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ON TORIC HYPERKÄHLER MANIFOLDS WITH COMPACT COMPLEX SUBMANIFOLDS

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Abstract

A *toric hyperkähler manifold* is defined as a hyperkähler quotient of the flat quaternionic space $\mathbf{H}^N$ by a subtorus of the real torus T^N. The purposes of this paper are to construct compact complex submanifolds of toric hyperkähler manifolds, and to show that our hyperkähler manifold is a resolution of singularities of an affine algebro-geometric quotient. We also show that these submanifolds are biholomorphic to Delzant spaces, which are Kähler quotients of $\mathbf{C}^N$ by subtori of T^N. Finally, we apply these results to determining whether complex structures on our hyperkähler manifold are equivalent.

1. Introduction

A Riemannian manifold is said to be *hyperkählerian* precisely when this manifold is equipped with three complex structures $\boldsymbol{I}$, $\boldsymbol{J}$, and $\boldsymbol{K}$ that satisfy the algebraic relations of the quaternions i, j, k and the Riemannian metric is Kählerian with respect to $\boldsymbol{I}$, $\boldsymbol{J}$, and $\boldsymbol{K}$. The flat quaternionic space $\mathbf{H}^N$ is an example of a hyperkähler manifold. We denote the Kähler form corresponding to the complex structure $\boldsymbol{I}$ (respectively $\boldsymbol{J}$, $\boldsymbol{K}$) by $\omega_{\boldsymbol{I}}$ (respectively $\omega_{\boldsymbol{J}}$, $\omega_{\boldsymbol{K}}$). There exists a way to construct a new hyperkähler manifold from an old one with a group action: the hyperkähler quotient method of Hitchin, Karlhede, Lindström, and Roček [6, §3.(D)]. Bielawski and Dancer defined a *toric hyperkähler manifold* as a hyperkähler quotient of $\mathbf{H}^N$ by a subtorus of $T^N := U(1)^N$ [2, §3]. Let K be a subtorus of T^N. Let $\mathfrak{k}$ be the Lie algebra of K, and let $\mathfrak{k}^*$ be the dual space of $\mathfrak{k}$. Set $\mathfrak{k}_{\mathbf{C}}^* := \mathfrak{k}^* \otimes \mathbf{C}$. We restrict the natural action of T^N on $\mathbf{H}^N$ to K. We use $\mu := (\mu_{\boldsymbol{I}}, \mu_{\mathbf{C}}) : \mathbf{H}^N \to \mathfrak{k}^* \times \mathfrak{k}_{\mathbf{C}}^*$ to denote the hyperkähler moment map for the action of K on $\mathbf{H}^N$. If $(\alpha, \beta) \in \mathfrak{k}^* \times \mathfrak{k}_{\mathbf{C}}^*$ is a regular value of μ and if K acts freely on $\mu^{-1}(\alpha, \beta)$, then we obtain the toric hyperkähler manifold

$$X(\alpha, \beta) := \mu^{-1}(\alpha, \beta)/K.$$

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The quotient group $T^n = T^N/K$ acts in the natural way on $X(\alpha,\beta)$. Let $\phi : X(\alpha,\beta) \to (\mathbf{R}^n)^* \times (\mathbf{C}^n)^*$ be the hyperkähler moment map for the action of T^n on $X(\alpha,\beta)$.

Let $K_{\mathbf{C}}$ be the complexification of K. Then the inclusion homomorphism $\mathbf{C}[\mu_{\mathbf{C}}^{-1}(\beta)]^{K_{\mathbf{C}}} \hookrightarrow \mathbf{C}[\mu_{\mathbf{C}}^{-1}(\beta)]$ induces an affine quotient map $p : \mu_{\mathbf{C}}^{-1}(\beta) \to \operatorname{Spm} \mathbf{C}[\mu_{\mathbf{C}}^{-1}(\beta)]^{K_{\mathbf{C}}} =: \mu_{\mathbf{C}}^{-1}(\beta)//K_{\mathbf{C}}$, and the morphism p induces a holomorphic mapping

$$\Psi : (X(\alpha,\beta), \boldsymbol{I}) \to \mu_{\mathbf{C}}^{-1}(\beta)//K_{\mathbf{C}}.$$

Let $\{u_1, \dots, u_N\}$ be the dual basis corresponding to the standard basis for $\mathbf{R}^N$, and let $\iota^* : (\mathbf{R}^N)^* \to \mathfrak{k}^*$ be the transpose of the inclusion mapping $\iota : \mathfrak{k} \to \mathbf{R}^N$. Let $\mathscr{V}$ be the set of all codimension one subspaces of $\mathfrak{k}^*$ generated by subsets of $\{\iota^* u_1, \dots, \iota^* u_N\}$. Set $\mathscr{V}_\beta := \{V \in \mathscr{V} \mid \beta \in V \otimes \mathbf{C}\}$. Bielawski and Dancer showed ([2, Theorem 5.1]) that, if $\mathscr{V}_\beta = \emptyset$, then the mapping Ψ is biholomorphic. On the other hand, we showed ([1, Theorem 3.3 and Proposition 3.4]) that, if $\mathscr{V}_\beta \neq \emptyset$, then $\mathbf{P}^1$ is embedded in $(X(\alpha,\beta), \boldsymbol{I})$. A result similar to that of us was obtained independently by Konno [8, Theorem 6.10]. Thus $(X(\alpha,\beta), \boldsymbol{I})$ is biholomorphic to an affine variety if and only if $\mathscr{V}_\beta = \emptyset$.

This paper consists of three parts.

The first part (§3) is devoted to the construction of compact complex submanifolds of $(X(\alpha,\beta), \boldsymbol{I})$. Suppose that $\mathscr{V}_\beta \neq \emptyset$. Let J be a subset of $\{1, \dots, N\}$ such that

(a) $\{\pi(e_j) \mid j \in J\}$ is a basis for $\pi(\mathbf{R}^N)$, and

(b) let $\beta_j \in \mathbf{C}$ $(j \in J^c)$ be such that $\beta = \sum_{j \in J^c} \beta_j \iota^* u_j$. Then

$$\{j \in J^c \mid \beta_j = 0\} \neq \emptyset.$$

Since $\mathscr{V}_\beta \neq \emptyset$, such a J exists. We associate with J a hyperplane arrangement $\mathscr{A}_J$ of $(\mathbf{R}^n)^*$. The main result of this part is the following

Theorem 1.1. *Let $\mathcal{F}$ be a bounded face of the arrangement $\mathscr{A}_J$.*

(i) *$\phi^{-1}(\mathcal{F} \times \{0\})$ is a compact complex submanifold of $(X(\alpha,\beta), \boldsymbol{I})$, isotropic with respect to the form $\omega_{\boldsymbol{J}} + \sqrt{-1}\omega_{\boldsymbol{K}}$, and invariant under the T^n-action.*

(ii) *The polytope $\mathcal{F}$ is Delzant, and $\phi^{-1}(\mathcal{F} \times \{0\})$ is biholomorphic to the Delzant space associated with $\mathcal{F}$.*

This construction not only produces the projective line $\mathbf{P}^1$ but also higher dimensional compact submanifolds (see Proposition 5.4). In the special case where $\beta = 0$, Bielawski and Dancer proved Theorem 1.1 [2, Theorem 6.5(ii), (iii)] (see also [4, Theorem 3.5(2), (3)]).

Now recall that a point $x \in \mu_{\mathbf{C}}^{-1}(\beta)$ is said to be *stable* for the action of $K_{\mathbf{C}}$ precisely when the orbit $x \cdot K_{\mathbf{C}}$ is a Zariski closed subset of $\mu_{\mathbf{C}}^{-1}(\beta)$ and the isotropy group of x is finite. Let $\mu_{\mathbf{C}}^{-1}(\beta)^s$ denote the set of all stable points for the $K_{\mathbf{C}}$-action, and set $U_\beta := p(\mu_{\mathbf{C}}^{-1}(\beta)^s)$. The second part of this paper (§4) is devoted to proving the following

Theorem 1.2. *The mapping Ψ is a resolution of singularities, that is,*

(i) *Ψ is proper and surjective,*

(ii) $\Psi^{-1}(U_\beta)$ *is a dense open subset of* $X(\alpha,\beta)$, *and*
(iii) Ψ *maps* $\Psi^{-1}(U_\beta)$ *biholomorphically onto* U_β.

To prove Part (i), we use the Transposition Theorem of Stiemke. For another proof of Part (i), see [9, Proposition 3.7]. Konno's proof is similar to that of [10, Proposition 3.10] or [13, Theorem 4.1(1)]. We state a criterion for stability in terms of the elements of $\mathscr{V}_\beta$. We use this criterion to show that $\mu_{\mathbf{C}}^{-1}(\beta)^s$ is nonempty.

The last part of this paper (§5) is devoted to discussing when complex structures on $X(\alpha,\beta)$ are equivalent. We regard S^2 as the unit sphere in $\mathbf{R}^3$. If $p := {}^t(p_1,p_2,p_3) \in S^2$, then $\boldsymbol{I}_p := p_1\boldsymbol{I} + p_2\boldsymbol{J} + p_3\boldsymbol{K}$ is also a complex structure on $X(\alpha,\beta)$. Set

$$\mathscr{C}_{(\alpha,\beta)} := \{p \in S^2 \mid (X(\alpha,\beta), \boldsymbol{I}_p) \text{ is not biholomorphic to an affine variety}\}.$$

Let $\#\mathscr{C}_{(\alpha,\beta)} = 2$. Then $\mathscr{C}_{(\alpha,\beta)} = \{p,-p\}$ for some $p \in S^2$. In the preceding paper, we showed that $\boldsymbol{I}_{p_1}$ and $\boldsymbol{I}_{p_2}$ are equivalent for each $p_1, p_2 \in S^2 \backslash \mathscr{C}_{(\alpha,\beta)}$ [1, Theorem 5.2(1)]. In this part, we show that $\boldsymbol{I}_p$ and $-\boldsymbol{I}_p$ are equivalent. Without the hypothesis that $\#\mathscr{C}_{(\alpha,\beta)} = 2$, however, $\boldsymbol{I}_{p_1}$ and $\boldsymbol{I}_{p_2}$ are not necessarily equivalent for each $p_1, p_2 \in \mathscr{C}_{(\alpha,\beta)}$. We use the results of Sections 3 and 4 to give an example that illustrates this point (Proposition 5.4). In this example, $\#\mathscr{C}_{(\alpha,\beta)}$ is equal to 8.

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2. The definition of toric hyperkähler manifold

In this section, we sketch the differential geometric construction of toric hyperkähler manifolds [2, §3].

Recall that the standard metric on $\mathbf{H}^N$ is hyperkählerian. Let $\{1,i,j,k\}$ be the standard basis for $\mathbf{H}$. Left multiplication by i (respectively j, k) defines a complex structure $\boldsymbol{I}$ (respectively $\boldsymbol{J}$, $\boldsymbol{K}$) on $\mathbf{H}^N$. This metric is Kählerian with respect to the complex structures $\boldsymbol{I}$, $\boldsymbol{J}$, and $\boldsymbol{K}$.

We identify $i \in \mathbf{H}$ with $\sqrt{-1} \in \mathbf{C}$. We define a mapping

$$\begin{aligned} \mathbf{C}^N \times \mathbf{C}^N &\to \mathbf{H}^N \\ (z^+, z^-) &\mapsto z^+ + z^- j. \end{aligned}$$

We use this mapping to identify $\mathbf{H}^N$ with $\mathbf{C}^N \times \mathbf{C}^N$. For $(z^+,z^-) \in \mathbf{C}^N \times \mathbf{C}^N$, we write $(z^+,z^-) = (z_1^+,\dots,z_N^+,z_1^-,\dots,z_N^-)$ with $z_j^+, z_j^- \in \mathbf{C}$ for each $j = 1,\dots,N$.

Let T^N be the real torus

$$T^N := \{t := (t_1,\dots,t_N) \in \mathbf{C}^N \mid |t_j| = 1 \text{ for each } j = 1,\dots,N\},$$

and let T^N act on the right on $\mathbf{H}^N$ by $(z^+,z^-) \cdot t = (z^+ \cdot t, z^- \cdot t^{-1})$. This action preserves the hyperkähler structure. Let $\{e_1,\dots,e_N\}$ be the standard basis for

$\mathbf{R}^N$, and let $\{u_1, \ldots, u_N\}$ be the corresponding dual basis. Then the hyperkähler moment map $\mu^0 := (\mu_{\boldsymbol{I}}^0, \mu_{\boldsymbol{J}}^0, \mu_{\boldsymbol{K}}^0) : \mathbf{H}^N \to (\mathbf{R}^N)^* \otimes \mathbf{R}^3$ for this action is given by

$$\mu_{\boldsymbol{I}}^0(z^+, z^-) = \frac{1}{2}\sum_{j=1}^{N}(|z_j^+|^2 - |z_j^-|^2)u_j \tag{2.1}$$

and

$$(\mu_{\boldsymbol{J}}^0 + \sqrt{-1}\mu_{\boldsymbol{K}}^0)(z^+, z^-) = -\sqrt{-1}\sum_{j=1}^{N} z_j^+ z_j^- u_j. \tag{2.2}$$

Note that the hyperkähler moment map is surjective.

