$$
$$- \sum_{\alpha,\beta=2}^{p}\{\mathrm{tr}[A_\alpha, A_\beta]^t[A_\alpha, A_\beta] + (\mathrm{tr}\ A_\alpha A_\beta)^2\} + (\mathrm{tr}\ A_1)(\mathrm{tr}\ A_1A_1^2)$$
$$- (\mathrm{tr}\ A_1A_1)^2 + \sum_{\alpha=1}^{p}\|\nabla^* A_\alpha\|^2 - \frac{1}{2}\Delta(\mathrm{tr}\ A_1^2).$$

We now establish the expression for $\Delta(\mathrm{tr}\ A_1^2)$. It follows from (11), that

$$(15)\qquad \frac{1}{2}\Delta(\mathrm{tr}\ A_\alpha^2) = \mathrm{tr}((\Delta' A_\alpha)A_\alpha) + \|\nabla A_\alpha\|^2.$$

Erbacher [E1] obtained the following expression for $\Delta' A_\alpha$:

$$(16)\qquad \Delta' A_\alpha = n\tilde{c}A_\alpha - \tilde{c}(\mathrm{tr}\ A_\alpha)I + \sum_\beta(\mathrm{tr}\ A_\beta)A_\alpha A_\beta - \sum_\beta(\mathrm{tr}\ A_\beta A_\alpha)A_\beta$$
$$+ \sum_\beta[A_\beta, A_\alpha A_\beta] + \sum_\beta A_\beta[A_\alpha, A_\beta] + \sum_{i,\beta}(\nabla_{E_i}S_{\alpha\beta})(E_i)A_\beta$$
$$+ 2\sum_{i,\beta}S_{\alpha\beta}(E_i)\nabla_{E_i}A_\beta - \sum_{i,\beta,\gamma}S_{\alpha\beta}(E_i)S_{\beta\gamma}(E_i)A_\gamma.$$

Substituting (16) in (15) and using the properties of the trace function, we obtain

$$(17)\qquad \frac{1}{2}\Delta f_\alpha = n\tilde{c}\ \mathrm{tr}\ A_\alpha^2 - \tilde{c}(\mathrm{tr}\ A_\alpha)^2 + \sum_\beta(\mathrm{tr}\ A_\beta)\ \mathrm{tr}(A_\alpha A_\beta A_\alpha)$$
$$- \sum_\beta(\mathrm{tr}\ A_\beta A_\alpha)^2 \sum_\beta \mathrm{tr}[A_\beta, A_\alpha A_\beta]A_\alpha + \sum_\beta \mathrm{tr}\ A_\beta[A_\alpha, A_\beta]A_\alpha$$
$$+ \sum_{i,\beta}(\nabla_{E_i}S_{\alpha\beta})(E_i)\ \mathrm{tr}\ A_\beta A_\alpha + 2\sum_{i,\beta}S_{\alpha\beta}(E_i)\ \mathrm{tr}(\nabla_{E_i}A_\beta)A_\alpha$$
$$- \sum_{i,\beta,\gamma}S_{\alpha\beta}(E_i)S_{\beta\gamma}(E_i)\ \mathrm{tr}\ A_\gamma A_\alpha + \|\nabla A_\alpha\|^2.$$

Now, consider $\alpha = 1$ in (17). It follows from (7), (8), (9), and the relation

$$[A_\beta, A_1A_\beta]A_1 = A_\beta A_1 A_\beta A_1 - A_1 A_\beta A_\beta A_1 = 0,$$

that

$$(18)\quad \frac{1}{2}\Delta(\mathrm{tr}\ A_1^2) = n\tilde{c}\ \mathrm{tr}\ A_1^2 - \tilde{c}(\mathrm{tr}\ A_1)^2 + (\mathrm{tr}\ A_1)(\mathrm{tr}\ A_1A_1A_1) - (\mathrm{tr}\ A_1A_1)^2$$
$$- \sum_{\beta=2}^{p}(\mathrm{tr}\ A_\beta A_1)^2 + \|\nabla A_1\|^2.$$

