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LAGRANGIAN SUBMANIFOLDS WITH CODIMENSION 1 TOTALLY GEODESIC FOLIATION IN COMPLEX PROJECTIVE SPACES

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Dedicated to Professor Yoshio Matsuyama on the occasion of his sixtieth birthday

1. Introduction

Let $(\tilde{M}, \omega)$ be a complex n-dimensional Kähler manifold with Kähler form ω, and let M be a real n-dimensional manifold. Then an immersion $x : M \to \tilde{M}$ is called *Lagrangian* if $x^*\omega = 0$ on M. Y. G. Oh defined [8] that a Lagrangian submanifold M in $\tilde{M}$ is *Hamiltonian minimal* (or *H-minimal*) if the volume of M is stationary for any compactly-supported Hamiltonian deformation of the Lagrangian immersion. The Hamiltonian minimality is characterized as the harmonicity of mean curvature form $\delta\alpha_H = 0$ by the first variational formula. It is important to study either minimal or H-minimal Lagrangian submanifolds in complex projective spaces $\mathbf{CP}^n$.

This paper is concerned with Lagrangian submanifolds in $\mathbf{CP}^n$ which are solutions of above variational problem, with some *symmetry*. Namely, we consider Lagrangian submanifolds which are obtained as a 1-parameter family of totally geodesic $\mathbf{RP}^{n-1}$ in $\mathbf{CP}^n$. To do that let $\mathscr{M}_n$ be the set of totally geodesic $\mathbf{RP}^{n-1}$ in $\mathbf{CP}^n$. Since the unitary group $U(n+1)$ acts on $\mathscr{M}_n$ transitively, $\mathscr{M}_n$ is a homogeneous space of $U(n+1)$. From a curve $\gamma : I \to \mathscr{M}_n$, we can construct a real n-dimensional submanifold M (which may have some singularities) with 1-parameter family of totally geodesic $\gamma(t) = \mathbf{RP}^{n-1}$ in $\mathbf{CP}^n$. First we will show that M is a Lagrangian submanifold on the open subset of regular points if and only if the corresponding curve γ in $\mathscr{M}_n$ is *horizontal* with respect to the natural fibration $\mathscr{M}_n \to \mathbf{CP}^n$ (Proposition 3.1).

Using this argument, we will see that minimal Lagrangian submanifold with 1-parameter family of totally geodesic $\mathbf{RP}^{n-1}$ in $\mathbf{CP}^n$ is totally geodesic (Theorem 4.1). Next we will show that for a Lagrangian submanifold with 1-parameter family of totally geodesic $\mathbf{RP}^{n-1}$ in $\mathbf{CP}^n$, its Hamiltonian minimality is expressed as a system of 2nd order ODE's for curves in S^3 (Proposition 4.1). As a special solution, if we take a curve γ in $\mathscr{M}_n$ as an orbit of 1-parameter subgroup of

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$U(n+1)$, then we have neither totally geodesic nor minimal Lagrangian submanifolds M^n in $\mathbf{CP}^n$ satisfying $\delta\alpha_H = 0$ (cf. Theorem 4.2). When $n \geq 3$, M^n must have some singularities, but when $n = 2$, M^2 is everywhere regular and flat, and the mean curvature vector $H \neq 0$ is parallel with respect to the normal connection. Such Lagrangian surface in $\mathbf{CP}^2$ was studied by Ogata [7].