Let K be a subtorus of T^N whose Lie algebra $\mathfrak{k} \subset \mathbf{R}^N$ is generated by rational vectors. Set $k := \dim K$. Let $\iota : \mathfrak{k} \to \mathbf{R}^N$ be the inclusion mapping, and let $\pi : \mathbf{R}^N \to \mathbf{R}^n := \mathbf{R}^N/\mathfrak{k}$ be the canonical projection. Then we obtain an exact sequence

$$0 \to \mathfrak{k} \xrightarrow{\iota} \mathbf{R}^N \xrightarrow{\pi} \mathbf{R}^n \to 0,$$

and, by duality, an exact sequence

$$0 \leftarrow \mathfrak{k}^* \xleftarrow{\iota^*} (\mathbf{R}^N)^* \xleftarrow{\pi^*} (\mathbf{R}^n)^* \leftarrow 0.$$

We now restrict the action of T^N on $\mathbf{H}^N$ to K. We set $\mathfrak{k}_{\mathbf{C}}^* := \mathfrak{k}^* \otimes \mathbf{C}$, and define $\mu_{\boldsymbol{I}} : \mathbf{H}^N \to \mathfrak{k}^*$ (respectively $\mu_{\mathbf{C}} : \mathbf{H}^N \to \mathfrak{k}_{\mathbf{C}}^*$) to be the mapping $\iota^* \circ \mu_{\boldsymbol{I}}^0$ (respectively $\iota^* \circ \mu_{\boldsymbol{J}}^0 + \sqrt{-1}\iota^* \circ \mu_{\boldsymbol{K}}^0$). Then the hyperkähler moment map for the action of K on $\mathbf{H}^N$ is $\mu := (\mu_{\boldsymbol{I}}, \mu_{\mathbf{C}}) : \mathbf{H}^N \to \mathfrak{k}^* \times \mathfrak{k}_{\mathbf{C}}^*$.

Definition 2.1 (Bielawski-Dancer). Let $(\alpha, \beta) \in \mathfrak{k}^* \times \mathfrak{k}_{\mathbf{C}}^*$ be a regular value of μ, and let K act freely on $\mu^{-1}(\alpha, \beta)$. Then we refer to the hyperkähler quotient

$$X(\alpha, \beta) := \mu^{-1}(\alpha, \beta)/K$$

as a *toric hyperkähler manifold.*

Remarks. (i) A toric hyperkähler manifold is not a toric manifold in the usual sense.

(ii) Suppose that $\pi(e_{j_0}) = 0$ for some $j_0 \in \mathbf{N}$ with $1 \le j_0 \le N$. Then the toric hyperkähler manifold $X(\alpha, \beta)$ is a hyperkähler quotient of $\mathbf{H}^{N-1}$ by $K \cap T^{N-1}$, where $\mathbf{H}^{N-1} = \{(z^+, z^-) \in \mathbf{H}^N \mid z_{j_0}^+ = z_{j_0}^- = 0\}$ and $T^{N-1} = \{t \in T^N \mid t_{j_0} = 1\}$.

Suppose that $\iota^* u_{j_0} = 0$ for some $j_0 \in \mathbf{N}$ with $1 \le j_0 \le N$. Then the subtorus K is a subgroup of T^{N-1}, and $X(\alpha, \beta)$ is the Cartesian product of $\mathbf{H}$ with a hyperkähler quotient of $\mathbf{H}^{N-1}$ by K.

These cases are not essential for our purposes. Thus we exclude these cases in this paper.

For $(z^+, z^-) \in \mu^{-1}(\alpha, \beta)$, we denote its equivalence class in $X(\alpha, \beta)$ by $[z^+, z^-]$.

A toric hyperkähler manifold $X(\alpha,\beta)$ is a non-compact connected manifold of real dimension $4n$. The standard metric on $\mathbf{H}^N$ and the complex structures $\boldsymbol{I}$, $\boldsymbol{J}$, and $\boldsymbol{K}$ descend to $X(\alpha,\beta)$, and the induced metric on $X(\alpha,\beta)$ is hyperkählerian.

The quotient group $T^n = T^N/K$ acts in the natural way on $X(\alpha,\beta)$, preserving the hyperkähler structure. Let $a \in (\mathbf{R}^N)^*$ and $b \in (\mathbf{C}^N)^*$ be such that $\iota^* a = \alpha$ and $\iota^* b = \beta$. Then the hyperkähler moment map $\phi_{a,b} := (\phi_{\boldsymbol{I}}^a, \phi_{\mathbf{C}}^b) : X(\alpha,\beta) \to (\mathbf{R}^n)^* \times (\mathbf{C}^n)^*$ for the natural action is given by

$$\phi_{\boldsymbol{I}}^a([z^+, z^-]) = \mu_{\boldsymbol{I}}^0(z^+, z^-) - a$$

and

$$\phi_{\mathbf{C}}^b([z^+, z^-]) = (\mu_{\boldsymbol{J}}^0 + \sqrt{-1}\mu_{\boldsymbol{K}}^0)(z^+, z^-) - b.$$

Remark. We use the monomorphism π^* to identify $(\mathbf{R}^n)^*$ with $\ker \iota^*$. Then, for each $(z^+, z^-) \in \mu^{-1}(\alpha,\beta)$, we have $\mu_{\boldsymbol{I}}^0(z^+, z^-) - a \in (\mathbf{R}^n)^*$ and $(\mu_{\boldsymbol{J}}^0 + \sqrt{-1}\mu_{\boldsymbol{K}}^0)(z^+, z^-) - b \in (\mathbf{C}^n)^*$.

In [2], Bielawski and Dancer gave necessary and sufficient conditions for a hyperkähler quotient $\mu^{-1}(\alpha,\beta)/K$ to be smooth or an orbifold. The following two propositions are due to them [2], partly based on results by Konno [7].

We first give necessary and sufficient conditions for $(\alpha,\beta) \in \mathfrak{k}^* \times \mathfrak{k}_{\mathbf{C}}^*$ to be a regular value of μ. Let $a \in (\mathbf{R}^N)^*$ and let $b \in (\mathbf{C}^N)^*$. For $j = 1, \dots, N$, set

$$\mathcal{H}(j,a) := \{x \in (\mathbf{R}^n)^* \mid \langle x, \pi(e_j)\rangle = -\langle a, e_j\rangle\},$$

a hyperplane in $(\mathbf{R}^n)^*$, and

$$\mathcal{H}_{\mathbf{C}}(j,b) := \{x \in (\mathbf{C}^n)^* \mid \langle x, \pi(e_j)\rangle = -\langle b, e_j\rangle\},$$

a hyperplane in $(\mathbf{C}^n)^*$. For each $j = 1, \dots, N$, the two closed half-spaces in $(\mathbf{R}^n)^*$ bounded by $\mathcal{H}(j,a)$ are

$$\mathcal{H}^+(j,a) := \{x \in (\mathbf{R}^n)^* \mid \langle x, \pi(e_j)\rangle \geq -\langle a, e_j\rangle\},$$
$$\mathcal{H}^-(j,a) := \{x \in (\mathbf{R}^n)^* \mid \langle x, \pi(e_j)\rangle \leq -\langle a, e_j\rangle\}.$$

Let $\mathscr{V}$ be the set of all codimension one subspaces of $\mathfrak{k}^*$ generated by subsets of $\{\iota^* u_1, \dots, \iota^* u_N\}$. For each $V \in \mathscr{V}$, set $V_{\mathbf{C}} := V \otimes \mathbf{C}$.

Proposition 2.2 (See [2, Theorems 3.2 and 3.3] and [7, Proposition 2.1]). *Let $a \in (\mathbf{R}^N)^*$ and $b \in (\mathbf{C}^N)^*$ be such that $\iota^* a = \alpha$ and $\iota^* b = \beta$. Then the following statements are equivalent*:

(i) *(α,β) is a regular value of μ*;
(ii) *$\bigcap_{j \in J} \mathcal{H}(j,a) \times \mathcal{H}_{\mathbf{C}}(j,b) = \emptyset$ for each subset J of $\{1, \dots, N\}$ with $\#J = n+1$*;
(iii) *for each $V \in \mathscr{V}$, we have either $\alpha \notin V$ or $\beta \notin V_{\mathbf{C}}$*. □

We denote the set of all regular values of μ by $(\mathfrak{k}^* \times \mathfrak{k}_{\mathbf{C}}^*)_{\mathrm{reg}}$.

We next give necessary and sufficient conditions for K to act freely on $\mu^{-1}(\alpha,\beta)$.

PROPOSITION 2.3 (See [7, Lemma 2.2 and Proposition 2.2]). *Suppose that* $\{\pi(e_1),\dots,\pi(e_n)\}$ *is a basis for* $\mathbf{R}^n$. *Let* A *be the matrix of* π *relative to the bases* $\{e_1,\dots,e_N\}$, $\{\pi(e_1),\dots,\pi(e_n)\}$. *Let* $(\alpha,\beta)\in(\mathfrak{k}^*\times\mathfrak{k}^*_{\mathbf{C}})_{\mathrm{reg}}$. *Then the following statements are equivalent*:

(i) *K acts freely on* $\mu^{-1}(\alpha,\beta)$;

(ii) $\{\pi(e_j)\mid j\in J\}$ *is a* $\mathbf{Z}$*-basis for* $\pi(\mathbf{Z}^N)$ *for each subset* J *of* $\{1,\dots,N\}$ *such that* $\{\pi(e_j)\mid j\in J\}$ *is a basis for* $\pi(\mathbf{R}^N)$;

(iii) *A is a totally unimodular matrix, that is, each square submatrix of A has determinant equal to* 0, $+1$, *or* -1. □

We consider only the case where a hyperkähler quotient $\mu^{-1}(\alpha,\beta)/K$ is smooth. So we suppose throughout this paper that Condition (ii) above holds.

A toric hyperkähler manifold $X(\alpha,\beta)$, the Kähler quotient of $\mu_{\mathbf{C}}^{-1}(\beta)$ by K, can be idetified as follows with the quotient of a suitable open subset of $\mu_{\mathbf{C}}^{-1}(\beta)$ by the complexified torus $K_{\mathbf{C}}$. We start with a basic definition.

DEFINITION 2.4. Let $(\alpha,\beta)\in(\mathfrak{k}^*\times\mathfrak{k}^*_{\mathbf{C}})_{\mathrm{reg}}$ and let $(z^+,z^-)\in\mu_{\mathbf{C}}^{-1}(\beta)$. We say that (z^+,z^-) is *α-stable* precisely when the orbit of $K_{\mathbf{C}}$ through (z^+,z^-) meets $\mu_{\boldsymbol{I}}^{-1}(\alpha)$.

We denote the set of all α-stable points of $\mu_{\mathbf{C}}^{-1}(\beta)$ by $\mu_{\mathbf{C}}^{-1}(\beta)^{\alpha-st}$.

Remark. By [8, Theorem 5.2(2)], this definition is equivalent to Konno's definition [8, Definition 5.1].

The set $\mu_{\mathbf{C}}^{-1}(\beta)^{\alpha-st}$ is $K_{\mathbf{C}}$-invariant. By definition, we have $\mu^{-1}(\alpha,\beta)\subset\mu_{\mathbf{C}}^{-1}(\beta)^{\alpha-st}$. Hence the inclusion $\mu^{-1}(\alpha,\beta)\subset\mu_{\mathbf{C}}^{-1}(\beta)^{\alpha-st}$ induces a natural mapping

$$X(\alpha,\beta)=\mu^{-1}(\alpha,\beta)/K\to\mu_{\mathbf{C}}^{-1}(\beta)^{\alpha-st}/K_{\mathbf{C}}.$$

By [8, Theorem 5.2], we can use the natural mapping to identify $(X(\alpha,\beta),\boldsymbol{I})$ with $\mu_{\mathbf{C}}^{-1}(\beta)^{\alpha-st}/K_{\mathbf{C}}$.

