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DIFFEOMORPHISMS ADMITTING SRB MEASURES AND THEIR REGULARITY

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Abstract

We are interested in the stochastic property of some "Anosov-like" system. In this paper we will treat a transitive and partially hyperbolic diffeomorphism f of a 2-dimensional torus with uniformly contracting direction, and show that if f is of C^2 and admits an SRB measure, then f is an Anosov diffeomorphism. In our proof we use the Pujals-Sambarino theorem for C^2 diffeomorphisms with dominated splitting. In the case of $C^{1+\alpha}$ the above statement is not true in general, i.e. we can construct a $C^{1+\alpha}$ counter example of Maneville-Pomeau type.

1. Introduction

We know that if f is a C^2-Anosov diffeomorphism of a compact manifold M, then f admits an *Sinai-Ruelle-Bowen measure* μ (or *SRB measure*), i.e., μ has absolutely continuous conditional measures on unstable manifolds (Sinai [23]). This measure μ is isomorphic to a Berunulli shift, and it has exponential decay of correlations for Hölder continuous functions, and furthermore satisfies that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(f^j x) = \int \varphi \, d\mu$$

for any continuous function $\varphi : M \to \mathbf{R}$ and for Lebesgue almost every $x \in M$. This result has been extended to Axiom A attractors by Bowen and Ruelle (e.g. [4]).

Let f be a $C^{1+\alpha}$-diffeomorphism of a 2-dimensional torus $\mathbf{T}^2$ $(0 < \alpha \le 1)$ and Γ be an f-invariant set of $\mathbf{T}^2$, i.e. $f(\Gamma) = \Gamma$. We say that f is *partially hyperbolic with contracting direction* on Γ if there exist a norm $\|\cdot\|$ on $\mathbf{T}^2$ and $0 < \lambda_1 < \lambda_2$ with $\lambda_1 < 1$ so that each $x \in \mathbf{T}^2$ has a $D_x f$-invariant decomposition $\mathrm{T}_x\mathbf{T}^2 = E_1(x) \oplus E_2(x)$ of subspaces $E_1(x)$ and $E_2(x)$ such that

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$$\|D_x f|_{E_1(x)}\| \le \lambda_1, \quad \|D_x f|_{E_2(x)}\| \ge \lambda_2 \tag{1.1}$$

(Here $D_x f$ denotes the derivative of f at x). Moreover, when $\lambda_2 > 1$, f is called *hyperbolic* on Γ (or Γ is a *hyperbolic set* for f). We call that f is an *Anosov diffeomorphism* if f is hyperbolic on the entire space $\mathbf{T}^2$. f is said to be *topologically transitive* if there exists a point $x \in \mathbf{T}^2$ such that its orbit $\{f^n(x)\}_{n \in \mathbf{Z}}$ is dense in $\mathbf{T}^2$.

The purpose of this paper is to show the following two theorems:

Theorem A. *Let f be a C^2-diffeomorphism of $\mathbf{T}^2$. Then f is an Anosov diffeomorphism if and only if the following holds:*
(1) *f is partially hyperbolic with contracting direction on $\mathbf{T}^2$,*
(2) *f is topologically transitive and*
(3) *f admits an SRB measure.*

Theorem A is false in the case when f is of $C^{1+\alpha}$ $(0 < \alpha < 1)$, i.e.

Theorem B. *For $\alpha \in (0,1)$ there exists a $C^{1+\alpha}$-diffeomorphism f such that*
(1) *f is partially hyperbolic with contracting direction on $\mathbf{T}^2$ but not an Anosov diffeomorphism,*
(2) *f is topologically transitive and*
(3) *f admits an SRB measure.*