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2. Preliminaries

First we recall about Hamiltonian deformation of Lagrangian submanifolds in Kähler manifolds, defined by Oh [8]. Let $\tilde{M}$ be a complex n-dimensional Kähler manifold with Kähler form ω, Riemann metric $\langle\,,\,\rangle$, and complex structure J. Let $x : M \to \tilde{M}$ be a Lagrangian immersion from a real n-dimensional manifold M to $\tilde{M}$, i.e., $\omega|_{TM} = 0$. For a vector field V along x, we define a 1-form α_V on M as $\alpha_V = \langle JV, \cdot\rangle|_{TM}$. Smooth family of embeddings $\iota_t : M \to P$ is called *Hamiltonian deformation* if for the variational vector field V, the 1-form α_V is exact. A Lagrangian submanifold M is *Hamiltonian minimal* (or *H-minimal*) if M is stationary for any Hamiltonian deformation. Oh [8] showed that when M is *compact*, M is H-minimal if and only if α_H is co-closed, i.e., $\delta\alpha_H = 0$ where H is the mean curvature vector field of M. We have

$$\delta\alpha_H = 0 \Leftrightarrow \operatorname{div} JH = 0. \tag{1}$$

Next we recall the Fubini-Study metric on the complex projective space $\mathbf{CP}^n$ (cf. [2, 4]). The Euclidean metric $\langle\,,\,\rangle$ on $\mathbf{C}^{n+1}$ is given by $\langle z, w\rangle = \mathrm{Re}({}^t z\bar{w})$ for $z, w \in \mathbf{C}^{n+1}$. The unit sphere S^{2n+1} in $\mathbf{C}^{n+1}$ is the principal fiber bundle over $\mathbf{CP}^n$ with the structure group S^1 and the Hopf fibration $\pi : S^{2n+1} \to \mathbf{CP}^n$. The tangent space of S^{2n+1} at a point z is

$$T_z S^{2n+1} = \{w \in C^{n+1} \mid \langle z, w\rangle = 0\}.$$

Let

$$T'_z = \{w \in C^{n+1} \mid \langle z, w\rangle = \langle iz, w\rangle = 0\}.$$

The distribution T'_z defines a connection in the principal fiber bundle $S^{2n+1}(\mathbf{CP}^n, S^1)$, because T'_z is complementary to the subspace $\{iz\}$ tangent to the fibre through z, and invariant under the S^1-action. Then the Fubini-Study metric g of constant holomorphic sectional curvature 4 is given by $g(X, Y) = \langle X^*, Y^*\rangle$, where $X, Y \in T_x\mathbf{CP}^n$, and X^*, Y^* are respectively their horizontal lifts at a point z with $\pi(z) = x$. The complex structure on T' defined by multiplication by $\sqrt{-1}$ induces a canonical complex structure J on $\mathbf{CP}^n$ through π_*.

3. Lagrangian submanifolds with 1-parameter family of totally geodesic $\mathbf{RP}^{n-1}$ in $\mathbf{CP}^n$

Let $\mathbf{CP}^n$ be the complex projective space with Fubini-Study metric of constant holomorphic sectional curvature 4 as §2. We will construct Lagrangian submanifolds M^n in $\mathbf{CP}^n$ with codimension 1 totally geodesic foliation such that each leaf is a part of totally geodesic $(n-1)$-dimensional real projective space $\mathbf{RP}^{n-1}$, from a curve in

$$\mathscr{M}_n = \{\mathbf{RP}^{n-1} \subset \mathbf{CP}^n\text{: totally geodesic}\}. \tag{2}$$

In [5] we showed that the space of totally geodesic $\mathbf{RP}^n$ in $\mathbf{CP}^n$ is naturally identified with Riemannian symmetric space $SU(n+1)/SO(n+1)$. Since $U(n+1)$ acts on $\mathscr{M}_n$ transitively, $\mathscr{M}_n$ is identified with the homogeneous space $U(n+1)/K$, where

$$K = \left\{ e^{i\theta}\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}; g_1 \in O(n), g_2 \in U(1), \theta \in \mathbf{R} \right\}.$$

We define a bi-invariant Riemannian metric $(\,,)$ on $U(n+1)$ as

$$(A, B) = \mathrm{Re}(\mathrm{trace}\, A^t \bar{B})/4, \quad A, B \in \mathfrak{u}(n+1).$$

Then $U(n+1)$-invariant Riemannian metric g on $\mathscr{M}_n$ is defined naturally such that the projection $\hat{\pi} : U(n+1) \to \mathscr{M}_n$ is a Riemannian submersion.