We end the section by giving a useful criterion for α-stability. This criterion is due to Konno [8]. For each $V\in\mathscr{V}$, fix $Y_V\in\mathfrak{k}$ such that

$$V=\{v\in\mathfrak{k}^*\mid\langle v,Y_V\rangle=0\}.$$

Let $(\alpha,\beta)\in(\mathfrak{k}^*\times\mathfrak{k}^*_{\mathbf{C}})_{\mathrm{reg}}$. Set $\mathscr{V}_\beta:=\{V\in\mathscr{V}\mid\beta\in V_{\mathbf{C}}\}$, and, for each $V\in\mathscr{V}_\beta$, set

$$J_V^+:=\{j\in\{1,\dots,N\}\mid\langle\iota^*u_j,Y_V\rangle\langle\alpha,Y_V\rangle>0\}$$

and

$$J_V^-:=\{j\in\{1,\dots,N\}\mid\langle\iota^*u_j,Y_V\rangle\langle\alpha,Y_V\rangle<0\}.$$

PROPOSITION 2.5 (See [8, Theorem 5.10]). *Let* $(z^+,z^-)\in\mu_{\mathbf{C}}^{-1}(\beta)$. *Then the following statements are equivalent*:

(i) $(z^+, z^-) \in \mu_{\mathbf{C}}^{-1}(\beta)^{\alpha-st}$;
(ii) *for each* $V \in \mathcal{V}_\beta$, *there exists* $j \in J_V^+ \cup J_V^-$ *such that either* $j \in J_V^+$ *with* $z_j^+ \neq 0$ *or* $j \in J_V^-$ *with* $z_j^- \neq 0$. □

3. A construction of compact complex submanifolds

Suppose that $(\alpha, \beta) \in (\mathfrak{k}^* \times \mathfrak{k}_{\mathbf{C}}^*)_{\text{reg}}$. In this section, we consider only the case where $(X(\alpha, \beta), \boldsymbol{I})$ is not biholomorphic to an affine variety. So we suppose that $\mathcal{V}_\beta \neq \emptyset$ (see [2, Theorem 5.1] and [1, Corollary 3.6]). The purpose of this section is to construct compact complex submanifolds of $(X(\alpha, \beta), \boldsymbol{I})$ that are invariant under the T^n-action. We denote the Kähler form corresponding to the complex structure $\boldsymbol{J}$ (respectively $\boldsymbol{K}$) by $\omega_{\boldsymbol{J}}$ (respectively $\omega_{\boldsymbol{K}}$). We show that these submanifolds are isotropic with respect to the form $\omega_{\boldsymbol{J}} + \sqrt{-1}\omega_{\boldsymbol{K}}$, and that these submanifolds are biholomorphic to Delzant spaces.

We first give a brief review of Delzant's construction of certain toric varieties from polytopes. We follow the exposition of Guillemin [5, Chapter 1 and Appendix 1].

Recall that a d-dimensional polytope $\mathcal{P}$ in $(\mathbf{R}^d)^*$ is said to be *Delzant* precisely when

(i) $\mathcal{P}$ is simple, that is, each vertex p of $\mathcal{P}$ is contained in precisely d edges of $\mathcal{P}$, and
(ii) for each vertex p of $\mathcal{P}$, there exists a $\mathbf{Z}$-basis $\{w_1, \dots, w_d\}$ for $(\mathbf{Z}^d)^*$ such that the d edges of $\mathcal{P}$ containing the vertex p lie on the rays $p + tw_i$, $0 \leq t < \infty$.

Let $\mathcal{P}$ be the Delzant polytope in $(\mathbf{R}^d)^*$ defined by a system of inequalities of the form

$$\langle x, a_j \rangle \geq \gamma_j, \quad (j = 1, \dots, m),$$

where $a_j \in \mathbf{Z}^d$ and $\gamma_j \in \mathbf{R}$ for each $j = 1, \dots, m$ and m is the number of facets of $\mathcal{P}$. Let $q : \mathbf{R}^m \to \mathbf{R}^d$ be a linear mapping for which $q(e_j) = a_j$ for each $j = 1, \dots, m$. Set $\mathfrak{l} := \ker q$ and let $i : \mathfrak{l} \to \mathbf{R}^m$ denote the inclusion mapping. Then we obtain an exact sequence

$$0 \to \mathfrak{l} \xrightarrow{i} \mathbf{R}^m \xrightarrow{q} \mathbf{R}^d \to 0,$$

and, by duality, an exact sequence

$$0 \leftarrow \mathfrak{l}^* \xleftarrow{i^*} (\mathbf{R}^m)^* \xleftarrow{q^*} (\mathbf{R}^d)^* \leftarrow 0.$$

Since $q(\mathbf{Z}^m) \subset \mathbf{Z}^d$, the mapping q induces a group homomorphism from T^m to T^d. Denoting by L the kernel of this homomorphism, we obtain an exact sequence

$$1 \to L \to T^m \to T^d \to 1$$

of abelian groups.

The natural action of T^m on $\mathbf{C}^m$ is Hamiltonian, and its moment map is

$$\nu^0 : \mathbf{C}^m \to (\mathbf{R}^m)^*, \quad (z_1, \dots, z_m) \mapsto \frac{1}{2} \sum_{j=1}^m |z_j|^2 u_j.$$

We restrict the action of T^m on $\mathbf{C}^m$ to L. The moment map for the action of L on $\mathbf{C}^m$ is $\nu := \imath^* \circ \nu^0 : \mathbf{C}^m \to \mathfrak{l}^*$. Set $\gamma := -\sum_{j=1}^m \gamma_j \imath^* u_j$. Then L acts freely on the level set $\nu^{-1}(\gamma)$. Reducing $\mathbf{C}^m$ with respect to the action of L, we obtain the Delzant space

$$X_{\mathcal{P}} := \nu^{-1}(\gamma)/L.$$

For $z \in \nu^{-1}(\gamma)$, we denote its equivalence class in $X_{\mathcal{P}}$ by $[z]$.

The quotient group $T^d = T^m/L$ acts in the natural way on $X_{\mathcal{P}}$. Set $c := -\sum_{j=1}^m \gamma_j u_j$. Then the moment map $\psi : X_{\mathcal{P}} \to (\mathbf{R}^d)^*$ for the natural action is given by

$$\psi([z]) = \nu^0(z) - c.$$

Remark. We use the monomorphism q^* to identify $(\mathbf{R}^d)^*$ with $\ker \imath^*$. Then, for each $z \in \nu^{-1}(\gamma)$, we have $\nu^0(z) - c \in (\mathbf{R}^d)^*$.

The Delzant space $X_{\mathcal{P}}$ can be identified as follows with the quotient of a suitable open subset of $\mathbf{C}^m$ by the complexified torus $L_{\mathbf{C}}$. For each subset J of $\{1, \dots, m\}$, set

$$\mathbf{C}_J^m := \{(z_1, \dots, z_m) \in \mathbf{C}^m \mid z_j = 0 \text{ if and only if } j \in J\}.$$

Each orbit in $\mathbf{C}^m$ of the complexified torus $T_{\mathbf{C}}^m$ is of the form $\mathbf{C}_J^m$ for some subset J of $\{1, \dots, m\}$. Now let $\mathcal{F}$ be a face of $\mathcal{P}$. Then, since $\mathcal{P}$ is simple, there exists a unique subset J of $\{1, \dots, m\}$ such that $\mathcal{F}$ is defined by a system of equalities

$$\langle x, a_j \rangle = \gamma_j, \quad (j \in J).$$

Let $\mathbf{C}_{\mathcal{F}}^m := \mathbf{C}_J^m$. Then

$$\mathbf{C}_{\mathcal{P}}^m := \bigcup_{\mathcal{F} \text{ face of } \mathcal{P}} \mathbf{C}_{\mathcal{F}}^m$$

is an open subset of $\mathbf{C}^m$. The set $\mathbf{C}_{\mathcal{P}}^m$ contains $\nu^{-1}(\gamma)$, and the inclusion $\nu^{-1}(\gamma) \subset \mathbf{C}_{\mathcal{P}}^m$ induces a natural mapping

$$X_{\mathcal{P}} = \nu^{-1}(\gamma)/L \to \mathbf{C}_{\mathcal{P}}^m / L_{\mathbf{C}}.$$

We can use the natural mapping to identify $X_{\mathcal{P}}$ with the orbit space $\mathbf{C}_{\mathcal{P}}^m / L_{\mathbf{C}}$.

Now we are ready to consider our main problem. We need some notation. Fix a subset J of $\{1, \dots, N\}$ such that

(a) $\{\pi(e_j) \mid j \in J\}$ is a basis for $\pi(\mathbf{R}^N)$, and

(b) let $\beta_j \in \mathbf{C}$ $(j \in J^c)$ be such that $\beta = \sum_{j \in J^c} \beta_j \imath^* u_j$. Then

$$J_0 := \{j \in J^c \mid \beta_j = 0\} \neq \emptyset.$$

Since $\mathscr{V}_\beta \neq \emptyset$, such a J exists. We can write $\alpha = \sum_{j \in J^c} \alpha_j \imath^* u_j$ for suitable $\alpha_j \in \mathbf{R}$. We set

$$a := \sum_{j \in J^c} \alpha_j u_j \quad \text{and} \quad b := \sum_{j \in J^c} \beta_j u_j.$$

We denote by Θ the set of all mappings from $J \cup J_0$ to $\{+,-\}$. Let $\varepsilon \in \Theta$. Then we define two mappings $\varepsilon_- : J \cup J_0 \to \{+,-\}$ and $\delta : J \cup J_0 \to \{1,-1\}$ by

$$\varepsilon_-(j) := \begin{cases} + & \text{for each } j \in J \cup J_0 \text{ with } \varepsilon(j) = -, \\ - & \text{for each } j \in J \cup J_0 \text{ with } \varepsilon(j) = +, \end{cases}$$

and

$$\delta(j) := \begin{cases} 1 & \text{for each } j \in J \cup J_0 \text{ with } \varepsilon(j) = +, \\ -1 & \text{for each } j \in J \cup J_0 \text{ with } \varepsilon(j) = -. \end{cases}$$

For each $\varepsilon \in \Theta$, let $\mathcal{P}_\varepsilon$ be the polyhedral set

$$\mathcal{P}_\varepsilon := \bigcap_{j \in J \cup J_0} \mathcal{H}^{\varepsilon(j)}(j,a).$$

Now we can state the theorem.

Theorem 3.1. *Let $\varepsilon \in \Theta$ and let $\mathcal{F}$ be a bounded face of $\mathcal{P}_\varepsilon$.*
(i) *$(\phi_{a,b})^{-1}(\mathcal{F} \times \{0\})$ is a compact complex submanifold of $(X(\alpha,\beta), \boldsymbol{I})$, isotropic with respect to the form $\omega_{\boldsymbol{J}} + \sqrt{-1}\omega_{\boldsymbol{K}}$, and invariant under the T^n-action.*
(ii) *The polytope $\mathcal{F}$ is Delzant, and $(\phi_{a,b})^{-1}(\mathcal{F} \times \{0\})$ is biholomorphic to the Delzant space $X_{\mathcal{F}}$.*

Remark. By the proof of Theorem 3.3 of [1], we see that $\mathcal{P}_\varepsilon$ possesses a bounded edge for some $\varepsilon \in \Theta$.

For the proof, we need

Proposition 3.2. *Let $[z^+, z^-] \in (\phi^b_{\mathbf{C}})^{-1}(0)$. Then, for each $j \in J \cup J_0$, the following holds*:
(i) *$[z^+, z^-] \in (\phi^a_{\boldsymbol{I}})^{-1}(\mathcal{H}^{\varepsilon(j)}(j,a))$ if and only if $z_j^{\varepsilon_-(j)} = 0$.*
(ii) *$[z^+, z^-] \in (\phi^a_{\boldsymbol{I}})^{-1}(\mathcal{H}(j,a))$ if and only if $z_j^+ = z_j^- = 0$.*

Proof. By assumption, we have

$$0 = \langle \pi^*(\phi^b_{\mathbf{C}}([z^+, z^-])) + b, e_j \rangle = -\sqrt{-1} z_j^+ z_j^- \tag{3.1}$$

for each $j \in J \cup J_0$. Since

$$\langle \pi^*(\phi^a_{\boldsymbol{I}}([z^+, z^-])) + a, e_j \rangle = \frac{1}{2}(|z_j^+|^2 - |z_j^-|^2)$$

for each $j \in J \cup J_0$, the assertions follow immediately from (3.1). □

Proof of Theorem 3.1. We may assume that $d := \dim \mathcal{F} \geq 1$.