Substituting (18) in (14) and using (9) and (13), we have that

$$\frac{1}{2}\Delta|T|^2 = \tilde{c}n|T|^2 + \sum_{\alpha=2}^{p}\{(\text{tr } A_1)(\text{tr } A_1A_\alpha^2) - (\text{tr } A_1A_\alpha)^2\}$$

$$- \sum_{\alpha,\beta=2}^{p}\{\text{tr}[A_\alpha, A_\beta]^t[A_\alpha, A_\beta] + (\text{tr } A_\alpha A_\beta)^2\} + \|\nabla^* A_1\|^2$$

$$+ \sum_{\alpha=2}^{p}\|\nabla^* A_\alpha\|^2 - \|\nabla A_1\|^2.$$

Then, it follows from (5) that (12) holds. This completes the proof of Lemma 1.1. □

Introducing the following notation

$$N(A_\alpha) = \text{tr } A_\alpha^t A_\alpha, \quad Z_{\alpha\beta} = \text{tr } A_\alpha A_\beta, \tag{19}$$

we can rewrite (12) as follows

$$\frac{1}{2}\Delta|T|^2 = \sum_{\alpha=2}^{p}\{(\text{tr } A_1)(\text{tr } A_1A_\alpha^2) - (\text{tr } A_1A_\alpha)^2\}$$

$$- \sum_{\alpha,\beta=2}^{p}\{N(A_\alpha A_\beta - A_\beta A_\alpha) + Z_{\alpha\beta}^2\} + \tilde{c}n|T|^2 + \sum_{\alpha=2}^{p}\|\nabla^* A_\alpha\|^2.$$

The following two algebraic results will be very useful for the proof of our main results.

LEMMA 2.2 [LL]. *Let $A_1, \dots, A_l$ be symmetric $n \times n$ matrices. Then,*

$$\sum_{\alpha,\beta=1}^{l}\{N(A_\alpha A_\beta - A_\beta A_\alpha) + Z_{\alpha\beta}^2\} \le \frac{3}{2}\left(\sum_{\alpha=1}^{l} N(A_\alpha)\right)^2, \tag{20}$$

where N and Z are defined by (19). *Equality holds if, and only if, one of the following conditions hold*:

1. $A_1 = \cdots = A_l = 0$;
2. *Only two of the matrices $A_1, \dots, A_l$ are nonzero matrices. Moreover, assuming $A_1 \neq 0$ and $A_2 \neq 0$, then $N(A_1) = N(A_2) := L$ and there exists an orthogonal matrix T such that*

$$T^t A_1 T = \sqrt{\frac{L}{2}}\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad T^t A_2 T = \sqrt{\frac{L}{2}}\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

LEMMA 2.3 (Chen [C]). *Let $a_1, \dots, a_n$ and b be $n+1$ real numbers, $n > 1$, which satisfy the following*

$$\left(\sum_{i=1}^{n} a_i\right)^2 \geq (n-1)\sum_{i=1}^{n} a_i^2 + b \quad (resp.\ >).$$

Then, for all $i \neq j$,

$$2a_i a_j \geq \frac{b}{n-1} \quad (resp.\ >)$$

3. Proof of the main results

We begin this section proving the following lemma:

LEMMA 3.1. *Let M^n be a submanifold $(n \geq 3)$ in $\tilde{M}^{n+p}$. Suppose the mean curvature vector H does not vanish and it is parallel in the normal bundle. Let $\xi_1, \dots, \xi_p$ be orthonormal vector fields in $T^{\perp}M$ such that $H = |H|\xi_1$. If the second fundamental form h of M satisfies*

$$\langle h\rangle^2 \leq \frac{n^2|H|^2}{n-1}, \tag{21}$$

then

$$\sum_{\alpha=2}^{p}\{(\operatorname{tr} A_1)(\operatorname{tr} A_1 A_\alpha^2) - (\operatorname{tr} A_1 A_\alpha)^2\} \geq n\frac{|T|^4}{2} \tag{22}$$

$$\sum_{\alpha,\beta=2}^{p}\{N(A_\alpha A_\beta - A_\beta A_\alpha) + Z_{\alpha\beta}^2\} \leq \frac{3}{2}\left(\sum_{\alpha=2}^{p} N(A_\alpha)\right)^2 = \frac{3}{2}|T|^4, \tag{23}$$

where N, Z and T are defined by (19) *and* (10). *Moreover, the Ricci curvature of M^n has a lower bound.*