The Lie algebra $\mathfrak{k}$ of K is written as

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \middle| A \in \mathfrak{o}(n) \right\} \oplus \left\{ \sqrt{-1}\begin{pmatrix} \alpha E_n & 0 \\ 0 & \beta \end{pmatrix} \middle| \alpha, \beta \in \mathbf{R} \right\},$$

where E_n denotes $n \times n$ identity matrix. If we put

$$\mathfrak{p} = \left\{ \begin{pmatrix} \sqrt{-1}B & \mathbf{z} \\ -\mathbf{z}^* & 0 \end{pmatrix} \middle| B \in \mathrm{Sym}(n, \mathbf{R}), \mathrm{trace}\, B = 0, \mathbf{z} \in \mathbf{C}^n \right\},$$

where $\mathrm{Sym}(n, \mathbf{R})$ denotes the set of $n \times n$ real symmetric matrices, then $\mathfrak{u}(n+1) = \mathfrak{k} + \mathfrak{p}$ is a direct sum decomposition of the Lie algebra of $U(n+1)$.

Let $\gamma : I \to \mathscr{M}_n$ be a regular curve and let $g : I \to U(n+1)$ be a lift of γ, where $I \subset \mathbf{R}$ denotes an interval. Then g is horizontal with respect to the Riemannian submersion $\hat{\pi} : U(n+1) \to \mathscr{M}_n$ if and only if for each $t \in I$, $g(t)^{-1}g'(t) \in \mathfrak{p}$. We define a map $\tilde{\Phi} : I \times S^{n-1} \to S^{2n+1} \subset \mathbf{C}^{n+1}$ as

$$\tilde{\Phi}(t, \mathbf{x}) = g(t)\begin{pmatrix} \mathbf{x} \\ 0 \end{pmatrix}, \quad (\mathbf{x} \in S^{n-1} \subset \mathbf{R}^n, 0 \in \mathbf{R}), \tag{3}$$

where g is a horizontal lift of γ. Then $\Phi : I \times \mathbf{RP}^{n-1} \to \mathbf{CP}^n$ is defined by

$$\Phi(t, [\mathbf{x}]) = [\tilde{\Phi}(t, \mathbf{x})] = \left[g(t)\begin{pmatrix} \mathbf{x} \\ 0 \end{pmatrix} \right], \tag{4}$$

where $[\mathbf{x}]$ (resp. $[\tilde{\Phi}(t, \mathbf{x})]$) denotes the image of the projection $S^{n-1} \to \mathbf{RP}^{n-1}$ (resp. Hopf fibration $S^{2n+1} \to \mathbf{CP}^n$). We note that the image of Φ is the union of 1-parameter family of totally geodesic $\mathbf{RP}^{n-1}$ and independent of a choice of horizontal lift $g(t)$ of $\gamma(t)$. The pullback of Maurer-Cartan form on $U(n+1)$ by g is written as

$$g(t)^{-1}g'(t) = \begin{pmatrix} \sqrt{-1}B(t) & \mathbf{z}(t) \\ -\mathbf{z}(t)^* & 0 \end{pmatrix} \in \mathfrak{p}. \tag{5}$$

Then the differential map of $\tilde{\Phi}$ is given by

$$\begin{aligned} d\tilde{\Phi}(\partial/\partial t) &= g'(t)\begin{pmatrix} \mathbf{x} \\ 0 \end{pmatrix} = g(t)\begin{pmatrix} \sqrt{-1}B(t)\mathbf{x} \\ -\mathbf{z}(t)^*\mathbf{x} \end{pmatrix}, \\ d\tilde{\Phi}(X) &= g(t)\begin{pmatrix} X \\ 0 \end{pmatrix}, \quad (X \in T_{\mathbf{x}}S^{n-1}). \end{aligned} \tag{6}$$