Let x_0 be a vertex of $\mathcal{F}$, and set $J' := \{ j \in J \cup J_0 \mid x_0 \in \mathcal{H}(j,a)\}$. Then, since $(\alpha,\beta) \in (\mathfrak{k}^* \times \mathfrak{k}^*_{\mathbf{C}})_{\mathrm{reg}}$, it follows from Proposition 2.2 that $\{\pi(e_j) \mid j \in J'\}$ is a basis for $\pi(\mathbf{R}^N)$. We can write

$$\alpha = \sum_{j \in \{1,\dots,N\} \setminus J'} \alpha'_j \iota^* u_j \quad \text{and} \quad \beta = \sum_{j \in \{1,\dots,N\} \setminus J'} \beta'_j \iota^* u_j$$

for suitable $\alpha'_j \in \mathbf{R}$ and for suitable $\beta'_j \in \mathbf{C}$. Setting

$$J'_0 := \{ j \in \{1,\dots,N\} \setminus J' \,|\, \beta'_j = 0 \},$$

we have

$$(J'_0)^c = (J_0)^c. \tag{3.2}$$

Hence $J \cup J_0 = J' \cup J'_0$, so that $J'_0 \neq \emptyset$. Thus the subset J' satisfies Conditions (a) and (b). Since $J \cup J_0 = J' \cup J'_0$, there exists a unique mapping $\varepsilon' : J' \cup J'_0 \to \{+,-\}$ such that $\varepsilon' = \varepsilon$. Set

$$a' := \sum_{j \in \{1,\dots,N\} \setminus J'} \alpha'_j u_j \quad \text{and} \quad b' := \sum_{j \in \{1,\dots,N\} \setminus J'} \beta'_j u_j.$$

Let $\mathcal{P}_{\varepsilon'}$ be the polyhedral set

$$\mathcal{P}_{\varepsilon'} := \bigcap_{j \in J' \cup J'_0} \mathcal{H}^{\varepsilon'(j)}(j, a').$$

Now let $T : (\mathbf{R}^n)^* \to (\mathbf{R}^n)^*$ be the translation for which $T(x) = x - x_0$ for each $x \in (\mathbf{R}^n)^*$. Since $\langle x_0, \pi(e_j) \rangle = \langle a' - a, e_j \rangle$ for each $j \in J'$ and $a' - a \in \ker \iota^*$, we have $a' - a = \pi^*(x_0)$. Hence we have $T(\mathcal{P}_\varepsilon) = \mathcal{P}_{\varepsilon'}$. Set $\mathcal{F}' := T(\mathcal{F})$. Then $\mathcal{F}'$ is a bounded face of $\mathcal{P}_{\varepsilon'}$. Note that the origin is a vertex of $\mathcal{F}'$. Now, since $a' - a = \pi^*(x_0)$, we have $T \circ \phi_{\boldsymbol{I}}^a = \phi_{\boldsymbol{I}}^{a'}$. On the other hand, since $b = b'$ by (3.2), we have $\phi_{\mathbf{C}}^b = \phi_{\mathbf{C}}^{b'}$. Hence we have $(\phi_{a,b})^{-1}(\mathcal{F} \times \{0\}) = (\phi_{a',b'})^{-1}(\mathcal{F}' \times \{0\})$. We may therefore assume that the origin is a vertex of $\mathcal{F}$.

For each $j \in J \cup J_0$, we set $\mathcal{H}_j := \mathcal{H}(j,a)$, $\mathcal{H}_j^+ := \mathcal{H}^+(j,a)$, and $\mathcal{H}_j^- := \mathcal{H}^-(j,a)$. We set $\phi_{\boldsymbol{I}} := \phi_{\boldsymbol{I}}^a$ and $\phi := \phi_{a,b}$. By rearranging the indices, we may assume that

$$J = \{1,\dots,d,d+k+1,\dots,N\} \quad \text{and} \quad J_0 = \{d+1,\dots,l\},$$

where $d < l \leq d+k$. Since $(\alpha,\beta) \in (\mathfrak{k}^* \times \mathfrak{k}^*_{\mathbf{C}})_{\mathrm{reg}}$, we have $\alpha_j \neq 0$ for each $j \in \mathbf{N}$ with $d < j \leq l$. Hence $0 \notin \mathcal{H}_j$ for each $j \in \mathbf{N}$ with $d < j \leq l$, so that, since $0 \in \mathcal{F}$, we have $\mathcal{F} \not\subset \mathcal{H}_j$ for each such j. Thus, by a suitable rearrangement of indices, we can write

$$\mathcal{F} = \bigcap_{j=1}^{m} \mathcal{H}_j^{\varepsilon(j)} \cap \bigcap_{j=d+k+1}^{N} \mathcal{H}_j,$$

where $d < m \leq l$ and

$$\mathcal{F} \neq \bigcap_{\substack{j=1 \\ j \neq i}}^{m} \mathcal{H}_j^{\varepsilon(j)} \cap \bigcap_{j=d+k+1}^{N} \mathcal{H}_j \quad \text{for each } i = 1,\dots,m. \tag{3.3}$$

(i) Since the canonical projection $X(\alpha,\beta) \to X(\alpha,\beta)/T^n$ is proper, ϕ is proper by [2, Theorem 3.1(i)]. Therefore, by assumption, $\phi^{-1}(\mathcal{F}\times\{0\})$ is compact; moreover, it is invariant under the T^n-action.

We set

$$M := \{(z^+, z^-) \in \mathbf{H}^N \mid z_j^{\varepsilon_-(j)} = 0 \ \ (1 \le j \le l),$$
$$-\sqrt{-1} z_j^+ z_j^- = \beta_j \ \ (l < j \le d+k), \quad z_j^+ = z_j^- = 0 \ \ (d+k < j \le N)\}.$$

Since $\beta_j \neq 0$ for each $j \in \mathbf{N}$ with $l < j \le d+k$, it follows that M is a complex submanifold of $(\mathbf{H}^N, \boldsymbol{I})$. Let $\rho : \mu^{-1}(\alpha,\beta) \to X(\alpha,\beta)$ be the canonical projection. By Proposition 3.2, we have

$$(\phi\circ\rho)^{-1}(\mathcal{F}\times\{0\}) = M \cap \mu_{\boldsymbol{I}}^{-1}(\alpha). \tag{3.4}$$

The restriction of $\mu_{\boldsymbol{I}}$ to M is the moment map for the induced action of K on M. Note that K acts freely on $M\cap\mu_{\boldsymbol{I}}^{-1}(\alpha)$. We obtain the Kähler quotient

$$(M\cap\mu_{\boldsymbol{I}}^{-1}(\alpha))/K = \phi^{-1}(\mathcal{F}\times\{0\}). \tag{3.5}$$

Hence $\phi^{-1}(\mathcal{F}\times\{0\})$ is a compact complex submanifold of $(X(\alpha,\beta), \boldsymbol{I})$ that is invariant under the T^n-action.

Now M is isotropic with respect to the holomorphic symplectic form on $\mathbf{H}^N$, and so $\phi^{-1}(\mathcal{F}\times\{0\})$ is also isotropic with respect to $\omega_{\boldsymbol{J}} + \sqrt{-1}\omega_{\boldsymbol{K}}$.

(ii) Let $A = (a_{ij})$ be the matrix of π relative to the bases $\{e_1,\dots,e_N\}$, $\{\pi(e_1),\dots,\pi(e_d),\pi(e_{d+k+1}),\dots,\pi(e_N)\}$. Then we have

$$K_{\mathbf{C}} = \left\{ (t_1,\dots,t_N) \in T_{\mathbf{C}}^N \mid \right.$$
$$\left. t_i = \prod_{j=d+1}^{d+k} t_j^{-a_{ij}} \ \ (1 \le i \le d), \quad t_i = \prod_{j=d+1}^{d+k} t_j^{-a_{i-k,j}} \ \ (d+k < i \le N) \right\}.$$

For each $j = 1,\dots,d$, let $\alpha_j := 0 \in \mathbf{R}$. For each $j = 1,\dots,m$, set $\tilde{a}_j := {}^t(a_{1j},\dots,a_{dj})$, and let $\tilde{\mathcal{H}}_j$ be the hyperplane

$$\tilde{\mathcal{H}}_j := \{x \in (\mathbf{R}^d)^* \mid \langle x, \tilde{a}_j\rangle = -\alpha_j\}$$

in $(\mathbf{R}^d)^*$. Then, for each $j = 1,\dots,m$, the two closed half-spaces in $(\mathbf{R}^d)^*$ bounded by $\tilde{\mathcal{H}}_j$ are

$$\tilde{\mathcal{H}}_j^+ := \{x \in (\mathbf{R}^d)^* \mid \langle x, \tilde{a}_j\rangle \ge -\alpha_j\},$$
$$\tilde{\mathcal{H}}_j^- := \{x \in (\mathbf{R}^d)^* \mid \langle x, \tilde{a}_j\rangle \le -\alpha_j\}.$$

Let $\tilde{\mathcal{F}}$ be the d-dimensional polyhedral set

$$\tilde{\mathcal{F}} := \bigcap_{j=1}^{m} \tilde{\mathcal{H}}_j^{\varepsilon(j)}.$$

Since $\mathcal{F}$ is bounded, the polyhedral set $\tilde{\mathcal{F}}$ is a polytope. By (3.3), we have

$$\tilde{\mathcal{F}} \neq \bigcap_{\substack{j=1 \\ j \neq i}}^{m} \tilde{\mathcal{H}}_j^{\varepsilon(j)} \quad \text{for each } i = 1, \dots, m.$$

The proof is divided into two parts. In Part A, we prove that the polytope $\tilde{\mathcal{F}}$ is Delzant. In Part B, we prove that $\phi^{-1}(\mathcal{F} \times \{0\})$ is biholomorphic to the Delzant space $X_{\tilde{\mathcal{F}}}$.

Part A. Since

$$(\alpha, \beta) \in (\mathfrak{k}^* \times \mathfrak{k}^*_{\mathbf{C}})_{\text{reg}} \quad \text{and} \quad \mathcal{F} \subset \bigcap_{j=d+k+1}^{N} \mathcal{H}_j,$$

Proposition 2.2 implies that each vertex of $\tilde{\mathcal{F}}$ is contained in precisely d facets. Thus $\tilde{\mathcal{F}}$ is simple. Let p be a vertex of $\tilde{\mathcal{F}}$, and let $\tilde{\mathcal{F}}_1, \dots, \tilde{\mathcal{F}}_d$ be d facets of $\tilde{\mathcal{F}}$ containing p. Then, for each $j = 1, \dots, d$, there exists the integer λ_j, $1 \leq \lambda_j \leq m$, such that $\tilde{\mathcal{F}}_j = \tilde{\mathcal{F}} \cap \tilde{\mathcal{H}}_{\lambda_j}$. Since $\tilde{a}_{\lambda_1}, \dots, \tilde{a}_{\lambda_d}$ are linearly independent, the matrix $\tilde{A} := (\tilde{a}_{\lambda_1}, \dots, \tilde{a}_{\lambda_d})$ is unimodular by Proposition 2.3. For each $i = 1, \dots, d$, let v_i be the ith row vector of $\tilde{A}^{-1}$. Then the matrix

$$\begin{pmatrix} \delta(\lambda_1) v_1 \\ \vdots \\ \delta(\lambda_d) v_d \end{pmatrix}$$

is also unimodular. Since the polytope $\tilde{\mathcal{F}}$ is simple, it follows that

$$\tilde{e}_i = \bigcap_{\substack{j=1 \\ j \neq i}}^{d} \tilde{\mathcal{F}}_j$$

is an edge of $\tilde{\mathcal{F}}$ for each $i = 1, \dots, d$. For each $i = 1, \dots, d$, the edge $\tilde{e}_i$ lies on the ray $p + t\delta(\lambda_i) v_i$, $0 \leq t < \infty$. Thus the polytope $\tilde{\mathcal{F}}$ is Delzant. Note that

$$L = \left\{ (t_1, \dots, t_m) \in T^m \mid t_i = \prod_{j=d+1}^{m} t_j^{-\delta(i)\delta(j)a_{ij}} \ (1 \leq i \leq d) \right\}.$$

Part B. By [8, Theorem 5.2(2)] and (3.5), we can naturally identify $\phi^{-1}(\mathcal{F} \times \{0\})$ with the orbit space $(M \cap \mu_{\mathbf{C}}^{-1}(\beta)^{\alpha - st})/K_{\mathbf{C}}$.