Proof. It follows from the hypothesis that (6), (7) and (9) hold. Therefore, for each $\alpha \geq 2$, A_1 and A_α can be simultaneously diagonalized. Let $\rho_1, \dots, \rho_n$ and $\rho_1^\alpha, \dots, \rho_n^\alpha$ be the eigenvalues of A_1 and A_α, respectively. We observe that for each fixed $\alpha \geq 2$,

$$(\operatorname{tr} A_1)(\operatorname{tr} A_1 A_\alpha^2) - (\operatorname{tr} A_1 A_\alpha)^2 = \left(\sum_{i=1}^{n}\rho_i\right)\left(\sum_{j=1}^{n}\rho_j(\rho_j^\alpha)^2\right) - \left(\sum_{i=1}^{n}\rho_i\rho_i^\alpha\right)\left(\sum_{j=1}^{n}\rho_j\rho_j^\alpha\right).$$

Therefore,

$$(\operatorname{tr} A_1)(\operatorname{tr} A_1 A_\alpha^2) - (\operatorname{tr} A_1 A_\alpha)^2 = \frac{1}{2}\sum_{i,j=1}^{n}\rho_i\rho_j(\rho_i^\alpha - \rho_j^\alpha)^2. \tag{24}$$

On the other hand, the hypothesis (21) is equivalent to

$$(n|H|)^2 \geq (n-1)\ \mathrm{tr}\ A_1^2 + (n-1)|T|^2.$$

Since (6) holds, we obtain the inequality

$$\left(\sum_{i=1}^{n} \rho_i\right)^2 \geq (n-1)\sum_{i=1}^{n}(\rho_i)^2 + (n-1)|T|^2. \tag{25}$$

It follows from Lemma 2.3 that

$$\rho_i\rho_j \geq \frac{|T|^2}{2}, \quad i \neq j. \tag{26}$$

Substituting the inequality (26) in (24), we obtain

$$\begin{aligned}(\mathrm{tr}\ A_1)(\mathrm{tr}\ A_1A_\alpha^2) - (\mathrm{tr}\ A_1A_\alpha^2) &\geq \frac{1}{4}|T|^2\sum_{i,j=1}^{n}(\rho_i^\alpha - \rho_j^\alpha)^2 \\ &= \frac{1}{2}|T|^2\left\{\sum_{i,j=1}^{n}(\rho_i^\alpha)^2 - \sum_{i,j=1}^{n}\rho_i^\alpha\rho_j^\alpha\right\} \\ &= \frac{|T|^2}{2}n\sum_{i=1}^{n}(\rho_i^\alpha)^2 - \frac{1}{2}|T|^2\left(\sum_{i=1}^{n}\rho_i^\alpha\right)^2.\end{aligned}$$

Therefore, for each $\alpha \geq 2$, since (7) holds, we have

$$(\mathrm{tr}\ A_1)(\mathrm{tr}\ A_1A_\alpha^2) - (\mathrm{tr}\ A_1A_\alpha^2) \geq \frac{|T|^2}{2}n\ \mathrm{tr}\ A_\alpha^2. \tag{27}$$

Summing over α, we obtain the inequality (22). The proof of (23) follows from Lemma 2.2.

We will now show that the Ricci curvature has a lower bound. For each fixed $\alpha \geq 2$, using (21) and the fact that $|T|^2 \leq \langle h\rangle^2$, we have

$$(n-1)\ \mathrm{tr}\ A_\alpha^2 - n^2|H|^2 \leq (n-1)|T|^2 - n^2|H|^2 \leq 0 = (\mathrm{tr}\ A_\alpha)^2.$$

This can be written as

$$0 = \left(\sum_{i=1}^{n}\rho_i^\alpha\right)^2 \geq (n-1)\sum_{i=1}^{n}(\rho_i^\alpha)^2 - n^2|H|^2.$$

Hence, it follows from Lemma 2.3 that

$$\rho_i^\alpha\rho_j^\alpha \geq \frac{-n^2|H|^2}{2(n-1)}, \quad i \neq j.$$

This inequality together with (26) implies that the sectional curvature of M^n has a lower bound. Consequently, the Ricci curvature of M^n has a lower bound. $\square$

In the next result we treat the case $\tilde{c} \geq 0$. We remark that, when $\tilde{c} = 0$, we obtain the result proved by Cheng and Nonaka [CN].

THEOREM 3.2. *Let M^n, $n \geq 3$, be a complete connected submanifold of $\tilde{M}^{n+p}(\tilde{c})$, $\tilde{c} \geq 0$. Suppose the mean curvature vector H does not vanish and it is parallel in the normal bundle. If the second fundamental form h of M satisfies*

$$\langle h \rangle^2 \leq \frac{n^2|H|^2}{n-1}, \tag{28}$$

then the codimension reduces to 1.