2. Definitions and preliminaries

Fix $\alpha \in (0,1]$ and let f be a $C^{1+\alpha}$-diffeomorphism of $\mathbf{T}^2$. Assume that f is partially hyperbolic and has contracting direction. Then f has the decomposition satisfying (1.1). Let μ be an f-invariant probability measure on $\mathbf{T}^2$. By Birkhoff's ergodic theorem there exist a set Y_μ with full μ-measure and real numbers $\chi_1(x) < \chi_2(x)$ $(x \in Y_\mu)$ which satisfy the following:

$$\lim_{n \to \pm\infty} \frac{1}{n} \log\|D_x f^n|_{E_i(x)}\| = \chi_i(x) \quad (i = 1, 2).$$

We call $\chi_i(x)$ $(i = 1, 2)$ the *Lyapunov exponents* of μ at $x \in Y_\mu$. By (1.1) we have

$$\chi_1(x) \le \log \lambda_1 < \log \lambda_2 \le \chi_2(x) \quad (x \in Y_\mu).$$

We say that μ is an *SRB measure* if (i) $\chi_2(x) > 0$ and (ii) μ has the conditional measures which are absolutely continuous w.r.t. the Lebesgue measures on unstable manifolds, which is defined as follows:

If $\chi_2(x) > 0$ for any $x \in Y_\mu$, then it is well known (see [16]) that there exists $\varepsilon_0 > 0$ sufficiently small and the *local unstable manifold* $W^u_{loc}(x)$ such that

$$f^{-1}(W^u_{loc}(x)) \subset W^u_{loc}(f^{-1}(x)),$$

for any $y \in W^u_{loc}(x)$ and $n \ge 0$

$$d^u(f^{-n}(x), f^{-n}(y)) \leq C(x) \exp((-\chi_2(x) + \varepsilon_0)n) d^u(x, y) \tag{2.1}$$

and $E_2(x) = T_x W^u_{loc}(x)$, where d^u denotes the Riemannian metric on $W^u_{loc}(x)$ and $C(x)$ satisfies

$$\lim_{n \to \pm\infty} \frac{1}{n} \log C(f^n(x)) = 0. \tag{2.2}$$

The *unstable manifold* $W^u(x)$ is defined by

$$W^u(x) = \bigcup_{n \geq 0} f^n(W^u_{loc}(f^{-n}(x))).$$

Since $\chi_1(x) < 0$ for any $x \in Y_\mu$, the *local stable manifold* $W^s_{loc}(x)$ exists ([16]) and satisfies

$$f(W^s_{loc}(x)) \subset W^s_{loc}(f(x)),$$

for any $y \in W^s_{loc}(x)$ and $n \geq 0$

$$d^s(f^n(x), f^n(y)) \leq D(x) \exp((\chi_1(x) + \varepsilon_0)n) d^s(x, y) \tag{2.3}$$

and $E_1(x) = T_x W^s_{loc}(x)$, where d^s denotes the Riemannian metric on $W^s_{loc}(x)$ and $D(x)$ satisfies $\lim_{n \to \pm\infty} (1/n) \log D(f^n(x)) = 0$. The *stable manifold* $W^s(x)$ is also defined.

Let $\mathscr{B}$ denote the Borel σ algebra of $\mathbf{T}^2$. For any measurable partition ξ of $\mathbf{T}^2$ we denote by $\mathscr{B}_\xi$ the set of all Borel subsets which consist of the unions of the elements of ξ. A measurable partition ξ of $\mathbf{T}^2$ defines a family of measures $\{\mu^\xi_x\}$ (μ-a.e.x) such that for μ-a.e.x and $B \in \mathscr{B}$, $\mu^\xi_x(B)$ is a $\mathscr{B}_\xi$-measurable function of x and

$$\mu(E \cap B) = \int_E \mu^\xi_x(B)\, d\mu(x) \quad (E \in \mathscr{B}_\xi).$$

If there exists a sequence $\{\xi_i\}_{i \geq 1}$ of countable measurable partitions such that

$$\xi_1 \leq \xi_2 \leq \cdots \leq \bigvee_{i \geq 1} \xi_i = \xi,$$

then $\mu^\xi_x(\xi(x)) = 1$ where $\xi(x)$ denotes an element of ξ containing x. The family of measures $\{\mu^\xi_x\}$ (μ-a.e.x) is said to be the *canonical system* of *conditional measures* of μ w.r.t. ξ (see [20]).

