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SIMONS-TYPE INEQUALITIES FOR THE COMPACT SUBMANIFOLDS IN THE SPACE OF CONSTANT CURVATURE*

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Abstract

For the compact submanifold M immersed in the standard Euclidean sphere S^{n+p} or the Euclidean space R^{n+p}, we obtain Simons-type inequalities about the first eigenvalue λ_1 and the squared norm of the second fundamental form S respectively. In particular, for the case of the ambient space is S^{n+p}, we need not the assumption that M is minimal. Following which, we obtain the estimate about the lower bound for S if it is constant respectively.

1. Introduction and main results

In this paper we shall be concerned with the Simons-type inequalities about the first eigenvalue λ_1 and the squared norm of the second fundamental form S for the compact submanifold immersed in the space with nonnegative constant curvature. Further, we obtain a lower bound for S provided that S is a constant. For the ambient space being the Euclidean space or the standard Euclidean sphere, we state as follows respectively.

For the compact orientable submanifold M immersed in the standard Euclidean sphere S^{n+p}, if M is minimal, we know the famous Simons inequalities ([12]): $\int_M S[(2-1/p)S-n]\,dV_M \geq 0$. If $0 \leq S \leq \dfrac{n}{2-\dfrac{1}{p}}$, then $S=0$ (M is totally geodesic) or $S = \dfrac{n}{2-\dfrac{1}{p}}$. Later, Chern, do Carmo and Kobayashi [3] further showed that the Veronese surface in S^4 and the submanifold $S^m\left(\sqrt{\dfrac{m}{n}}\right) \times S^{n-m}\left(\sqrt{\dfrac{n-m}{n}}\right)$ in S^{n+1} are the only compact minimal submanifolds of dimension n in S^{n+p} satisfying $S = \dfrac{n}{2-\dfrac{1}{p}}$.

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Simons inequalities and its corollary make people try to improve the estimate of the upper bound for S and study the rigidity of the associated submanifolds. For the well-known results about the above questions, we refer to [11] and [7]. Towards the other direction, there is a natural question about the distribution of values for S. That is, for the compact minimal submanifold M immersed in S^{n+p}, if the squared norm of the second fundamental form S of M is a constant, then whether the set of values for S is discrete? What is the next value for S? These questions have not been resolved completely (c.f. [9] or [13]). The known result due to Leung ([5]) showed that $S \geq n - \lambda_1$, where λ_1 stands for the first non-zero eigenvalue of the Laplacian operator $\triangle$ on M. Recently, Barbosa and Barros [1] improved Leung's gap for compact minimal hypersurface $M \subset S^{n+1}$ by showing that there is a rational constant $k \in \left[\frac{n}{n-1}, n\right]$ depending either on h or on the first eigenfunction of $\triangle$ such that $S \geq k\frac{n-1}{n}(n - \lambda_1)$.

In this paper, without assuming that M is minimal, we obtain a Simons-type integral inequality concerning the squared norm of the second fundamental form S of the submanifold M immersed in the standard Euclidean sphere S^{n+p} and the first non-zero eigenvalue λ_1 of the Laplacian operator $\triangle$ on M and the gradient of the associated eigenfunction f. Further, under the assumption that S is a constant, we obtain a lower bound for S. It should be noted that our results need not provide that M is minimal and extends the results of [1] to the higher codimension. Our proof still make use of Bochner formula similar to [1], but we use different method to estimate the Ricci curvature of submanifold M, it is in this process that we drop the assumption for M to be minimal. Now we will announce our result according to the next theorem.

THEOREM 1.1. *Let M be an n-dimensional compact orientable submanifold immersed in the standard Euclidean sphere S^{n+p}. Let f be an eigenfunction associated to λ_1, then*

$$\int_M \left[\frac{\sqrt{n-1}}{2}S - \frac{(n-1)(n-\lambda_1)}{n}\right]|\nabla f|^2 \, dV_M \geq 0.$$

In particular, if S is a constant, then $S \geq \dfrac{2\sqrt{n-1}(n-\lambda_1)}{n}$.