The horizontal part $\mathscr{H}X$ of $X \in T_{\mathbf{z}}\mathbf{CP}^n$ with respect to the Hopf fibration $S^{2n+1} \to \mathbf{CP}^n$ is given by $\mathscr{H}X = X - \langle X, \sqrt{-1}\mathbf{z}\rangle\sqrt{-1}\mathbf{z}$. Hence we have

$$\begin{aligned} \mathscr{H}\ d\tilde{\Phi}(\partial/\partial t) &= g(t)\begin{pmatrix} \sqrt{-1}(B(t)\mathbf{x} - \langle B(t)\mathbf{x}, \mathbf{x}\rangle\mathbf{x}) \\ -\mathbf{z}(t)^*\mathbf{x} \end{pmatrix}, \\ \mathscr{H}\ d\tilde{\Phi}(X) &= d\tilde{\Phi}(X). \end{aligned} \tag{7}$$

With respect to the complex structure J on $\mathbf{CP}^n$, Φ is a Lagrangian immersion on the open subset of regular points of Φ if and only if $J\mathscr{H}\ d\tilde{\Phi}(\partial/\partial t) \perp d\tilde{\Phi}(X)$ for any $X \in T_{\mathbf{x}}S^{n-1}$. By (6) and (7), this condition is equivalent to $B(t)\mathbf{x} = \langle B(t)\mathbf{x}, \mathbf{x}\rangle$ for any $\mathbf{x} \in S^{n-1}$. Since $B(t)$ is a symmetric matrix and trace $B(t) = 0$, we see that Φ is a Lagrangian immersion on the open subset of regular points if and only if $B(t) \equiv 0$.

For $\mathbf{RP}^{n-1} \in \mathscr{M}_n$, there exists unique complex projective hyperplane $\mathbf{CP}^{n-1} (\subset \mathbf{CP}^n)$ which contains $\mathbf{RP}^{n-1}$, and we have a Riemannian submersion

$$\tilde{\pi} : \mathscr{M}_n \to \mathbf{CP}^n, \quad \mathbf{RP}^{n-1} \mapsto \mathbf{CP}^{n-1}, \tag{8}$$

where we identify a complex line in $\mathbf{C}^{n+1}$ and its dual complex projective hyperplane in $\mathbf{CP}^n$. If $\gamma(t)$ be a regular curve in $\mathscr{M}_n$ and if $g(t)$ is its horizontal lift to $U(n+1)$, then γ is horizontal with respect to the fibration $\mathscr{M}_n \to \mathbf{CP}^n$ if and only if $B(t) \equiv 0$ in (5). From the above argument, we obtain

Proposition 3.1. *Let $\gamma : I \to \mathscr{M}_n$ be a regular curve and let $g : I \to U(n+1)$ be a horizontal lift with respect to the Riemannian submersion $U(n+1) \to \mathscr{M}_n$. Then the map $\Phi : I \times \mathbf{RP}^{n-1} \to \mathbf{CP}^n$ is a Lagrangian immersion on the subset of regular points if and only if γ is horizontal with respect to the fibration $\tilde{\pi} : \mathscr{M}_n \to \mathbf{CP}^n$.*

4. Results

Let $\gamma(s)$ be a regular curve in $\mathscr{M}_n$ with unit speed and suppose that γ is horizontal with respect to the fibration (8) $\tilde{\pi} : \mathscr{M}_n \to \mathbf{CP}^n$. Then for a horizontal lift $g(s)$ of γ to $U(n+1)$, according to Proposition 3.1 we have

$$g(s)^{-1}g'(s) = \begin{pmatrix} 0 & \mathbf{z}(s) \\ -\mathbf{z}(s)^* & 0 \end{pmatrix} \in \mathfrak{p}, \quad \mathbf{z}(s) \in S^{2n-1} \subset \mathbf{C}^n. \tag{9}$$