Proof. Since H does not vanish, we can choose orthonormal vector fields $\xi_1, \dots, \xi_p$, normal to M, such that $\xi_1 = H/|H|$. Hence, (6) and (7) hold. We consider $|T|^2$ defined by (10). It follows from Lemma 2.1 that (12) holds. Using the notation introduced in (19), we have that

$$\begin{aligned}\frac{1}{2}\Delta|T|^2 = &\sum_{\alpha=2}^{p}\{(\mathrm{tr}\,A_1)(\mathrm{tr}\,A_1A_\alpha^2) - (\mathrm{tr}\,A_1A_\alpha)^2\} \\ &- \sum_{\alpha,\beta=2}^{p}\{N(A_\alpha A_\beta - A_\beta A_\alpha) + Z_{\alpha\beta}^2\} + \tilde{c}n|T|^2 + \sum_{\alpha=2}^{p}\|\nabla^* A_\alpha\|^2.\end{aligned}$$

Motivated by (22) and (23), we now define P_1 and P_2, as

$$P_1 = \sum_{\alpha=2}^{p}\{(\mathrm{tr}\,A_1)(\mathrm{tr}\,A_1A_\alpha^2) - (\mathrm{tr}\,A_1A_\alpha)^2\}, \tag{29}$$

$$P_2 = \sum_{\alpha,\beta=2}^{p}\{N(A_\alpha A_\beta - A_\beta A_\alpha) + Z_{\alpha\beta}^2\}. \tag{30}$$

Then

$$\frac{1}{2}\Delta|T|^2 = P_1 - P_2 + \tilde{c}n|T|^2 + \sum_{\alpha=2}^{p}\|\nabla^* A_\alpha\|^2. \tag{31}$$

It follows from Lemma 3.1 that $P_1 \geq n|T|^4/2$, and $P_2 \leq 3|T|^4/2$. Hence,

$$P_1 - P_2 \geq \frac{(n-3)}{2}|T|^4. \tag{32}$$

Therefore, since $\tilde{c} \geq 0$, it follows that

$$\frac{1}{2}\Delta|T|^2 \geq P_1 - P_2 \geq \frac{(n-3)}{2}|T|^4. \tag{33}$$

From Lemma 3.1, the Ricci curvature of M^n has a lower bound and by hypothesis $|T|^2$, is bounded from above by $n^2|H|^2/(n-1)$. Therefore, the Generalized Maximum Principle (see [O], [Y]), applied to the function $|T|^2$, implies that there is a sequence $\{x_k\}$ of points on M, such that

$$\lim_{k\to\infty} |T|^2(x_k) = \sup|T|^2, \tag{34}$$

$$\lim_{k\to\infty} \sup \Delta|T|^2(x_k) \le 0. \tag{35}$$

Then, using (35), (33) and (34), we obtain

$$0 \ge \lim_{k\to\infty} \sup \Delta|T|^2(x_k) \ge (n-3)\left(\lim_{k\to\infty} |T|^2(x_k)\right)^2 = (n-3)(\sup|T|^2)^2 \ge 0.$$

Therefore,

$$(n-3)(\sup|T|^2)^2 = 0.$$

If $n \ge 4$, then we must have $|T|^2 = 0$ on M. Hence, for each $\alpha \ge 2$, $A_\alpha = 0$ and consequently, the first normal space is generated $\xi_1(x)$. We conclude from Erbacher's theorem [E2], that the codimension of the immersion reduces to 1.

Now, suppose $n = 3$. From (33) we have $\Delta|T|^2 \ge 0$. So, it follows from (35) that

$$\lim_{k\to\infty} \sup \Delta|T|^2(x_k) = 0. \tag{36}$$

Observe that the hypothesis (28) implies that the sequence $\{h_{ji}^\alpha(x_k)\}_{k=1}^\infty$ is bounded for each j, i and α. Therefore there is a convergent subsequence $\{h_{ji}^\alpha(x_{k_r})\}_{k_r=1}^\infty$. Define

$$\bar{A}_\alpha = \lim_{k_r\to\infty} A_\alpha(x_{k_r}).$$

Restricting the inequality (33) to the sequence $\{x_{k_r}\}_{k_r=1}^\infty$, taking the limit and denoting by $\overline{P_1}$ and $\overline{P_2}$ the limits taken in P_1 and P_2, respectively, we obtain using (35) and (34) that

$$0 \ge \lim_{k_r\to\infty} \sup \Delta|T|^2(x_{k_r}) \ge 2(\overline{P_1} - \overline{P_2}) \ge (n-3)(\sup|T|^2)^2 \ge 0.$$