Coming into the case of the ambient space being the Euclidean space R^{n+p}, denote also the squared norm of the second fundamental form of M by S, the Laplacian operator on the functions space $C^\infty(M)$ by $\triangle$, the associated first non-zero eigenvalue by λ_1. Reilly [10] obtained a Simons-type integral inequality concerning S and λ_1: $\int_M (S - \lambda_1)\, \mathrm{d}V_M \geq 0$, using Newton inequality and Hsiung-Minkowski formula. In this paper we give a very concise proof with the spectral resolution. The result is re-stated as follows:

THEOREM 1.2. *Let $x : M \to R^{n+p}$ be an isometric immersion. Denote by S, λ_1 the squared norm of the second fundamental form and the first non-zero eigenvalue respectively, then we have*

$$\int_M (S - \lambda_1)\, dV_M \geq 0.$$

In particular, if S is a constant, then $S \geq \lambda_1$.

2. Proof of Theorem 1.1

We recall now the Bochner formula (c.f. [13]), which states that for any differentiable function $f : M \to \mathbf{R}$,

$$\frac{1}{2}\triangle(|\nabla f|^2) = \mathrm{Ric}(\nabla f, \nabla f) + \langle \nabla f, \nabla(\triangle f)\rangle + |\mathrm{Hess}\, f|^2, \tag{2.1}$$

where Ric denote the Ricci tensor of M, and for any smooth tangent vector fields X, Y,

$$\langle \nabla f, X\rangle = X(f), \quad \mathrm{Hess}\, f(X, Y) = \langle \nabla_X(\nabla f), Y\rangle, \quad \triangle f = \mathrm{tr}(\mathrm{Hess}\, f).$$

For a bilinear form T, the norm of T considered here is the Euclidean, which is given by $|T|^2 = \mathrm{tr}(TT^t)$.

Since M is compact and $\triangle f + \lambda_1 f = 0$, integrating (2.1) on M we get

$$\int_M \mathrm{Ric}(\nabla f, \nabla f)\, \mathrm{d}V_M + \int_M |\mathrm{Hess}\, f|^2\, \mathrm{d}V_M - \lambda_1 \int_M |\nabla f|^2\, \mathrm{d}V_M = 0. \tag{2.2}$$

In the following, we will estimate the first and the second parts on the left hand side of (2.2) respectively.

Considering the Hessian part, let I denotes the identity operator on the tangent bundle TM of M, for any $t \in \mathbf{R}$, we have

$$|\mathrm{Hess}\, f - tfI|^2 = |\mathrm{Hess}\, f|^2 - 2tf\triangle f + nt^2 f^2.$$

Then

$$\int_M |\mathrm{Hess}\, f - tfI|^2\, \mathrm{d}V_M = \int_M |\mathrm{Hess}\, f|^2\, \mathrm{d}V_M + \left(2t + \frac{n}{\lambda_1}t^2\right)\int_M |\nabla f|^2\, \mathrm{d}V_M. \tag{2.3}$$

In particular, putting $t = -\dfrac{\lambda_1}{n}$ into (2.3), we get

$$\begin{aligned} \int_M |\mathrm{Hess}\, f|^2\, \mathrm{d}V_M &= \int_M \left|\mathrm{Hess}\, f + \frac{\lambda_1}{n} fI\right|^2 \mathrm{d}V_M + \frac{\lambda_1}{n}\int_M |\nabla f|^2\, \mathrm{d}V_M \\ &\geq \frac{\lambda_1}{n}\int_M |\nabla f|^2\, \mathrm{d}V_M. \end{aligned} \tag{2.4}$$

Moreover, the equality holds if and only if M is isometric to the sphere $S^n\left(\sqrt{\frac{\lambda_1}{n}}\right)$ (c.f. [8]).

In order to estimate the Ricci curvature part, we recall a main theorem in [6]: Let M be an n-dimensional immersed submanifold in $(n+p)$-dimensional Riemannian manifold N. Let Ric, S and H denote the functions that assign to each point of M the minimum Ricci curvature, the square length of the second fundamental form, and the mean curvature of M respectively. If all the sectional curvatures of N are bounded below by a constant c, then

$$\mathrm{Ric} \geq \frac{n-1}{n}\left\{nc + nH^2 - \|\varphi\|^2 - \frac{n-2}{\sqrt{n-1}}\sqrt{nH^2}\|\varphi\|\right\},$$

where $\|\varphi\|^2 = S - nH^2$. Therefore, when the ambient space is the standard Euclidean sphere S^{n+p} we get

$$\mathrm{Ric} \geq \frac{n-1}{n}\left\{n + nH^2 - \|\varphi\|^2 - \frac{n-2}{\sqrt{n-1}}\sqrt{nH^2}\|\varphi\|\right\}.$$

Let us consider the following quadratic form with eigenvalues $\pm\frac{n}{2\sqrt{n-1}}$:

$$F(x,y) = x^2 - \frac{n-2}{\sqrt{n-1}}xy - y^2.$$

By using an orthogonal transformation

$$\left\{\begin{aligned} u &= \frac{1}{\sqrt{2n}}[(1+\sqrt{n-1})x + (1-\sqrt{n-1})y], \\ v &= \frac{1}{\sqrt{2n}}[-(1-\sqrt{n-1})x + (1+\sqrt{n-1})y], \end{aligned}\right. \tag{2.5}$$

we get $F(x,y) = F(x(u,v), y(u,v)) = \frac{n}{2\sqrt{n-1}}(u^2 - v^2)$.