In this case the vector tangent to $\tilde{\Phi}$,

$$d\tilde{\Phi}(\partial/\partial s) = -g(s)\begin{pmatrix} 0 \\ \mathbf{z}(s)^*\mathbf{x} \end{pmatrix} \tag{10}$$

in (6) is horizontal with respect to the Hopf fibration $\pi : S^{2n+1} \to \mathbf{CP}^n$. We define a quadratic form $G(s,\cdot)$ on $\mathbf{R}^n$ as

$$G(s,\mathbf{x}) = |\mathbf{z}(s)^*\mathbf{x}|^2 = {}^t\mathbf{x}(\mathrm{Re}(\mathbf{z}(s)\mathbf{z}(s)^*))\mathbf{x}.$$

Then the metric on $I \times \mathbf{RP}^{n-1}$ which is induced by $\Phi : I \times \mathbf{RP}^{n-1} \to \mathbf{CP}^n$ is written as

$$\begin{aligned} \langle \partial/\partial s, \partial/\partial s\rangle &= G(s,\mathbf{x}), \\ \langle \partial/\partial s, X\rangle &= 0, \quad (X \in T_{[\mathbf{x}]}\mathbf{RP}^{n-1}) \end{aligned} \tag{11}$$

and for tangent vectors in $T_{[\mathbf{x}]}\mathbf{RP}^{n-1}$, the induced metric $\Phi^*\langle\ ,\ \rangle$ is same as the standard metric on $\mathbf{RP}^{n-1}$. Hence Φ is regular at $(s,[\mathbf{x}]) \in I \times \mathbf{RP}^{n-1}$ if and only if $G(s,\mathbf{x}) \neq 0$. By (6), (7) and (10), on a regular point $(s,[\mathbf{x}])$ of Φ, the normal space is written by

$$\begin{aligned} T^{\perp}_{\Phi(s,[\mathbf{x}])}(I \times \mathbf{RP}^{n-1}) = {} & \left\{ d\pi\left(g(s)\begin{pmatrix} \sqrt{-1}X \\ 0 \end{pmatrix}\right) \,\middle|\, X \in T_{\mathbf{x}}S^{n-1} \right\} \\ & \oplus \mathbf{R}\sqrt{-1}\, d\pi\left(g(s)\begin{pmatrix} 0 \\ \mathbf{z}(s)^*\mathbf{x} \end{pmatrix}\right). \end{aligned}$$

Let σ be the second fundamental tensor of the Lagrangian immersion Φ on the open subset of regular points in $I \times \mathbf{RP}^{n-1}$. Since $\mathbf{RP}^{n-1}$ is totally geodesic in $\mathbf{CP}^n$, we have

$$\sigma(X,Y) = 0 \quad \text{for } X, Y \in T_{[\mathbf{x}]}\mathbf{RP}^{n-1}. \tag{12}$$

By (9) and (10), we obtain

$$\begin{aligned} D_{d\tilde{\Phi}(\partial/\partial s)}\, d\tilde{\Phi}(\partial/\partial s) &= -g'(s)\begin{pmatrix} 0 \\ \mathbf{z}(s)^*\mathbf{x} \end{pmatrix} - g(s)\begin{pmatrix} 0 \\ \mathbf{z}'(s)^*\mathbf{x} \end{pmatrix} \\ &= -g(s)\begin{pmatrix} \mathbf{z}(s)\mathbf{z}(s)^*\mathbf{x} \\ \mathbf{z}'(s)^*\mathbf{x} \end{pmatrix}, \end{aligned} \tag{13}$$

where D denotes the Euclidean covariant differentiation on $\mathbf{C}^{n+1}$. Also (3) implies that

$$D_{d\tilde{\Phi}(\partial/\partial s)}\, d\tilde{\Phi}(\partial/\partial s) \perp \sqrt{-1}\tilde{\Phi}(s,\mathbf{x})$$

and is horizontal with respect to the Hopf fibration $S^{2n+1} \to \mathbf{CP}^n$. By taking the normal component of (13), we get

$$\sigma(\partial/\partial s, \partial/\partial s) = d\pi\left(\sqrt{-1}g(s)\begin{pmatrix} -\mathrm{Im}(\mathbf{z}(s)\mathbf{z}(s)^*)\mathbf{x} \\ {}^t\mathbf{x}\,\mathrm{Im}(\mathbf{z}'(s)\mathbf{z}(s))\mathbf{x}/{}^t\mathbf{z}(s)\mathbf{x} \end{pmatrix}\right). \tag{14}$$