This implies $\overline{P_1} = \overline{P_2}$. Hence, when we take the limit in (22) and (23), we have the following equalities

$$\sum_{\alpha=2}^{p} \{\mathrm{tr}\, \bar{A}_1 \,\mathrm{tr}\, \bar{A}_1\bar{A}_\alpha^2 - (\mathrm{tr}\, \bar{A}_1\bar{A}_\alpha)^2\} = \frac{3}{2} \sup|T|^2 \sum_{\alpha=2}^{p} \mathrm{tr}\, \bar{A}_\alpha^2 \tag{37}$$

$$\sum_{\alpha,\beta=2}^{p} \{N(\bar{A}_\alpha\bar{A}_\beta - \bar{A}_\beta\bar{A}_\alpha) + \bar{Z}_{\alpha\beta}^2\} = \frac{3}{2}\left(\sum_{\alpha=2}^{p} N(\bar{A}_\alpha)\right)^2. \tag{38}$$

From Lemma 2.2, applied to (38), we have either

a) $\bar{A}_2 = \cdots = \bar{A}_p = 0$; or

b) Only two matrices among $\bar{A}_2, \dots, \bar{A}_p$ are not zero. In this case, we may assume, without loss of generality, $\bar{A}_2 \neq 0$ and $\bar{A}_3 \neq 0$. Moreover, there is an orthogonal matrix T such that

$$T^t \bar{A}_2 T = \sqrt{\frac{\bar{L}}{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad T^t \bar{A}_3 T = \sqrt{\frac{\bar{L}}{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{39}$$

where $\bar{L} = \operatorname{tr} \bar{A}_2^2 = N(\bar{A}_2) = N(\bar{A}_3)$.

If a) occurs, then we have

$$\sup|T|^2 = \sum_{\alpha=2}^{p} \operatorname{tr} \bar{A}_\alpha^2 = 0.$$

Therefore, $|T|^2 = 0$ on M and the codimension reduces to 1.

We will now prove that b) cannot occur. Suppose, by contradiction, that b) occurs. For each $\alpha = 2, \dots, p$, as we saw in (24), we have

$$\begin{aligned} \operatorname{tr} A_1 \operatorname{tr} A_1 A_\alpha^2 - (\operatorname{tr} A_1 A_\alpha)^2 &= \frac{1}{2} \sum_{i,j=1}^{n} \rho_i \rho_j (\rho_i^\alpha - \rho_j^\alpha)^2 \geq \frac{1}{4} |T|^2 \sum_{i,j=1}^{n} (\rho_i^\alpha - \rho_j^\alpha)^2 \\ &= \frac{3}{2} |T|^2 \operatorname{tr} A_\alpha^2 \end{aligned}$$

where we used (26) and (27). Hence, restricting this inequality to the sequence $\{x_{k_r}\}$ and taking the limit, we get from (37) that

$$\sum_{i,j=1}^{n} \bar{\rho}_i \bar{\rho}_j (\bar{\rho}_i^\alpha - \bar{\rho}_j^\alpha)^2 = \frac{1}{2} \sup|T|^2 \sum_{i,j=1}^{n} (\bar{\rho}_i^\alpha - \bar{\rho}_j^\alpha)^2, \tag{40}$$

where $\bar{\rho}_i = \lim_{k_r \to \infty} \rho_i(x_{k_r})$ e $\bar{\rho}_i^\alpha = \lim_{k_r \to \infty} \rho_i^\alpha(x_{k_r})$. Moreover, from (39) we have $\bar{\rho}_i^2 \neq \bar{\rho}_j^2$, for $i \neq j$. Then, from (26) and (40) we obtain

$$\bar{\rho}_i \bar{\rho}_j = \frac{1}{2} \sup|T|^2, \quad \text{for } i \neq j. \tag{41}$$

Now, we observe that

$$\begin{aligned} \operatorname{tr} A_1^2 - n|H|^2 &= \operatorname{tr} A_1^2 - 2n|H|^2 + n|H|^2 \\ &= \sum_{i=1}^{3} (\rho_i)^2 - 2|H| \sum_{i=1}^{3} \rho_i + \sum_{i=1}^{3} |H|^2 \\ &= \sum_{i=1}^{3} (\rho_i - |H|)^2 \geq 0. \end{aligned}$$