(a) We construct a holomorphic mapping $f : \phi^{-1}(\mathcal{F} \times \{0\}) \to X_{\tilde{\mathcal{F}}}$. Let $(z^+, z^-) \in M \cap \mu_{\boldsymbol{I}}^{-1}(\alpha)$. Then we have $(z_1^{\varepsilon(1)}, \dots, z_m^{\varepsilon(m)}) \in \nu^{-1}(\gamma)$. Since $\nu^{-1}(\gamma) \subset \mathbf{C}^m_{\tilde{\mathcal{F}}}$, we have

$$(z_1^{\varepsilon(1)}, \dots, z_m^{\varepsilon(m)}) \in \mathbf{C}^m_{\tilde{\mathcal{F}}}. \tag{3.6}$$

Set $\varepsilon(j) := +$ and $\delta(j) := 1$ for each $j \in \mathbf{N}$ with $l < j \leq d + k$. Then we have the following

Claim 1. *For each* $j \in \mathbf{N}$ *with* $m < j \leq d + k$, *we have* $z_j^{\varepsilon(j)} \neq 0$.

Proof. Since $-\sqrt{-1} z_j^+ z_j^- = \beta_j \neq 0$ for each $j \in \mathbf{N}$ with $l < j \leq d + k$, we have $z_j^{\varepsilon(j)} \neq 0$ for each such j.

We show that

$$\mathcal{F} \cap \mathcal{H}_j = \emptyset \quad \text{for each } j \in \mathbf{N} \text{ with } m < j \leq l. \tag{3.7}$$

Suppose that $\mathcal{F} \cap \mathcal{H}_{j_0} \neq \emptyset$ for some $j_0 \in \mathbf{N}$ with $m < j_0 \leq l$, and seek a contradiction. Then $\mathcal{F} \cap \mathcal{H}_{j_0}$ is a face of $\mathcal{F}$, so that $\mathcal{F} \cap \mathcal{H}_{j_0}$ is a polytope. Let x be a vertex of $\mathcal{F} \cap \mathcal{H}_{j_0}$. Then x is a vertex of $\mathcal{F}$. Hence there exists $J_1 \subset \{1, \dots, m\}$ such that $\# J_1 = d$ and $x \in \bigcap_{j \in J_1} \mathcal{H}_j$, and so

$$x \in \bigcap_{j \in J_1 \cup \{j_0\}} \mathcal{H}_j \cap \bigcap_{j = d+k+1}^{N} \mathcal{H}_j =: \mathcal{Q}.$$

But, by Proposition 2.2, we have $\mathcal{Q} = \emptyset$; we have therefore arrived at a contradicion. Hence we obtain (3.7).

We now prove that $z_j^{\varepsilon(j)} \neq 0$ for each $j \in \mathbf{N}$ with $m < j \leq l$. Since $z_j^{\varepsilon_-(j)} = 0$ for each $j \in \mathbf{N}$ with $m < j \leq l$, it follows from Part (ii) of Proposition 3.2, (3.4), and (3.7) that $z_j^{\varepsilon(j)} \neq 0$ for each $j \in \mathbf{N}$ with $m < j \leq l$. □

It follows from (3.6) and Claim 1 that

$$z := \left(z_1^{\varepsilon(1)} \prod_{j=m+1}^{d+k} (z_j^{\varepsilon(j)})^{a_{1j}\delta(j)\delta(1)}, \dots, z_d^{\varepsilon(d)} \prod_{j=m+1}^{d+k} (z_j^{\varepsilon(j)})^{a_{dj}\delta(j)\delta(d)}, \right. \\ \left. z_{d+1}^{\varepsilon(d+1)}, \dots, z_m^{\varepsilon(m)} \right) \tag{3.8}$$

is also in $\mathbf{C}^m_{\tilde{\mathcal{F}}}$. Hence we can define a mapping

$$M \cap \mu_{\boldsymbol{I}}^{-1}(\alpha) \to \mathbf{C}^m_{\tilde{\mathcal{F}}}$$
$$(z^+, z^-) \mapsto z.$$

This mapping induces a holomorphic mapping

$$f : \phi^{-1}(\mathcal{F} \times \{0\}) = (M \cap \mu_{\boldsymbol{I}}^{-1}(\alpha))/K \to \mathbf{C}^m_{\tilde{\mathcal{F}}}/L_{\mathbf{C}} = X_{\tilde{\mathcal{F}}}.$$

It is easy to check that the mapping f is well-defined.

(b) We next construct the inverse of f. Let $z = (z_1, \dots, z_m) \in \nu^{-1}(\gamma)$. Set

(1) $z_j^{\varepsilon(j)} := z_j$ and $z_j^{\varepsilon_-(j)} := 0$ for each $j = 1, \dots, m$,

(2) $z_j^{\varepsilon(j)} := 1$ and $z_j^{\varepsilon_-(j)} := 0$ for each $j = m+1, \dots, l$,

(3) $z_j^+ := 1$ and $z_j^- := \sqrt{-1}\beta_j$ for each $j = l+1, \dots, d+k$, and
(4) $z_j^+ := z_j^- := 0$ for each $j = d+k+1, \dots, N$.
Then

$$(z^+, z^-) \in M \subset \mu_{\mathbf{C}}^{-1}(\beta); \tag{3.9}$$

moreover, we have the following

Claim 2. *The point (z^+, z^-) is α-stable.*

Proof. We can write $\psi([z]) = \sum_{j=1}^d c_j u_j$ for suitable $c_1, \dots, c_d \in \mathbf{R}$. Let $\{v_j \mid j \in J\}$ be the dual basis of $\{\pi(e_j) \mid j \in J\}$. Set $v := \sum_{j=1}^d c_j v_j$. Then $v \in \mathcal{F}$. By [2, Theorem 3.1(i)], there exists $[w^+, w^-] \in X(\alpha, \beta)$ such that $\phi([w^+, w^-]) = (v, 0)$. Setting $w := (w_1^{\varepsilon(1)}, \dots, w_m^{\varepsilon(m)})$, we have $w \in \nu^{-1}(\gamma)$. For each $j = 1, \dots, d$, we have

$$\langle \psi([w]), e_j \rangle = \langle \psi([w]), \delta(j) q(e_j) \rangle = \frac{1}{2}\delta(j)|w_j^{\varepsilon(j)}|^2. \tag{3.10}$$

On the other hand, we have, for each $j = 1, \dots, d$,

$$\begin{aligned} \langle \psi([z]), e_j \rangle &= \langle v, \pi(e_j) \rangle \\ &= \langle \phi_{\boldsymbol{I}}([w^+, w^-]), \pi(e_j) \rangle \\ &= \frac{1}{2}(|w_j^+|^2 - |w_j^-|^2). \end{aligned} \tag{3.11}$$

It therefore follows from (3.4) that

$$\langle \psi([z]), e_j \rangle = \frac{1}{2}\delta(j)|w_j^{\varepsilon(j)}|^2 \quad \text{for each } j = 1, \dots, d.$$

Hence, by (3.10), we have $\psi([z]) = \psi([w])$, and so there exists $t \in T^m$ such that $z = t \cdot w$. Thus, since the point (w^+, w^-) is α-stable, it follows from (3.4), (1), (2), (3), and Proposition 2.5 that the point (z^+, z^-) is also α-stable. □

By (3.9) and Claim 2, we can define a mapping

$$\begin{aligned} \nu^{-1}(\gamma) &\to M \cap \mu_{\mathbf{C}}^{-1}(\beta)^{\alpha-st} \\ z &\mapsto (z^+, z^-). \end{aligned}$$

This mapping induces a holomorphic mapping

$$g : X_{\tilde{\mathcal{F}}} = \nu^{-1}(\gamma)/L \to (M \cap \mu_{\mathbf{C}}^{-1}(\beta)^{\alpha-st})/K_{\mathbf{C}} = \phi^{-1}(\mathcal{F} \times \{0\}).$$

It is easy to check that the mapping g is well-defined and that $f \circ g = \mathrm{Id}_{X_{\tilde{\mathcal{F}}}}$ and $g \circ f = \mathrm{Id}_{\phi^{-1}(\mathcal{F} \times \{0\})}$.

Thus f is biholomorphic, as required. This completes the proof of Theorem 3.1. □

4. Resolution of singularities

We use [11] as a reference for basic facts about algebro-geometric quotients.

Suppose that $(\alpha,\beta)\in(\mathfrak{k}^*\times\mathfrak{k}^*_{\mathbf{C}})_{\mathrm{reg}}$. Then the inclusion homomorphism $\mathbf{C}[\mu_{\mathbf{C}}^{-1}(\beta)]^{K_{\mathbf{C}}}\hookrightarrow\mathbf{C}[\mu_{\mathbf{C}}^{-1}(\beta)]$ induces an affine quotient map

$$p:\mu_{\mathbf{C}}^{-1}(\beta)\to\mathrm{Spm}\,\mathbf{C}[\mu_{\mathbf{C}}^{-1}(\beta)]^{K_{\mathbf{C}}}=:\mu_{\mathbf{C}}^{-1}(\beta)/\!/K_{\mathbf{C}}.$$

The morphism p is given by generators of $\mathbf{C}[\mu_{\mathbf{C}}^{-1}(\beta)]^{K_{\mathbf{C}}}$. Let the affine variety $\mu_{\mathbf{C}}^{-1}(\beta)/\!/K_{\mathbf{C}}$ be equipped with the (usual) Euclidean topology. Then the composite mapping

$$\mu^{-1}(\alpha,\beta)\overset{\subset}{\to}\mu_{\mathbf{C}}^{-1}(\beta)\overset{p}{\to}\mu_{\mathbf{C}}^{-1}(\beta)/\!/K_{\mathbf{C}}$$

induces a holomorphic mapping

$$\Psi:(X(\alpha,\beta),\boldsymbol{I})=(\mu^{-1}(\alpha,\beta)/K,\boldsymbol{I})\to\mu_{\mathbf{C}}^{-1}(\beta)/\!/K_{\mathbf{C}}.$$

The purpose of this section is to prove that the mapping Ψ is a resolution of singularities (Theorem 4.6).

In this section, we use the fact that $\mu_{\mathbf{C}}^{-1}(\beta)$ is irreducible for each $\beta\in\mathfrak{k}^*_{\mathbf{C}}$. Since $\pi(e_j)\neq 0$ for each $j=1,\dots,N$, this fact follows immediately from the following proposition. This proposition is due to C. Nakayama.

PROPOSITION 4.1. *Let R be an integral domain, and let $a_1,\dots,a_t\in R\backslash\{0\}$. Then*

$$A:=R[z_1^+,\dots,z_t^+,z_1^-,\dots,z_t^-]/(z_1^+z_1^--a_1,\dots,z_t^+z_t^--a_t)$$

is also an integral domain.

Proof. Since the natural ring homomorphism $R\to A$ is injective, we may assume that $t=1$. Consider the ring homomorphism $g:R_1:=R[z_1^+,z_1^-]\to R[z_1^+,1/z_1^+]$ for which $g(h)=h$ for each $h\in R[z_1^+]$ and $g(z_1^-)=a_1/z_1^+$. We show that $\ker g=\langle z_1^+z_1^--a_1\rangle_{R_1}$. Let $h\in\ker g$. Then $h\in\langle z_1^--a_1/z_1^+\rangle_{R_1[1/z_1^+]}$. Hence there exist $n\in\mathbf{N}$ and $f\in R_1$ such that $(z_1^+)^nh=(z_1^+z_1^--a_1)f$. Thus, since $a_1\neq0$ and z_1^+ is prime element of R_1, we have $f\in\langle z_1^+\rangle_{R_1}$. Hence $h\in\langle z_1^+z_1^--a_1\rangle_{R_1}$, and so $\ker g\subset\langle z_1^+z_1^--a_1\rangle_{R_1}$. The reverse inclusion is immediate from the definition of g. Thus A is an integral domain. □

First, we prove the following

PROPOSITION 4.2. *The mapping Ψ is proper and surjective.*

Proof. Suppose that Ψ is not proper, and look for a contradiction. Then $p|_{\mu^{-1}(\alpha,\beta)}$ is not proper. Therefore there exists a compact subset $C\subset\mu_{\mathbf{C}}^{-1}(\beta)/\!/K_{\mathbf{C}}$ such that $(p|_{\mu^{-1}(\alpha,\beta)})^{-1}(C)$ is non-compact. Hence we can choose an unbounded sequence $\{z_\nu\}_{\nu\in\mathbf{N}}$ in $(p|_{\mu^{-1}(\alpha,\beta)})^{-1}(C)$. For each $\nu\in\mathbf{N}$, we write z_ν as $z_\nu=(z_{\nu,1}^+,\dots,z_{\nu,N}^+,z_{\nu,1}^-,\dots,z_{\nu,N}^-)$. We set

$$J_\infty^+:=\left\{j\in\{1,\dots,N\}\mid\lim_{\nu\to\infty}|z_{\nu,j}^+|=+\infty\right\}$$

and

$$J_\infty^- := \left\{ j \in \{N+1, \dots, 2N\} \mid \lim_{\nu\to\infty} |z_{\nu, j-N}^-| = +\infty \right\}.$$

We may assume that

(a) $J_\infty^+ \cup J_\infty^- \neq \emptyset$;

(b) the sequence $\{z_{\nu,j}^+\}_{\nu\in\mathbf{N}}$ is bounded for each $j \in (J_\infty^+)^c$; and

(c) the sequence $\{z_{\nu,j-N}^-\}_{\nu\in\mathbf{N}}$ is bounded for each $j \in (J_\infty^-)^c$.