We assume that a measurable partition ξ^u of $\mathbf{T}^2$ is *subordinate to the W^u-foliation*, i.e., ξ^u satisfies that (1) $\xi^u(x) \subset W^u(x)$ and (2) $\xi^u(x)$ contains an open set in $W^u(x)$ for μ-a.e.x. Let $\{\mu^u_x\}$ (μ-a.e.x) denote a canonical system of conditional measures of μ w.r.t. ξ^u and m^u_x denote the Lebesgue measure on $W^u(x)$. If μ^u_x is absolutely continuous w.r.t. m^u_x for μ-a.e.x ($\mu^u_x \ll m^u_x$), then we say that μ has the conditional measures which are absolutely continuous w.r.t. the Lebesgue measures on unstable manifolds (see [13]). If μ is an SRB measure, then so does every element in the ergodic decomposition of μ for a set of μ-full measure. If $\mu_x^u \ll m_x^u$ (μ-a.e.x), then we know that $\mu_x^u \sim m_x^u|_{\xi^u(x)}$ (see [13], [14], [7]).

3. Proof of Theorem A

Let f be a C^2-partially hyperbolic diffeomorphism with contracting direction on $\mathbf{T}^2$ and μ be an f-invariant probability measure on $\mathbf{T}^2$. Let us denote $I_\varepsilon = [-\varepsilon, \varepsilon]$ for any $0 < \varepsilon \le 1$ and $\mathrm{Emb}^2(I_1, \mathbf{T}^2)$ as the set of C^2-embeddings of I_1 into $\mathbf{T}^2$ equipped with the C^2-metric. We can find (see [9]) two continuous maps $\phi^s : \mathbf{T}^2 \to \mathrm{Emb}^2(I_1, \mathbf{T}^2)$ and $\phi^{cu} : \mathbf{T}^2 \to \mathrm{Emb}^2(I_1, \mathbf{T}^2)$ such that for any $0 < \varepsilon \le 1$ the *local stable* and *center unstable manifolds* $\tilde{W}_\varepsilon^s(x) = \phi^s(x)(I_\varepsilon)$ and $\tilde{W}_\varepsilon^{cu}(x) = \phi^{cu}(x)(I_\varepsilon)$ satisfies the following:

(i) $T_x \tilde{W}_\varepsilon^s(x) = E_1(x)$ and $T_x \tilde{W}_\varepsilon^{cu}(x) = E_2(x)$,

(ii) for any $\varepsilon_1 \in (0,1)$, $f(\tilde{W}_{\varepsilon_1}^s(x)) \subset \tilde{W}_{\varepsilon_1}^s(f(x))$,

(iii) for any $\varepsilon_1 \in (0,1)$, there exists $\varepsilon_2 \in (0,1)$ such that $f^{-1}(\tilde{W}_{\varepsilon_2}^{cu}(x)) \subset \tilde{W}_{\varepsilon_1}^{cu}(f^{-1}(x))$.

Thus there exists $\delta > 0$ such that if $d(x,y) < \delta$ (d denotes the Riemannian metric on $\mathbf{T}^2$), then $\tilde{W}_\varepsilon^s(x)$ and $\tilde{W}_\varepsilon^{cu}(y)$ have a single transverse intersection point, so we write

$$[x,y] = \tilde{W}_\varepsilon^s(x) \cap \tilde{W}_\varepsilon^{cu}(y) \quad (x, y \in \mathbf{T}^2 \text{ with } d(x,y) < \delta). \tag{3.1}$$

We denote by $\tilde{d}^s$ and $\tilde{d}^{cu}$ the Riemannian metric on $\tilde{W}_\varepsilon^s(x)$ and $\tilde{W}_\varepsilon^{cu}(x)$ respectively. Let $B(x,r)$ be the ball centered at x with radius r. Since $\varepsilon_0 > 0$ is small enough, without loss of generality we can assume that the diameater of $\tilde{W}_\varepsilon^{cu}(x)$ (respectively $\tilde{W}_\varepsilon^s(x)$) is greater than $W_{loc}^u(x)$ (respectively $W_{loc}^s(x)$) for $x \in Y_\mu$.