THEOREM 4.1. *Let $\gamma : I \to \mathscr{M}_n$ be a regular curve and suppose that γ is horizontal with respect to the fibration $\tilde{\pi} : \mathscr{M}_n \to \mathbf{CP}^n$. If the corresponding map $\Phi : I \times \mathbf{RP}^{n-1} \to \mathbf{CP}^n$ is a minimal Lagrangian immersion on the regular points, then Φ is totally geodesic.*

Proof. By (11), (12) and (14), $\Phi : I \times \mathbf{RP}^{n-1} \to \mathbf{CP}^n$ is a minimal immersion on the regular points if and only if $\operatorname{Im}(\mathbf{z}(s)\mathbf{z}(s)^*) = 0$ and ${}^t\mathbf{x}\operatorname{Im}(\mathbf{z}'(s)\mathbf{z}(s)^*)\mathbf{x} = 0$ hold for any $\mathbf{x} \in S^{n-1}$. The former equation yields that $\mathbf{z}(s) = e^{\sqrt{-1}\theta(s)}\mathbf{y}(s)$ for some $\theta : I \to S^1$ and $\mathbf{y} : I \to S^{n-1} \subset \mathbf{R}^n$. Then $\operatorname{Im}(\mathbf{z}'(s)\mathbf{z}(s)^*) = \theta'(s)\mathbf{y}(s){}^t\mathbf{y}(s)$ and latter equation implies that $\theta(s)$ is constant. By (3), we can see that $\Phi(I \times \mathbf{RP}^{n-1}) \subset \mathbf{RP}^n$ and Φ is totally geodesic. □

Next to study the condition for which $\Phi : I \times \mathbf{RP}^{n-1} \to \mathbf{CP}^n$ is Hamiltonian minimal, we will calculate div JH in terms of (1). By (11) and (14), the mean curvature vector of Φ is $H = G(s,\mathbf{x})^{-1}\sigma(\partial/\partial s, \partial/\partial s)$ and the tangent vector field JH along Φ is written as

$$JH = \frac{1}{G(s,\mathbf{x})}\, d\pi\left(g(s)\begin{pmatrix} \operatorname{Im}(\mathbf{z}(s)\mathbf{z}(s)^*)\mathbf{x} \\ -{}^t\mathbf{x}\operatorname{Im}(\mathbf{z}'(s)\mathbf{z}(s))\mathbf{x}/{}^t\mathbf{z}(s)\mathbf{x}\end{pmatrix}\right). \tag{15}$$

For a real $n \times n$ matrix A, we denote a quadratic form on $\mathbf{R}^n$ as

$$Q(A,\mathbf{x}) = {}^t\mathbf{x}A\mathbf{x}. \tag{16}$$

Then by using (13), we get

$$\begin{aligned}\langle \nabla_{\partial/\partial s}(JH), \partial/\partial s\rangle &= -\langle D_{d\tilde{\Phi}(\partial/\partial s)}\, d\tilde{\Phi}(\partial/\partial s), d\tilde{\Phi}(JH)\rangle \\ &= G(s,\mathbf{x})^{-1}\{Q(\operatorname{Im}(\mathbf{z}''(s)\mathbf{z}(s)^*),\mathbf{x}) + Q(\operatorname{Im}(\mathbf{z}'(s)\mathbf{z}'(s)^*),\mathbf{x}) \\ &\quad + Q(\operatorname{Re}(\mathbf{z}(s)\mathbf{z}(s)^*),\mathbf{x})Q(\operatorname{Im}(\mathbf{z}(s)\mathbf{z}(s)^*),\mathbf{x})\} \\ &\quad - 3G(s,\mathbf{x})^{-2}Q(\operatorname{Re}(\mathbf{z}'(s)\mathbf{z}(s)^*),\mathbf{x})Q(\operatorname{Im}(\mathbf{z}'(s)\mathbf{z}(s)^*),\mathbf{x}),\end{aligned}$$