Then, we define

$$|U|^2 := \mathrm{tr}\ A_1^2 - n|H|^2 \geq 0. \tag{42}$$

One can check that

$$\sum_{i\neq j} \rho_i\rho_j = [\mathrm{tr}\ A_1]^2 - \mathrm{tr}\ A_1^2.$$

Since, for $n = 3$ we have

$$\begin{aligned}[\mathrm{tr}\ A_1]^2 - \mathrm{tr}\ A_1^2 &= 6|H|^2 - [\mathrm{tr}\ A_1^2 - 3|H|^2] \\ &= 6|H|^2 - |U|^2.\end{aligned}$$

Therefore, we have the equality

$$\sum_{i\neq j} \rho_i\rho_j = 6|H|^2 - |U|^2. \tag{43}$$

Restricting to the sequence $\{x_{k_r}\}$, taking the limit and using (41), we obtain

$$\begin{aligned}6|H|^2 &= \lim_{k_r\to\infty} |U|^2(x_{k_r}) + \sum_{i\neq j} \bar{\rho}_i\bar{\rho}_j \\ &= \lim_{k_r\to\infty} |U|^2(x_{k_r}) + \sum_{i\neq j} \frac{1}{2} \sup|T|^2.\end{aligned}$$

Hence,

$$\lim_{k_r\to\infty} |U|^2(x_{k_r}) + 3 \sup|T|^2 = 6|H|^2. \tag{44}$$

On the other hand, applying the limit to the inequality (25), using (41) and Lemma 2.3, we have

$$\left(\sum_{i=1}^n \bar{\rho}_i\right)^2 = (n-1)\sum_{i=1}^n (\bar{\rho}_i)^2 + (n-1)\sup|T|^2. \tag{45}$$

We observe that,

$$\sum_{i=1}^n \bar{\rho}_i = \mathrm{tr}\ \bar{A}_1 = n|H|, \quad \sum_{i=1}^n (\bar{\rho}_i)^2 = \mathrm{tr}\ \bar{A}_1^2 \quad \mathrm{e} \quad \sup|T|^2 = \sum_{\alpha=2}^p \mathrm{tr}\ \bar{A}_\alpha^2.$$

So (45) for $n = 3$ reduces to

$$(3|H|)^2 = 2\ \mathrm{tr}\ \bar{A}_1^2 + 2\sum_{\alpha=2}^p \mathrm{tr}\ \bar{A}_\alpha^2 = 2 \lim_{k_r\to\infty} \langle h\rangle^2(x_{k_r}),$$

that becomes

$$\lim_{k_r\to\infty}\langle h\rangle^2(x_{k_r}) = \frac{9}{2}|H|^2. \tag{46}$$

From (42), we have $\langle h\rangle^2 = |U|^2 + |T|^2 + n|H|^2$. Hence, it follows from (46) that

$$\lim_{k_r\to\infty}|U|^2(x_{k_r}) + \sup|T|^2 + 3|H|^2 = \frac{9}{2}|H|^2,$$

that is,

$$\sup|T|^2 = \frac{3}{2}|H|^2 - \lim_{k_r\to\infty}|U|^2(x_{k_r}). \tag{47}$$

Substituting (47) in (44), we have

$$\begin{aligned}6|H|^2 &= \lim_{k_r\to\infty}|U|^2(x_{k_r}) + 3\left(\frac{3}{2}|H|^2 - \lim_{k_r\to\infty}|U|^2(x_{k_r})\right)\\ &= -2\lim_{k_r\to\infty}|U|^2(x_{k_r}) + \frac{9}{2}|H|^2.\end{aligned}$$

Whence it follows that

$$\lim_{k_r\to\infty}|U|^2(x_{k_r}) = -\frac{3}{4}|H|^2 < 0.$$

This is a contradiction. Therefore, b) cannot occur. This concludes the proof of the theorem. □

Corollary 3.3. *Let M^n be a complete connected submanifold ($n \geq 3$) of the sphere $S^{n+p}(1)$. Suppose the mean curvature vector H does not vanish and it is parallel in the normal bundle. If the second fundamental form h of M satisfies*

$$\langle h\rangle^2 \leq \frac{n^2|H|^2}{n-1},$$

then M^n is totally umbilic in $S^{n+1}(1)$.