LEMMA 3.1. *For any $x \in Y_\mu$, $W_{loc}^u(x)$ is contained in $\tilde{W}_\varepsilon^{cu}(x)$ and $W_{loc}^s(x)$ is contained in $\tilde{W}_\varepsilon^s(x)$.*

Proof. Let $C(x)$ be as in (2.1) and δ be as in (3.1). Firstly we prove that $W_{loc}^u(x) \cap B(x, \delta C(x)^{-1}) \subset \tilde{W}_\varepsilon^{cu}(x)$ for $x \in Y_\mu$. To do so, assume that there exist $x \in Y_\mu$ and $y \in W_{loc}^u(x) \cap B(x, \delta C(x)^{-1}) \setminus \tilde{W}_\varepsilon^{cu}(x)$. By (2.1) we have $d^u(f^{-n}(y), f^{-n}(x)) < \delta$ for $n \ge 1$ and then define

$$[f^{-n}(y), f^{-n}(x)] = \tilde{W}_\varepsilon^s(f^{-n}(y)) \cap \tilde{W}_\varepsilon^{cu}(f^{-n}(x)) \quad (n \ge 1).$$

Since $y \ne x$ and $[f^{-n}(y), f^{-n}(x)] = f^{-n}[y,x]$, we have

$$f^{-n}([y,x]) \in \tilde{W}_\varepsilon^s(f^{-n}(y)), \quad f^{-n}([y,x]) \ne f^{-n}(y) \quad (n \ge 1).$$

Since f is uniformly contracting along E_1, we have $\|D_z f^{-n}|_{E_1(z)}\| \ge \lambda_1^{-n}$ ($z \in \tilde{W}_\varepsilon^s(y)$) and then

$$\varepsilon > \tilde{d}^s(f^{-n}([y,x]), f^{-n}(y)) \ge \lambda_1^{-n} \tilde{d}^s([y,x], y) \quad (n \ge 1).$$

This is a contradiction.

(2.1) and (2.2) ensure the existence of $i \geq 1$ such that

$$f^{-i}(W^u_{loc}(x)) \subset W^u_{loc}(f^{-i}(x)) \cap B(f^{-i}(x), \delta C(f^{-i}(x))^{-1}) \subset \tilde{W}^{cu}_\varepsilon(f^{-i}(x))$$

for any $x \in Y_\mu$. Thus we have that $W^u_{loc}(x) \cap B(x, \varepsilon) \subset \tilde{W}^{cu}_\varepsilon(x)$ for any $x \in Y_\mu$.

Next we use the similar argument to prove the last part of the lemma. Assume that $W^s_{loc}(x) \cap B(x, \delta D(x)^{-1}) \not\subset \tilde{W}^s_\varepsilon(x)$ for some $x \in Y_\mu$. Here $D(x)$ be as in (2.3). Then there exists $y \in W^s_{loc}(x) \cap B(x, \delta D(x)^{-1}) \setminus \tilde{W}^s_\varepsilon(x)$. From this, we have $d^s(f^n(y), f^n(x)) < \delta$ for $n \geq 1$ and define

$$[f^n(y), f^n(x)] = \tilde{W}^s_\varepsilon(f^n(y)) \cap \tilde{W}^{cu}_\varepsilon(f^n(x)) \quad (n \geq 1).$$