where ∇ denotes the Levi-Civita connection on $I \times \mathbf{RP}^{n-1}$ induced by $\Phi : I \times \mathbf{RP}^{n-1} \to \mathbf{CP}^n$. For $X \in T_{[\mathbf{x}]}\mathbf{RP}^{n-1}$, we obtain

$$\begin{aligned}\langle \nabla_X(JH), X\rangle &= -2G(s,\mathbf{x})^{-2}\{{}^t\mathbf{x}\operatorname{Re}(\mathbf{z}(s)\mathbf{z}(s)^*)X\,{}^tX\operatorname{Im}(\mathbf{z}(s)\mathbf{z}(s)^*)\mathbf{x}\} \\ &\quad + G(s,\mathbf{x})^{-1}\,{}^tX\operatorname{Im}(\mathbf{z}(s)\mathbf{z}(s)^*)X.\end{aligned}$$

Hence we obtain

$$\begin{aligned}\operatorname{div}(JH) &= G(s,\mathbf{x})^{-2}\{Q(\operatorname{Im}(\mathbf{z}''(s)\mathbf{z}(s)^*),\mathbf{x}) - \frac{1}{2}Q(\operatorname{Im}(\overline{\mathbf{z}(s)}\,{}^t\mathbf{z}(s)\mathbf{z}(s)\mathbf{z}(s)^*),\mathbf{x})\} \\ &\quad - 3G(s,\mathbf{x})^{-3}Q(\operatorname{Re}(\mathbf{z}'(s)\mathbf{z}(s)^*),\mathbf{x})Q(\operatorname{Im}(\mathbf{z}'(s)\mathbf{z}(s)^*),\mathbf{x}).\end{aligned} \tag{17}$$

We consider the case $n = 2$. Then the curve $\mathbf{z}(s)$ in S^3 given by (9) and a vector $\mathbf{x}$ in S^1 given by (3) are respectively written as

$$\mathbf{z}(s) = \begin{pmatrix} z_1(s) \\ z_2(s) \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix},$$

for some $\theta \in \mathbf{R}/2\pi\mathbf{Z}$. Then by reducing (17) to a common denominator, we see that $\operatorname{div} JH = 0$ is equivalent to a homogeneous algebraic equation of order 4 with variables $\cos\theta$ and $\sin\theta$ whose coefficients are independent to s. Hence

Proposition 4.1. *Let $\gamma : I \to \mathcal{M}_2$ be a regular curve and suppose that γ is horizontal with respect to the fibration $\tilde{\pi} : \mathcal{M}_2 \to \mathbf{CP}^2$. Then on the regular points, the first variational formula $\delta\alpha_H = 0$ of the corresponding Lagrangian immersion $\Phi : I \times \mathbf{RP}^1 \to \mathbf{CP}^2$ with respect to Hamiltonian deformations is written as a system of 5 ODE's of second order for curves $\mathbf{z}(s)$ in S^3.*