Proof. If $\tilde{c} = 1$ in Theorem 3.2, then M^n is contained in the sphere $S^{n+1}(1)$ and it has constant mean curvature. Let $\{E_1, \dots, E_n\}$ be an orthonormal basis of T_pM for which the second fundamental form A is diagonal. Denote by $\rho_1, \rho_2, \dots, \rho_n$ the eigenalues of A. Then, Gauss equation can be written

$$K(E_i, E_j) - 1 = \rho_i\rho_j, \quad i \neq j$$

where $K(E_i, E_j)$ denotes the secctional curvature. It follows from (26) that M^n has secctional curvature greater than or equal to 1. We conclude, using Theorem 2 proved by Nomizu and Smyth in [NS], that M^n is totally umbilic in $S^{n+1}(1)$. □

A result analogous to Theorem 3.2, when $\tilde{c} < 0$, needs an additional condition (see (48)) that is trivially satisfied when $\tilde{c} \geq 0$.

THEOREM 3.4. *Let M^n, $n \geq 3$, be a complete and connected submanifold of the hyperbolic space $\mathbf{H}^{n+p}(\tilde{c})$, $\tilde{c} < 0$. Suppose the mean curvature vector does not vanish and it is parallel in the normal bundle. Let $\xi_1, \xi_2, \dots, \xi_p$ be orthonormal vector fields normal to M such that $H = |H|\xi_1$. If the second fundamental form h of M satisfies*

$$\langle h \rangle^2 \leq \frac{n^2|H|^2}{n-1}$$

and

$$\tilde{c}n|T|^2 + \sum_{\alpha=2}^{p} \|\nabla^* A_\alpha\|^2 \geq 0, \tag{48}$$

where A_α is the second fundamental form associated to ξ_α, $|T|^2$ and ∇^* are defined by (10) *and* (5), *then the codimension reduces to* 1.

Proof. The proof starts with the same arguments used in the proof of Theorem 3.2. We obtain as in (31) that

$$\frac{1}{2}\Delta|T|^2 = P_1 - P_2 + \tilde{c}n|T|^2 + \sum_{\alpha=2}^{p} \|\nabla^* A_\alpha\|^2$$

where P_1 and P_2 where defined in (29) and (30) and moreover the inequality (32) holds. Using the hypothesis (48) and (32), we have

$$\frac{1}{2}\Delta|T|^2 \geq P_1 - P_2 \geq \frac{(n-3)}{2}|T|^4.$$

Now the proof follows with the arguments used in Theorem 3.2 for the case $\tilde{c} \geq 0$. □

We will now prove, that a compact submanifold with non-negative Ricci curvature and satisfying the hypothesis of Theorem 3.4 is a geodesic sphere in $\mathbf{H}^{n+1}(-1)$. This is a submanifold, whose points are at a fixed distance, from a given point. Such hipersurfaces are totally umbilic [MB].

THEOREM 3.5. *Let M^n be a compact connected submanifold of $\mathbf{H}^{n+p}(-1)$. Suppose that the mean curvature vector H does not vanish and it is parallel in the normal bundle and that the Ricci curvature is non-negative. Let $\xi_1, \dots, \xi_p$ be orthonormal vector fields normal to M such that $H = |H|\xi_1$. If the second fundamental form h of M satisfies*

$$\langle h \rangle^2 \leq \frac{n^2|H|^2}{n-1}$$

and

$$\tilde{c}n|T|^2 + \sum_{\alpha=2}^{p} \|\nabla^* A_\alpha\|^2 \geq 0,$$

then M^n is a geodesic sphere.

Proof. It follows from Theorem 3.4 that M^n is a hypersurface of $\mathbf{H}^{n+1}(-1)$. The Ricci curvature of M^n is non-negative by hypothesis. Since the mean curvature vector is parallel in the normal bundle, we have that the mean curvature of M^n in $\mathbf{H}^{n+1}(-1)$ is constant. We conclude the proof by using a theorem, proved by Morvan and Bao-Qiang [MB] that any compact hypersurface of $\mathbf{H}^{n+1}(-1)$ with non-negative Ricci curvature and constant mean curvature is a geodesic sphere. □

We conclude this section with our next result, where we consider an alternative hypothesis (see (50)), to the condition (48) of Theorem 3.4, that also allows us to controll the sign of the laplacian of $|T|^2$. In this case, as we will see, the codimension is reduced to 3.