Let $x = \sqrt{nH^2}$, $y = \|\varphi\|$, then $x^2 + y^2 = S$. It follows from (2.5) that $x^2 + y^2 = u^2 + v^2$. Therefore

$$\begin{aligned} F(x,y) &= \frac{n}{2\sqrt{n-1}}(u^2 - v^2) = -\frac{n}{2\sqrt{n-1}}(v^2 + u^2 - 2u^2) \\ &\geq -\frac{n}{2\sqrt{n-1}}(v^2 + u^2) \\ &= -\frac{n}{2\sqrt{n-1}}S. \end{aligned}$$

Hence, we have $\mathrm{Ric} \geq (n-1) - \frac{\sqrt{n-1}}{2}S$. Furthermore, we obtain

$$\int_M \mathrm{Ric}(\nabla f, \nabla f)\, \mathrm{d}V_M \geq \int_M \left[(n-1) - \frac{\sqrt{n-1}}{2} S\right] |\nabla f|^2 \, \mathrm{d}V_M. \tag{2.6}$$

Substituting (2.4) and (2.6) into (2.2), we have

$$0 \geq \int_M \left[\frac{\lambda_1}{n} + (n-1) - \frac{\sqrt{n-1}}{2} S - \lambda_1\right] |\nabla f|^2 \, \mathrm{d}V_M,$$

that is

$$\int_M \left[\frac{\sqrt{n-1}}{2} S - \frac{(n-1)(n-\lambda_1)}{n}\right] |\nabla f|^2 \, \mathrm{d}V_M \geq 0,$$

which completes the proof of Theorem 1.1.

3. Proof of Theorem 1.2

Before proving the theorem 1.2, we need some necessary preliminaries. We shall make use of the following convention on the ranges of indices:

$$1 \leq A, B, C \cdots \leq n+p; \quad 1 \leq i, j, k \cdots \leq n; \quad n+1 \leq \alpha, \beta, \gamma \cdots \leq n+p.$$

We choose a local field of orthonormal frames $\{e_i, e_\alpha\}$ in R^{n+p} such that, restricted to M, the vectors $\{e_i\}$ are tangent to M and the remaining vectors $\{e_\alpha\}$ are normal to M. Let A_α denote the shape operator in the direction e_α, $h_{ij}^\alpha = \langle A_\alpha e_i, e_j \rangle$. Let $\{\omega_A\}$ be the field of dual frames. For R^{n+p}, we have

$$\mathrm{d}x = \sum_A \omega_A e_A, \quad \mathrm{d}e_A = \sum_B \omega_{AB} e_B.$$

Restricted to M, we get

$$\begin{cases} \mathrm{d}x = \sum_i \omega_i e_i, \\ \mathrm{d}e_i = \sum_j \omega_{ij} e_j + \sum_\alpha \omega_{i\alpha} e_\alpha, \quad \omega_{i\alpha} = \sum_j h_{ij}^\alpha \omega_j. \end{cases}$$

Let $\boldsymbol{a}$ be a fixed vector in R^{n+p}, for isometric immersion $x : M \to R^{n+p}$, we can define the height function about $\boldsymbol{a}$ on M, $g(q) := \langle \boldsymbol{a}, x(q) \rangle$, $q \in M$, then

$$\mathrm{d}g = \sum_i \langle \boldsymbol{a}, e_i \rangle \omega_i := \sum_i g_i \omega_i.$$

Since

$$\sum_j g_{ij} \omega_j = \mathrm{d}g_i + \sum_j g_j \omega_{ji} = \sum_{\alpha, j} \langle \boldsymbol{a}, e_\alpha \rangle h_{ij}^\alpha \omega_j,$$

we have $g_{ij} = \sum_\alpha h_{ij}^\alpha \langle \boldsymbol{a}, e_\alpha \rangle$. Therefore

$$\triangle g = \sum_i g_{ii} = \sum_{\alpha, i} h_{ii}^\alpha \langle \boldsymbol{a}, e_\alpha \rangle = n \langle \boldsymbol{a}, \vec{H} \rangle. \tag{3.1}$$