By rearranging the indices, we may assume that $\{\iota^* u_1, \dots, \iota^* u_k\}$ is a basis for $\mathfrak{k}^*$. Let $P = (p_{ij})$ be the matrix of ι^* relative to the bases $\{u_1, \dots, u_N\}$, $\{\iota^* u_1, \dots, \iota^* u_k\}$. By Proposition 2.3, the matrix P is integral. Let $\hat{P}$ be obtained from the matrix $(P|-P)$ by replacing the jth column of $(P|-P)$ by 0 for each $j \in (J_\infty^+)^c \cup (J_\infty^-)^c$.

For real row vectors $a = (a_1, \dots, a_m)$ and $b = (b_1, \dots, b_m)$, we write $a \geq b$ precisely when $a_j \geq b_j$ for each $j = 1, \dots, m$. We show that there does not exist $y \in \mathbf{R}^k$ with ${}^t y\hat{P} \geq 0$ and ${}^t y\hat{P} \neq 0$. Suppose that such a y exists, and seek a contradiction. Let $q := (q_1, \dots, q_{2N}) := {}^t y\hat{P}$. Then, by (2.1) and the definition of μ_I, there exist c, c_i, $d_j \in \mathbf{R}$ $(i \in (J_\infty^+)^c, j \in (J_\infty^-)^c)$ such that

$$\sum_{i\in(J_\infty^+)^c} c_i |z_{\nu,i}^+|^2 + \sum_{j\in(J_\infty^-)^c} d_j |z_{\nu,j-N}^-|^2 + \sum_{i=1}^{N} q_i |z_{\nu,i}^+|^2 + \sum_{j=N+1}^{2N} q_j |z_{\nu,j-N}^-|^2 = c$$

for each $\nu \in \mathbf{N}$. For each $\nu \in \mathbf{N}$, we set

$$x_\nu := \sum_{i\in(J_\infty^+)^c} c_i |z_{\nu,i}^+|^2 + \sum_{j\in(J_\infty^-)^c} d_j |z_{\nu,j-N}^-|^2$$

and

$$y_\nu := \sum_{i=1}^{N} q_i |z_{\nu,i}^+|^2 + \sum_{j=N+1}^{2N} q_j |z_{\nu,j-N}^-|^2.$$

It is clear from Conditions (b) and (c) of the hypotheses that the sequence $\{x_\nu\}_{\nu\in\mathbf{N}}$ is bounded. It follows from the definition of $\hat{P}$ that $q_j = 0$ for each $j \in (J_\infty^+)^c \cup (J_\infty^-)^c$, so that, since $q \geq 0$ and $q \neq 0$, there exists $j \in J_\infty^+ \cup J_\infty^-$ such that $q_j > 0$. Hence we have $\lim_{\nu\to\infty} y_\nu = +\infty$. Thus we have $\lim_{\nu\to\infty}(x_\nu + y_\nu) = +\infty$. This is a contradiction. Hence there does not exist $y \in \mathbf{R}^k$ with ${}^t y\hat{P} \geq 0$ and ${}^t y\hat{P} \neq 0$.

Thus, since $\hat{P}$ is a rational matrix, it follows from the Transposition Theorem of Stiemke [14, p. 95] that there exists a vector $m = {}^t(m_1, \dots, m_{2N}) \in \mathbf{Z}^{2N}$ such that $m_j > 0$ for each $j = 1, \dots, 2N$ and $\hat{P}m = 0$. Setting

$$f := \prod_{i\in J_\infty^+} (z_i^+)^{m_i} \prod_{j\in J_\infty^-} (z_{j-N}^-)^{m_j},$$

we have $\lim_{\nu\to\infty} |f(z_\nu)| = +\infty$. On the other hand, since $\hat{P}m = 0$, the monomial f is $K_{\mathbf{C}}$-invariant. Thus, since $p(z_\nu) \in C$ for each $\nu \in \mathbf{N}$, the sequence $\{f(z_\nu)\}_{\nu\in\mathbf{N}}$

is bounded. But this contradicts the fact that $\lim_{\nu\to\infty}|f(z_\nu)| = +\infty$. Hence Ψ is proper.

We next prove that Ψ is surjective. It follows from Proposition 2.5 that $\mu_{\mathbf{C}}^{-1}(\beta)^{\alpha-st}$ is a nonempty Zariski open subset of $\mu_{\mathbf{C}}^{-1}(\beta)$. Thus, since $\mu_{\mathbf{C}}^{-1}(\beta)$ is irreducible by Proposition 4.1, the set $\mu_{\mathbf{C}}^{-1}(\beta)^{\alpha-st}$ is Zariski dense in $\mu_{\mathbf{C}}^{-1}(\beta)$. Thus, denoting the Zariski closure of a set X by $\mathrm{cl}^*(X)$, we have

$$\begin{aligned}\mu_{\mathbf{C}}^{-1}(\beta)//K_{\mathbf{C}} &= p(\mathrm{cl}^*(\mu_{\mathbf{C}}^{-1}(\beta)^{\alpha-st}))\\ &\subset \mathrm{cl}^*(p(\mu_{\mathbf{C}}^{-1}(\beta)^{\alpha-st})) \subset \mu_{\mathbf{C}}^{-1}(\beta)//K_{\mathbf{C}}.\end{aligned}$$

Hence

$$\mu_{\mathbf{C}}^{-1}(\beta)//K_{\mathbf{C}} = \mathrm{cl}^*(p(\mu_{\mathbf{C}}^{-1}(\beta)^{\alpha-st})).$$

For a subset X of $\mu_{\mathbf{C}}^{-1}(\beta)//K_{\mathbf{C}}$, we denote by $\mathrm{cl}(X)$ the closure of X in the Euclidean topology on $\mu_{\mathbf{C}}^{-1}(\beta)//K_{\mathbf{C}}$. Since $p(\mu_{\mathbf{C}}^{-1}(\beta)^{\alpha-st})$ is constructible [12, Corollary 2, p. 51], it follows from [12, Corollary 1, p. 60] that

$$\mathrm{cl}^*(p(\mu_{\mathbf{C}}^{-1}(\beta)^{\alpha-st})) = \mathrm{cl}(p(\mu_{\mathbf{C}}^{-1}(\beta)^{\alpha-st})).$$

Now Ψ is closed, since Ψ is proper. Hence

$$\mu_{\mathbf{C}}^{-1}(\beta)//K_{\mathbf{C}} = \mathrm{cl}(p(\mu_{\mathbf{C}}^{-1}(\beta)^{\alpha-st})) = \mathrm{cl}(\mathrm{Im}\,\Psi) = \mathrm{Im}\,\Psi.$$

This completes the proof of Proposition 4.2. □

Suppose that $\beta \in \mathfrak{k}_{\mathbf{C}}^*$. Recall ([11, Definition 5.12]) that a point $x \in \mu_{\mathbf{C}}^{-1}(\beta)$ is said to be *stable* for the action of $K_{\mathbf{C}}$ precisely when

(i) the orbit $x \cdot K_{\mathbf{C}}$ is a Zariski closed subset of $\mu_{\mathbf{C}}^{-1}(\beta)$, and

(ii) the isotropy group of x is finite.

Let $\mu_{\mathbf{C}}^{-1}(\beta)^s$ denote the set of all stable points for the $K_{\mathbf{C}}$-action, and set $U_\beta := p(\mu_{\mathbf{C}}^{-1}(\beta)^s)$. The stable set $\mu_{\mathbf{C}}^{-1}(\beta)^s \subset \mu_{\mathbf{C}}^{-1}(\beta)$ and its image $U_\beta \subset \mu_{\mathbf{C}}^{-1}(\beta)//K_{\mathbf{C}}$ are Zariski open sets [11, Proposition 5.15].

The following proposition is useful in the rest of this section.

Proposition 4.3. *Let* $(\alpha,\beta) \in (\mathfrak{k}^* \times \mathfrak{k}_{\mathbf{C}}^*)_{\mathrm{reg}}$. *Then*

$$\mu_{\mathbf{C}}^{-1}(\beta)^s \subset \mu_{\mathbf{C}}^{-1}(\beta)^{\alpha-st}.$$

Proof. Let $x \in \mu_{\mathbf{C}}^{-1}(\beta)^s$. Then, by Proposition 4.2, there exists $y \in \mu_{\mathbf{C}}^{-1}(\beta)^{\alpha-st}$ with $p(x) = p(y)$. It therefore follows from [11, Theorem 5.16] that $x \cdot K_{\mathbf{C}} = y \cdot K_{\mathbf{C}}$. Since the set $\mu_{\mathbf{C}}^{-1}(\beta)^{\alpha-st}$ is $K_{\mathbf{C}}$-invariant, we have $x \in \mu_{\mathbf{C}}^{-1}(\beta)^{\alpha-st}$. Hence $\mu_{\mathbf{C}}^{-1}(\beta)^s \subset \mu_{\mathbf{C}}^{-1}(\beta)^{\alpha-st}$. □

Since, by Definition 2.4, the variety $\mu_{\mathbf{C}}^{-1}(\beta)$ is smooth at each point of $\mu_{\mathbf{C}}^{-1}(\beta)^{\alpha-st}$, and since $K_{\mathbf{C}}$ acts freely on $\mu_{\mathbf{C}}^{-1}(\beta)^{\alpha-st}$ [8, Theorem 5.2(1)], it follows from Proposition 4.3 and [11, Corollary 9.52] that U_β is smooth. Hence, by

Proposition 4.3 again and [11, Theorem 5.16], the mapping Ψ maps $\Psi^{-1}(U_\beta)$ biholomorphically onto U_β. Hence

(4.1) *The exceptional set* $X(\alpha,\beta)\backslash\Psi^{-1}(U_\beta)$ *contains every compact complex submanifold of* $(X(\alpha,\beta),\boldsymbol{I})$.

We state a criterion for stability in terms of the elements of $\mathscr{V}_\beta$.

PROPOSITION 4.4. *Let* $(\alpha,\beta)\in(\mathfrak{k}^*\times\mathfrak{k}^*_{\mathbf{C}})_{\mathrm{reg}}$, *and let* $(z^+,z^-)\in\mu_{\mathbf{C}}^{-1}(\beta)^{\alpha-st}$. *Then* $(z^+,z^-)\in\mu_{\mathbf{C}}^{-1}(\beta)^{s}$ *if and only if*

(4.2) *For each* $V\in\mathscr{V}_\beta$, *there exists* $j\in J_V^+\cup J_V^-$ *such that either* $j\in J_V^+$ *with* $z_j^-\neq 0$ *or* $j\in J_V^-$ *with* $z_j^+\neq 0$.

Proof. Set

$$J_1 := \{j\,|\,1\leq j\leq N, z_j^+\neq 0,\ \text{and}\ z_j^-\neq 0\},$$
$$J_2 := \{j\,|\,1\leq j\leq N, z_j^+\neq 0,\ \text{and}\ z_j^-= 0\},$$
$$J_3 := \{j\,|\,1\leq j\leq N, z_j^+= 0,\ \text{and}\ z_j^-\neq 0\}.$$

Let $\mathbf{R}_{>0}$ (respectively $\mathbf{R}_{<0}$) denote the set of positive (respectively negative) real numbers. Let

$$\sigma := \sum_{j\in J_1}\mathbf{R}\iota^*u_j+\sum_{j\in J_2}\mathbf{R}_{>0}\iota^*u_j+\sum_{j\in J_3}\mathbf{R}_{<0}\iota^*u_j.$$

We first show that $\dim\sigma=k$. We suppose that $\dim\sigma<k$ and look for a contradiction. Then there exists $V\in\mathscr{V}$ such that

$$\sum_{j\in J_1\cup J_2\cup J_3}\mathbf{R}\iota^*u_j\subset V.$$

Since $(z^+,z^-)\in\mu_{\mathbf{C}}^{-1}(\beta)^{\alpha-st}$, it follows from (2.1), (2.2), and the definition of μ that

$$\alpha\in\sum_{j\in J_1\cup J_2\cup J_3}\mathbf{R}\iota^*u_j\quad\text{and}\quad\beta\in\sum_{j\in J_1\cup J_2\cup J_3}\mathbf{C}\iota^*u_j.$$

Hence we have $\alpha\in V$ and $\beta\in V_{\mathbf{C}}$. On the other hand, since $(\alpha,\beta)\in(\mathfrak{k}^*\times\mathfrak{k}^*_{\mathbf{C}})_{\mathrm{reg}}$, it follows from Proposition 2.2 that either $\alpha\notin V$ or $\beta\notin V_{\mathbf{C}}$. This is a contradiction. Hence

$$\dim\sigma=k. \tag{4.3}$$

Now let (z^+,z^-) satisfy Condition (4.2); since $(-\alpha,\beta)\in(\mathfrak{k}^*\times\mathfrak{k}^*_{\mathbf{C}})_{\mathrm{reg}}$ by Proposition 2.2, we can deduce from Proposition 2.5 that $(z^+,z^-)\in\mu_{\mathbf{C}}^{-1}(\beta)^{(-\alpha)-st}$. Hence, by [8, Definition 5.1], we have $\alpha\in\sigma\cap(-\sigma)$. In particular, $\sigma\cap(-\sigma)\neq\emptyset$, and so $\sigma\cap(-\sigma)$ is a subspace of $\mathfrak{k}^*$. Thus, since σ is an open subset of $\mathfrak{k}^*$ by (4.3), we have $\mathfrak{k}^*=\sigma\cap(-\sigma)$. Hence $\mathfrak{k}^*=\sigma$.