Theorem 3.6. *Let M^n, $n \geq 4$, be a complete connected submanifold of $\mathbf{H}^{n+p}(\tilde{c})$, with $\tilde{c} < 0$ and $p \geq 3$. Suppose that the mean curvature vector H does not vanish and it is parallel in the normal bundle. Let $\xi_1, \dots, \xi_p$ orthonormal vector fields normal to M such that $H = |H|\xi_1$. Furthermore, suppose that the first normal space is invariant by parallel translation with respect to the normal connection. If the second fundamental form h of M satisfies*

$$\langle h \rangle^2 \leq \frac{n^2|H|^2}{n-1} \tag{49}$$

and

$$|T|^2 \geq -\frac{2\tilde{c}n}{n-3}, \tag{50}$$

where $|T|^2$ is defined by (10), *then the codimension reduces to* 3.

Proof. With the same arguments used in the proof of Theorem 3.2. We obtain as in (31) that

$$\frac{1}{2}\Delta|T|^2 = P_1 - P_2 + \tilde{c}n|T|^2 + \sum_{\alpha=2}^{p} \|\nabla^* A_\alpha\|^2$$

where P_1 and P_2 where defined in (29) and (30). Moreover, it follows from (22) and (23) that

$$P1 \geq n\frac{|T|^4}{2}, \quad P2 \leq \frac{3}{2}\left(\sum_{\alpha=2}^{p} N(A_\alpha)\right)^2 = \frac{3}{2}|T|^4, \tag{51}$$

and the inequality (32) holds. Using (32), we have

$$\frac{1}{2}\Delta|T|^2 \geq \left[\frac{(n-3)}{2}|T|^2 + \tilde{c}n\right]|T|^2. \tag{52}$$

On the other hand, since (49) holds, by the Generalized Maximum Principle applied to $|T|^2$, we have the existence of a sequence $\{z_k\}$ of points of M such that

$$\lim_{k\to\infty} |T|^2(z_k) = \sup|T|^2 \quad \lim_{k\to\infty} \sup \Delta|T|^2(z_k) \leq 0. \tag{53}$$

Then, it follows from (52), (53) and the hypothesis (50), that

$$\begin{aligned} 0 \geq \lim_{k\to\infty} \sup \Delta|T|^2(z_k) &\geq 2 \lim_{k\to\infty}\left[\frac{(n-3)}{2}|T|^2(z_k) + \tilde{c}n\right]|T|^2(z_k) \\ &= 2\left[\frac{(n-3)}{2} \sup|T|^2 + \tilde{c}n\right] \sup|T|^2 \geq 0. \end{aligned}$$

Therefore,

$$\left[\frac{(n-3)}{2} \sup|T|^2 + \tilde{c}n\right] \sup|T|^2 = 0.$$

From (50), we have that $\sup|T|^2 > 0$. Hence, $\sup|T|^2 = -2\tilde{c}n/(n-3)$, and it follows from (50) that

$$|T|^2 \equiv -\frac{2\tilde{c}n}{n-3}. \tag{54}$$

Therefore, we have

$$0 = \frac{1}{2}\Delta|T|^2 \geq P_1 - P_2 + \tilde{c}n|T|^2 \geq \left[\frac{(n-3)}{2}|T|^2 + \tilde{c}n\right]|T|^2 = 0.$$

Hence, we have the equalities

$$P_1 - P_2 + \tilde{c}n|T|^2 = 0$$

and

$$\frac{(n-3)}{2}|T|^4 + \tilde{c}n|T|^2 = 0,$$

that imply

$$P_1 - P_2 = \frac{(n-3)}{2}|T|^4. \tag{55}$$

From (51), we have $P_2 \leq 3|T|^4/2$. We claim that equality holds. In fact, otherwise from (51), we would contradict (55). Therefore,

$$P_2 = 3|T|^4/2 = \frac{3}{2}\left(\sum_{\alpha=2}^{p} N(A_\alpha)\right)^2.$$

Therefore, at a given point $y_0 \in M$, Lemma 2.2 implies, that either a) $A_\alpha(y_0) = 0$, for all $\alpha \geq 2$ or b) only two among the matrices $A_\alpha(y_0)$ are not zero. In this case, we may assume wihout loss of generality, that $A_2(y_0) \neq 0$ and $A_3(y_0) \neq 0$.

We observe that a) cannot occur, since otherwise, we would have $|T|^2(y_0) = 0$, which contradicts (54). Then, b) must occur. Hence, the first normal space is generated by $\xi_1(y_0)$, $\xi_2(y_0)$ and $\xi_3(y_0)$ and by hypothesis, it is invariant by parallel translation of the normal connection. Therefore, it follows from Erbacher's theorem [E2], that the codimension reduces to 3. □

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