In addition, we denote the Laplacian operator on the functions space $C^\infty(M)$ by $\triangle$. Then $\triangle$ has discrete eigenvalues (c.f. [13]): $0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \lambda_k \leq \cdots \to \infty$. For any $f \in C^\infty(M)$, denoted by f_t the projection of f onto the eigen-space $V_t = \{f \in C^\infty(M) \mid \triangle f = -\lambda_t f\}$, then we have the L^2-spectral resolution,

$$f = f_0 + \sum_{t \geq 1} f_t, \quad \triangle f_t = -\lambda_t f_t.$$

Proof of Theorem 1.2. Let $x : M \to R^{n+p}$ be isometric immersion, considering the spectral resolution,

$$x = x_0 + \sum_{t \geq 1} x_t, \quad \triangle x_t = -\lambda_t x_t.$$

Noticing that for $t \neq s$, $\int_M \langle x_t, x_s \rangle \, \mathrm{d}V_M = 0$, and let $b_t = \int_M \langle x_t, x_t \rangle \, \mathrm{d}V_M$, then

$$\int_M \langle \triangle x, x \rangle \, \mathrm{d}V_M = -\sum_{t \geq 1} \lambda_t b_t, \tag{3.2}$$

$$\int_M \langle \triangle x, \triangle x \rangle \, \mathrm{d}V_M = \sum_{t \geq 1} \lambda_t^2 b_t. \tag{3.3}$$

Combining (3.2) and (3.3), we obtain

$$\int_M \langle \triangle x, \triangle x \rangle \, \mathrm{d}V_M + \lambda_1 \int_M \langle \triangle x, x \rangle \, \mathrm{d}V_M = \sum_{t \geq 1} (\lambda_t - \lambda_1) \lambda_t b_t \geq 0. \tag{3.4}$$

On the other hand, (3.1) leads to $\triangle x = n\vec{H}$, so

$$\int_M \langle \triangle x, \triangle x \rangle \, \mathrm{d}V_M = n^2 \int_M H^2 \, \mathrm{d}V_M. \tag{3.5}$$

For the isometric immersion $x : M \to R^{n+p}$, we also know $\int_M (1 + \langle x, \vec{H} \rangle) \, \mathrm{d}V_M = 0$ (c.f. [2]) and

$$\int_M \langle \triangle x, x \rangle \, \mathrm{d}V_M = \int_M \langle n\vec{H}, x \rangle \, \mathrm{d}V_M = -n \int_M \mathrm{d}V_M. \tag{3.6}$$

Therefore, using (3.4), (3.5), (3.6) and the well-known inequality $S \geq nH^2$ we get

$$
\begin{aligned}
0 &\leq \int_M \langle \triangle x, \triangle x \rangle \, \mathrm{d}V_M + \lambda_1 \int_M \langle \triangle x, x \rangle \, \mathrm{d}V_M \\
&= n^2 \int_M H^2 \, \mathrm{d}V_M - n\lambda_1 \int_M \mathrm{d}V_M \\
&\leq \int_M nS \, \mathrm{d}V_M - n\lambda_1 \int_M \mathrm{d}V_M \\
&= n \int_M (S - \lambda_1) \, \mathrm{d}V_M,
\end{aligned}
$$

which completes the proof of Theorem 1.2.

Final remarks. For the n-dimensional isometric immersed submanifold M in the hyperbolic space $H^{n+p}(-1)$, EI Soufi and Ilias ([4]) showed that $\lambda_1 V(M) \leq n \int_M (H^2 - 1) \, \mathrm{d}V_M$, where $V(M)$ stands for the volume of M. Following which, we immediately obtain that $\int_M (S - n - \lambda_1) \, \mathrm{d}V_M \geq 0$, and if S is a constant, then $S \geq n + \lambda_1$.

In addition, for the submanifolds in the Euclidean space and the hyperbolic space, we can still make use of the estimate of Ricci curvature in the proof of Theorem 1.1. But we can not obtain the similar Simons-type inequalities. Also the method in the proof of Theorem 1.2 can not be used to deal with the submanifolds in the sphere and the hyperbolic space. This is the reason why we prove theorem 1.1 and 1.2 in different way.

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