For each $v \in \mathfrak{k}^*$, we define a function $l_v : \mathfrak{k} \to \mathbf{R}$ by

$$l_v(X) = \langle v, X \rangle + \frac{1}{4}\sum_{j=1}^{N}(|z_j^+|^2 e^{-2\langle \iota^* u_j, X \rangle} + |z_j^-|^2 e^{2\langle \iota^* u_j, X \rangle}).$$

CLAIM 1. *Let $v \in \mathfrak{k}^*$, and let $X \in \mathfrak{k}$ be such that $\langle v, X \rangle \neq 0$. Then we have*

$$\lim_{t \to +\infty} l_v(tX) = +\infty.$$

Proof. The proof of the claim is the same as that of Claim 5.9 of [8] except for obvious modifications.

We have

$$l_v(tX) = t\langle v, X \rangle + \frac{1}{4}\sum_{j=1}^{N}(|z_j^+|^2 e^{-2t\langle \iota^* u_j, X \rangle} + |z_j^-|^2 e^{2t\langle \iota^* u_j, X \rangle}). \tag{4.4}$$

If $\langle v, X \rangle > 0$, then the claim holds by (4.4). Suppose that $\langle v, X \rangle < 0$. Since $\sigma = \mathfrak{k}^*$, we can write

$$v = \sum_{j \in J_1} c_j^{(1)} \iota^* u_j + \sum_{j \in J_2} c_j^{(2)} \iota^* u_j + \sum_{j \in J_3} c_j^{(3)} \iota^* u_j,$$

where $c_j^{(1)} \in \mathbf{R}$ for each $j \in J_1$, $c_j^{(2)} \in \mathbf{R}_{>0}$ for each $j \in J_2$, and $c_j^{(3)} \in \mathbf{R}_{<0}$ for each $j \in J_3$. Thus, since $\langle v, X \rangle < 0$, there exists $j \in J_1 \cup J_2 \cup J_3$ such that either

$$j \in J_1 \cup J_2 \text{ with } \langle \iota^* u_j, X \rangle < 0 \quad \text{or} \quad j \in J_1 \cup J_3 \text{ with } \langle \iota^* u_j, X \rangle > 0.$$

Hence, by (4.4), we have

$$\lim_{t \to +\infty} l_v(tX) = +\infty. \qquad \square$$

Suppose that the orbit $(z^+, z^-) \cdot K_{\mathbf{C}} \subset \mu_{\mathbf{C}}^{-1}(\beta)$ is not Zariski closed, and seek a contradiction. By [3, Lemma 3.4], there exists an element $(w^+, w^-) \in (\mathbf{C}^N \times \mathbf{C}^N) \backslash \{(z^+, z^-)\}$ and a one-parameter subgroup $\lambda : \mathbf{G}_m \to K_{\mathbf{C}}$ such that

$$(z^+, z^-) \cdot \lambda(x) \to (w^+, w^-) \quad \text{as } x \to 0. \tag{4.5}$$

We can write the one-parameter subgroup λ in the form

$$x \in \mathbf{C}^* \mapsto (x^{m_1}, \dots, x^{m_N}) \in K_{\mathbf{C}}$$

with $m_1, \dots, m_N \in \mathbf{Z}$. Setting $X := {}^t(m_1, \dots, m_N)$, we have $X \in \mathfrak{k} \backslash \{0\}$. Thus there exists an element $v \in \mathfrak{k}^*$ such that $\langle v, X \rangle < 0$. By Claim 1, we have $\lim_{t \to +\infty} l_v(tX) = +\infty$. On the other hand, since

$$\lim_{t \to +\infty} \sum_{j=1}^{N}(|z_j^+|^2 e^{-2t\langle \iota^* u_j, X \rangle} + |z_j^-|^2 e^{2t\langle \iota^* u_j, X \rangle}) = \sum_{j=1}^{N}(|w_j^+|^2 + |w_j^-|^2)$$

by (4.5), using $\langle v, X \rangle < 0$, we have $\lim_{t\to+\infty} l_v(tX) = -\infty$. This is a contradiction. Hence the orbit $(z^+, z^-) \cdot K_{\mathbf{C}}$ is Zariski closed. Thus, since $K_{\mathbf{C}}$ acts freely on $\mu_{\mathbf{C}}^{-1}(\beta)^{\alpha-st}$ [8, Theorem 5.2(1)], we have $(z^+, z^-) \in \mu_{\mathbf{C}}^{-1}(\beta)^s$.

Conversely, suppose that $(z^+, z^-) \in \mu_{\mathbf{C}}^{-1}(\beta)^s$; since $(-\alpha, \beta) \in (\mathfrak{k}^* \times \mathfrak{k}_{\mathbf{C}}^*)_{\mathrm{reg}}$ by Proposition 2.2, we can deduce from Proposition 4.3 that $(z^+, z^-) \in \mu_{\mathbf{C}}^{-1}(\beta)^{(-\alpha)-st}$. Thus, by Proposition 2.5, we see that (z^+, z^-) satisfies Condition (4.2). □

We use this criterion to prove the following

Proposition 4.5. *Let* $\beta \in \mathfrak{k}_{\mathbf{C}}^*$. *Then*

$$\mu_{\mathbf{C}}^{-1}(\beta)^s \neq \emptyset.$$

Proof. Let $\alpha \in \mathfrak{k}^*$ be such that $(\alpha, \beta) \in (\mathfrak{k}^* \times \mathfrak{k}_{\mathbf{C}}^*)_{\mathrm{reg}}$. If $\mathscr{V}_\beta = \emptyset$, then $\mu_{\mathbf{C}}^{-1}(\beta)^{\alpha-st} = \mu_{\mathbf{C}}^{-1}(\beta)$ by Proposition 2.5. Thus, since $K_{\mathbf{C}}$ acts freely on $\mu_{\mathbf{C}}^{-1}(\beta)^{\alpha-st}$ [8, Theorem 5.2(1)], it follows from [11, Corollary 5.14] that $\mu_{\mathbf{C}}^{-1}(\beta)^s = \mu_{\mathbf{C}}^{-1}(\beta)$. Hence, since $\mu_{\mathbf{C}}^{-1}(\beta)$ is nonempty, so is $\mu_{\mathbf{C}}^{-1}(\beta)^s$; we therefore suppose that $\mathscr{V}_\beta \neq \emptyset$.

Then, for each $V \in \mathscr{V}_\beta$, there exists $j_V \in J_V^+ \cup J_V^-$. Let $b \in (\mathbf{C}^N)^*$ be such that $\iota^* b = \beta$. Fix $x_0 \in (\mathbf{C}^n)^*$ such that $x_0 \notin \mathcal{H}_{\mathbf{C}}(j_V, b)$ for each $V \in \mathscr{V}_\beta$. By [2, Theorem 3.1(i)], there exists $[z^+, z^-] \in X(\alpha, \beta)$ such that $\phi_{\mathbf{C}}^b([z^+, z^-]) = x_0$. For each $V \in \mathscr{V}_\beta$, we have $z_{j_V}^+ z_{j_V}^- \neq 0$. Indeed, if $z_{j_V}^+ z_{j_V}^- = 0$ for some $V \in \mathscr{V}_\beta$, then we obtain

$$\begin{aligned} \langle \pi^*(x_0) + b, e_{j_V} \rangle &= \langle \pi^*(\phi_{\mathbf{C}}^b([z^+, z^-])) + b, e_{j_V} \rangle \\ &= -\sqrt{-1} z_{j_V}^+ z_{j_V}^- \\ &= 0. \end{aligned}$$

Thus $x_0 \in \mathcal{H}_{\mathbf{C}}(j_V, b)$. This is a contradiction. Hence, since $(z^+, z^-) \in \mu_{\mathbf{C}}^{-1}(\beta)^{\alpha-st}$, it follows from Proposition 4.4 that $(z^+, z^-) \in \mu_{\mathbf{C}}^{-1}(\beta)^s$. In particular, $\mu_{\mathbf{C}}^{-1}(\beta)^s \neq \emptyset$. □

Since $\mu_{\mathbf{C}}^{-1}(\beta) /\!/ K_{\mathbf{C}}$ is irreducible by Proposition 4.1, it follows from Proposition 4.5 that the set U_β is Zariski dense in $\mu_{\mathbf{C}}^{-1}(\beta) /\!/ K_{\mathbf{C}}$.

We summarise our discussions in the following

Theorem 4.6. *The mapping* Ψ *is a resolution of singularities, that is,*
(i) Ψ *is proper and surjective,*
(ii) $\Psi^{-1}(U_\beta)$ *is a dense open subset of* $X(\alpha, \beta)$, *and*
(iii) Ψ *maps* $\Psi^{-1}(U_\beta)$ *biholomorphically onto* U_β. □

5. Equivalence of complex structures

Let $(\alpha, \beta) \in (\mathfrak{k}^* \times \mathfrak{k}_{\mathbf{C}}^*)_{\mathrm{reg}}$. We can write $\beta = \beta_1 + \sqrt{-1}\beta_2$ for suitable $\beta_1, \beta_2 \in \mathfrak{k}^*$. We regard S^2 as the unit sphere in $\mathbf{R}^3$. If $p := {}^t(p_1, p_2, p_3) \in S^2$, then

$\boldsymbol{I}_p := p_1\boldsymbol{I} + p_2\boldsymbol{J} + p_3\boldsymbol{K}$ is also a complex structure on $X(\alpha,\beta)$. Set

$$\mathscr{C}_{(\alpha,\beta)} := \{p \in S^2 \mid (X(\alpha,\beta),\boldsymbol{I}_p) \text{ is not biholomorphic to an affine variety}\}.$$

Let $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ be complex structures on $X(\alpha,\beta)$. We say that $\boldsymbol{I}_1$ is equivalent to $\boldsymbol{I}_2$ and write $\boldsymbol{I}_1 \sim \boldsymbol{I}_2$, precisely when $(X(\alpha,\beta),\boldsymbol{I}_1)$ is biholomorphic to $(X(\alpha,\beta),\boldsymbol{I}_2)$.

In this section, we discuss when two complex structures $\boldsymbol{I}_p$ and $\boldsymbol{I}_q$ with $p, q \in \mathscr{C}_{(\alpha,\beta)}$ are equivalent.

We first give a sufficient condition for a complex structure $\boldsymbol{I}_p$ to be equivalent to the conjugate complex structure $-\boldsymbol{I}_p$.

Proposition 5.1. *Suppose that either $\beta_1 = 0$ or $\beta_2 = 0$. Let $p \in \mathscr{C}_{(\alpha,\beta)}$. Then $\boldsymbol{I}_p \sim -\boldsymbol{I}_p$.*

Proof. We provide a proof for the case where $\beta_1 = 0$; the other case is similar.