Let $\gamma : I \to \mathcal{M}_n$ be a unit speed *horizontal* curve with respect to $\mathcal{M}_n \to \mathbf{CP}^n$. And let $g(s)$ be a horizontal lift of $\gamma(s)$ to $U(n+1)$. Then $\gamma(s)$ is an orbit of a 1-parameter subgroup of $U(n+1)$ if and only if the vector valued function $\mathbf{z} : I \to S^{2n-1} \subset \mathbf{C}^n$ given by (9) is constant. In this case, for $\mathbf{z} \equiv \mathbf{z}(s)$ we have

$$g(s) = \exp s \begin{pmatrix} 0 & \mathbf{z} \\ -\mathbf{z}^* & 0 \end{pmatrix} \tag{18}$$

and (17) is written as

$$-2 \operatorname{div}(JH) = G(s, \mathbf{x})^{-2} Q(\operatorname{Im}(\bar{\mathbf{z}}^t\mathbf{z}\mathbf{z}\mathbf{z}^*), \mathbf{x}).$$

Now we determine Lagrangian submanifolds given by 1-parameter family of totally geodesic $\mathbf{RP}^{n-1}$ in $\mathbf{CP}^n$ satisfying the first variational formula $\delta\alpha_H = 0$, in the case that the corresponding curve γ in $\mathcal{M}_n$ is an orbit of 1-parameter subgroup of $U(n+1)$.

Theorem 4.2. *For $\mathbf{z} \in S^{2n-1} \subset \mathbf{C}^n$, let $g(s) = \exp s \begin{pmatrix} 0 & \mathbf{z} \\ -\mathbf{z}^* & 0 \end{pmatrix}$ be a 1-parameter subgroup of $U(n+1)$, and let $\gamma(s)$ be an orbit of $g(s)$ in $\mathcal{M}_n$. Then the corresponding Lagrangian immersion $\Phi : I \times \mathbf{RP}^{n-1} \to \mathbf{CP}^n$ is Hamiltonian minimal if and only if $\mathbf{z}$ satisfies one of the following conditions*:

(i) *There exists $\mathbf{x} \in S^{n-1} \subset \mathbf{R}^n$ and $\theta \in \mathbf{R}$ such that $\mathbf{z} = e^{\sqrt{-1}\theta}\mathbf{x}$. In this case, $\Phi(I \times \mathbf{RP}^{n-1}) \subset \mathbf{RP}^n$ and Φ is totally geodesic.*

(ii) *$\mathbf{z}$ is an isotopic vector, i.e., ${}^t\mathbf{z}\mathbf{z} = 0$.*

In fact, $\operatorname{Im}(\bar{\mathbf{z}}^t\mathbf{z}\mathbf{z}\mathbf{z}^*) = 0$ implies that either (i) $\operatorname{Re}\mathbf{z}$ and $\operatorname{Im}\mathbf{z}$ are linearly dependent, or (ii) $|\operatorname{Re}\mathbf{z}| = |\operatorname{Im}\mathbf{z}|$ and $\operatorname{Re}\mathbf{z} \perp \operatorname{Im}\mathbf{z}$.

In (ii) of Theorem 4.2, when $n \geq 3$, Φ always has some singularities and when $n = 2$, Φ is everywhere regular and the Lagrangian surface $\Phi(I \times \mathbf{RP}^1)$ has the following properties: (a) flat, i.e., the Gauss curvature $K = 0$, (b) the mean curvature vector field H is parallel with respect to the normal connection, and $H \neq 0$. Ogata (Chapter 5 in [7]) proved: (i) Let $M^2[K]$ be an oriented 2-

dimensional Riemannian manifold of constant Gaussian curvature K and let $x : M^2[K] \to \mathbf{CP}^2$ be an isometric immersion such that the mean curvature vector field H is parallel and not zero. Then x is Lagrangian and $K = 0$. (ii) Let $x : \mathbf{R}^2 \to \mathbf{CP}^2$ be an isometric immersion with non-zero parallel mean curvature vector field H. Then $x(\mathbf{R}^2)$ is an orbit of the Abelian Lie subgroup G of $U(3)$. So Hamiltonian minimal Lagrangian surfaces in $\mathbf{CP}^2$ obtained by (ii) of Theorem 4.2 are included in the examples that were given by T. Ogata's paper.

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