Since $\beta_1 = 0$, it follows from [1, Theorem 3.3] that $p_2 = 0$. Let $q_1, q_3 \in \mathbf{R}$ be such that the matrix

$$P := \begin{pmatrix} p_1 & 0 & p_3 \\ 0 & 1 & 0 \\ q_1 & 0 & q_3 \end{pmatrix}$$

is an element in $SO(3)$. Then we have

$$P\begin{pmatrix} \alpha \\ 0 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} p_1\alpha + p_3\beta_2 \\ 0 \\ q_1\alpha + q_3\beta_2 \end{pmatrix}.$$

Hence, if we set

$$\alpha' := p_1\alpha + p_3\beta_2 \quad \text{and} \quad \beta' := \sqrt{-1}(q_1\alpha + q_3\beta_2),$$

then it follows from [1, Theorem 3.2(2)] that

$$(X(\alpha,\beta),\boldsymbol{I}_p) \cong (X(\alpha',\beta'),\boldsymbol{I}).$$

Similarly, we have

$$(X(\alpha,\beta),-\boldsymbol{I}_p) \cong (X(-\alpha',\beta'),\boldsymbol{I}).$$

We can define a biholomorphic map

$$\begin{gathered} (X(\alpha',\beta'),\boldsymbol{I}) \to (X(-\alpha',\beta'),\boldsymbol{I}) \\ [z^+, z^-] \mapsto [z^-, z^+]. \end{gathered}$$

Hence we have $\boldsymbol{I}_p \sim -\boldsymbol{I}_p$. □

Corollary 5.2. *Let $\#\mathscr{C}_{(\alpha,\beta)} = 2$. Then $\mathscr{C}_{(\alpha,\beta)} = \{p, -p\}$ for some $p \in S^2$, and $\boldsymbol{I}_p \sim -\boldsymbol{I}_p$.*

Proof. By [1, Theorem 3.3], we have $\mathscr{C}_{(\alpha,\beta)} = \{p, -p\}$ for some $p \in S^2$. It therefore follows from [1, Theorem 3.2(2)] that there exists $\alpha' \in \mathfrak{k}^*$ such that $(X(\alpha,\beta), \boldsymbol{I}_p)$ (respectively $(X(\alpha,\beta), -\boldsymbol{I}_p)$) is biholomorphic to $(X(\alpha',0), \boldsymbol{I})$ (respectively $(X(\alpha',0), -\boldsymbol{I})$). Thus, by [1, Theorem 3.3] and Proposition 5.1, we have $\boldsymbol{I}_p \sim -\boldsymbol{I}_p$. □

Example 5.3. Let $\beta = 0$. Then, by [1, Theorem 3.3], we have $\mathscr{C}_{(\alpha,0)} = \{e_1, -e_1\}$. Hence we have $\boldsymbol{I} \sim -\boldsymbol{I}$ (see also [1, Example 4.1]).

In general, $\boldsymbol{I}_p$ and $\boldsymbol{I}_q$ need not be equivalent for each $p, q \in \mathscr{C}_{(\alpha,\beta)}$. We use the results of Sections 3 and 4 to give such an example.

Let K be the subtorus of T^5 whose Lie algebra $\mathfrak{k} \subset \mathbf{R}^5$ is generated by $e_1 + e_4$, $e_2 + e_5$, and $e_3 + e_4 + e_5$. Then $\{\pi(e_4), \pi(e_5)\}$ is a basis for $\mathbf{R}^2$. Thus Condition (iii) in Proposition 2.3 holds. Set

$$\alpha := \iota^* u_3 \quad \text{and} \quad \beta := \iota^* u_1 - \iota^* u_2.$$

Then it follows from Proposition 2.2 that $(\alpha, \beta) \in (\mathfrak{k}^* \times \mathfrak{k}^*_{\mathbf{C}})_{\text{reg}}$. We obtain the toric hyperkähler manifold $X(\alpha, \beta)$ of complex dimension four. We set

$$p_1 := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad p_2 := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad p_3 := \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad p_4 := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

By [1, Theorem 3.3], we have

$$\mathscr{C}_{(\alpha,\beta)} = \{\pm p_1, \pm p_2, \pm p_3, \pm p_4\}.$$

Proposition 5.4. *We have*
(i) $\boldsymbol{I}_{p_i} \sim -\boldsymbol{I}_{p_i}$ *for each* $i = 1, 2, 3, 4$;
(ii) $\boldsymbol{I}_{p_3} \sim \boldsymbol{I}_{p_4}$;
(iii) $\boldsymbol{I}_{p_i} \not\sim \boldsymbol{I}_{p_j}$ *for each* $i, j = 1, 2, 3$ *with* $i \neq j$.

Proof. Set

$$P := \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Then the matrix P is an element in $SO(3)$. We have

$$P \begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \iota^* u_1 - \iota^* u_2 - \iota^* u_3 \\ \iota^* u_1 - \iota^* u_2 + \iota^* u_3 \\ 0 \end{pmatrix}.$$

Hence, if we set

$$\alpha' := \frac{1}{\sqrt{2}} (\iota^* u_1 - \iota^* u_2 - \iota^* u_3) \quad \text{and} \quad \beta' := \frac{1}{\sqrt{2}} (\iota^* u_1 - \iota^* u_2 + \iota^* u_3),$$

then it follows from [1, Theorem 3.2(2)] that

$$(X(\alpha,\beta),\boldsymbol{I}_{p_3}) \cong (X(\alpha',\beta'),\boldsymbol{I}).$$

Similarly, we have

$$(X(\alpha,\beta),\boldsymbol{I}_{p_2}) \cong (X(\beta,\alpha),\boldsymbol{I}) \quad \text{and} \quad (X(\alpha,\beta),\boldsymbol{I}_{p_4}) \cong (X(\beta',\alpha'),\boldsymbol{I}).$$

(i) The claim follows immediately from Proposition 5.1.

(ii) Let $(z^+,z^-) \in \mu^{-1}(\alpha',\beta')$. Set

$$w_1^{\pm} := \pm z_2^{\mp}, \quad w_2^{\pm} := \pm z_1^{\mp}, \quad w_3^{\pm} := \pm z_3^{\mp}, \quad w_4^{\pm} := \pm z_5^{\mp}, \quad w_5^{\pm} := \pm z_4^{\mp}.$$

Then we have $(w^+,w^-) \in \mu^{-1}(\beta',\alpha')$. Hence we can define a biholomorphic map

$$\begin{aligned}(X(\alpha',\beta'),\boldsymbol{I}) &\to (X(\beta',\alpha'),\boldsymbol{I})\\ [z^+,z^-] &\mapsto [w^+,w^-].\end{aligned}$$

Thus we have $\boldsymbol{I}_{p_3} \sim \boldsymbol{I}_{p_4}$.

(iii) First, we use Theorem 3.1 to construct compact complex submanifolds of $(X(\alpha,\beta),\boldsymbol{I})$. Now set $a := u_3$ and $b := u_1 - u_2$. Let

$$\mathcal{P}_1 := \bigcap_{j=3}^{5} \mathcal{H}^+(j,a)$$

(see Figure 1). Then, since $\mathcal{P}_1$ is an isosceles right triangle, the space $X_{\mathcal{P}_1}$ is $\mathbf{P}^2$. Thus, by Theorem 3.1, the submanifold $X_1 := \phi_{a,b}^{-1}(\mathcal{P}_1 \times \{0\})$ is biholomorphic to $\mathbf{P}^2$. Set

$$\begin{aligned}M_1 := \{(z^+,z^-) \in \mathbf{H}^5 \mid z_3^- = z_4^- = z_5^- = 0,\\ -\sqrt{-1}z_1^+z_1^- = 1, -\sqrt{-1}z_2^+z_2^- = -1\} \cap \mu_{\boldsymbol{I}}^{-1}(\alpha).\end{aligned}$$

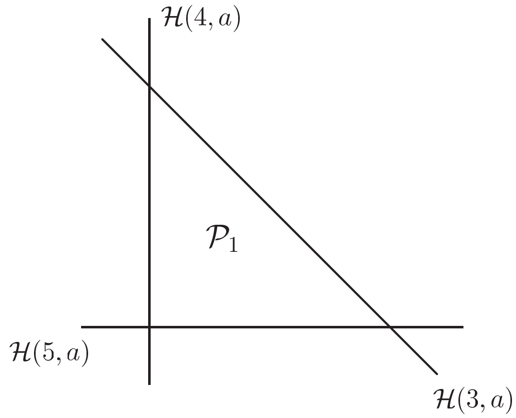

FIGURE 1

It follows from (3.5) that

$$X_1 = M_1/K. \tag{5.1}$$

Now take the basis $\{\iota^* u_3, \iota^* u_4, \iota^* u_5\}$ for $\mathfrak{k}^*$. We have $\beta = \iota^* u_4 - \iota^* u_5$. We set $b' := u_4 - u_5$. Let $\mathcal{P}_2 := \mathcal{H}^-(1,a) \cap \mathcal{H}^-(2,a) \cap \mathcal{H}^+(3,a)$. Then, since $\mathcal{P}_2$ is an isosceles right triangle, the submanifold $X_2 := \phi_{a,b'}^{-1}(\mathcal{P}_2 \times \{0\})$ is also biholomorphic to $\mathbf{P}^2$. Set

$$\begin{aligned} M_2 := \{(z^+, z^-) \in \mathbf{H}^5 \mid z_1^+ = z_2^+ = z_3^- = 0,\\ -\sqrt{-1} z_4^+ z_4^- = 1, -\sqrt{-1} z_5^+ z_5^- = -1\} \cap \mu_{\boldsymbol{I}}^{-1}(\alpha). \end{aligned}$$

It follows from (3.5) that

$$X_2 = M_2/K. \tag{5.2}$$

Since $M_1 \cap M_2 = \emptyset$, we have $X_1 \cap X_2 = \emptyset$.

Next, we use Proposition 4.4 to determine the exceptional set $X(\alpha,\beta) \backslash \Psi^{-1}(U_\beta)$. Let V_1 and V_2 be the following two-dimensional subspaces of $\mathfrak{k}^*$:

$$V_1 := \operatorname{span}\{\iota^* u_1, \iota^* u_2\} \quad \text{and} \quad V_2 := \operatorname{span}\{\iota^* u_4, \iota^* u_5\}.$$

Then we have $\mathscr{V}_\beta = \{V_1, V_2\}$. We set

$$Y_1 := e_3 + e_4 + e_5 \quad \text{and} \quad Y_2 := e_3 - e_1 - e_2.$$

For each $j = 1, 2$, we have $Y_j \in \mathfrak{k}$ and $V_j = \{v \in \mathfrak{k}^* \mid \langle v, Y_j \rangle = 0\}$. Hence we have

$$J_{V_1}^+ = \{3,4,5\}, \quad J_{V_1}^- = \emptyset, \quad J_{V_2}^+ = \{3\}, \quad J_{V_2}^- = \{1,2\}.$$

By (4.1), Proposition 4.4, (5.1), and (5.2), we have

$$X(\alpha,\beta) \backslash \Psi^{-1}(U_\beta) = X_1 \amalg X_2 \cong \mathbf{P}^2 \amalg \mathbf{P}^2. \tag{5.3}$$

Next, we determine the exceptional set $X(\alpha',\beta') \backslash \Psi^{-1}(U_{\beta'})$. Let V be the two-dimensional subspace $V := \operatorname{span}\{\iota^* u_2, \iota^* u_4\}$ of $\mathfrak{k}^*$. Then we have $\mathscr{V}_{\beta'} = \{V\}$. We can prove

$$X(\alpha',\beta') \backslash \Psi^{-1}(U_{\beta'}) \cong \mathbf{P}^2 \tag{5.4}$$

in a way similar to that just used for (5.3).

Finally, we construct a compact complex submanifold of $(X(\beta,\alpha), \boldsymbol{I})$. Let

$$\mathcal{P}_3 := \bigcap_{j=1,4} \mathcal{H}^+(j,b) \cap \bigcap_{j=2,5} \mathcal{H}^-(j,b)$$

(see Figure 2). Then, since $\mathcal{P}_3$ is a square, the space $X_{\mathcal{P}_3}$ is $\mathbf{P}^1 \times \mathbf{P}^1$. Thus, by Theorem 3.1, the submanifold $X_3 := \phi_{b,a}^{-1}(\mathcal{P}_3 \times \{0\})$ is biholomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$.

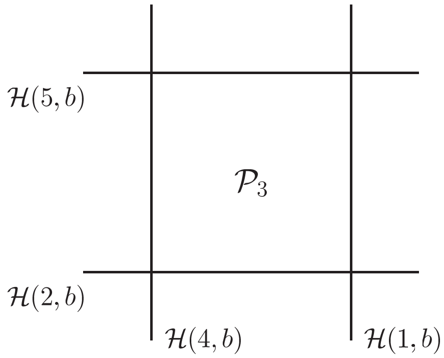

FIGURE 2

Hence, by (4.1), we have

$$\mathbf{P}^1 \times \mathbf{P}^1 \cong X_3 \subset X(\beta,\alpha) \backslash \Psi^{-1}(U_\alpha). \tag{5.5}$$

The claim follows from (4.1), (5.3), (5.4), and (5.5). □

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