Notice that $[y, x] \neq x$ because of $\tilde{W}^s_\varepsilon(y) \cap \tilde{W}^s_\varepsilon(x) = \emptyset$, and (2.3) shows that

$$\begin{aligned}
(3.2) \qquad \tilde{d}^u(f^n([y,x]), f^n(x)) &\leq \tilde{d}^s(f^n([y,x]), f^n(y)) + d^s(f^n(y), f^n(x)) \\
&\leq \lambda_1^n \tilde{d}^s([y,x], y) + D(x) \exp((\chi_1(x) + \varepsilon_0)n) d^s(x, y) \\
&\quad\ (n \geq 1) \\
&\to 0 \quad (n \to \infty).
\end{aligned}$$

By the first statement of the lemma, $\tilde{W}^{cu}_\varepsilon(f^n(x)) \supset W^u_{loc}(f^n(x)) \cap B(f^n(x), \varepsilon)$ and, since $f^n(W^u_{loc}(x) \cap B(x, \varepsilon))$ is expanding along E_2, it doesn't happen that $\tilde{d}^u(f^n([y,x]), f^n(x)) \to 0$ $(n \to \infty)$. This contradicts (3.2). So we have $W^s_{loc}(x) \cap B(x, \delta D(x)^{-1}) \subset \tilde{W}^s_\varepsilon(x)$ for any $x \in Y_\mu$. Then as in the proof of the first statement, we have that $W^s_{loc}(x) \cap B(x, \varepsilon) \subset \tilde{W}^s_\varepsilon(x)$ for any $x \in Y_\mu$. □

We say that $I \subset \tilde{W}^{cu}_\varepsilon(x)$ is an *interval* if there exist $y, z \in I_\varepsilon$ $(y < z)$ such that $I = \phi^{cu}(x)([y, z])$. For $0 < \varepsilon' < \varepsilon$ we identify $\tilde{W}^{cu}_{\varepsilon'}(x)$ with $I_{\varepsilon'} \subset \mathbf{R}$ if there is no confusion. For any fixed point p, without loss of generality we can assume that all the eigenvalues of $D_p f$ are positive (by replacing f by f^2 if necessary).

LEMMA 3.2 ([11] Lemma 4.1). *Let $p \in \mathbf{T}^2$ be a fixed point satisfying*

$$\|D_p f|_{E_2(p)}\| = 1, \quad f^{-1}(\tilde{W}^{cu}_{\varepsilon'}(p)) \subset \tilde{W}^{cu}_{\varepsilon'}(p)$$

for some $0 < \varepsilon' < \varepsilon$. Then, for any interval $J \subset \tilde{W}^{cu}_{\varepsilon'}(p)$ containing p,

$$\sum_{i=0}^{\infty} \ell(f^{-i}(J)) = \infty.$$

Here $\ell(I)$ denotes the length of I.

By $\|D_p f|_{E_1(x)}\| < \lambda_1$ for any $x \in \mathbf{T}^2$, the following statement is a result of Pujals-Sambarino ([18]).

LEMMA 3.3 ([18] Corollary 3.5). *Assume that $p \in \mathbf{T}^2$ is a fixed point such that*

$$f^{-1}(\tilde{W}^{cu}_{\varepsilon'}(p)) \subset \tilde{W}^{cu}_{\varepsilon'}(p)$$

for some $0 < \varepsilon' < \varepsilon$ *and fix an interval* $J \subset \tilde{W}^{cu}_{\varepsilon'}(p)$. *Then there exists* $L = L(J, \varepsilon) > 0$ *such that*

$$L^{-1}\ell(f^{-n}(J)) \leq \ell([f^{-n}(J), q]) \leq L\ell(f^{-n}(J))$$

for any $q \in \tilde{W}^{s}_{\varepsilon'}(p)$ *and* $n \geq 0$.

Remark 3.4. If f is topologically transitive, then any fixed point of f satisfies the condition of Lemma 3.3.

Let Γ be an f-invariant compact set. We say that f has a *dominated splitting* on Γ if there exist $C > 0$ and $0 < \lambda < 1$ such that each $x \in \Gamma$ is decomposed $\mathrm{T}_x\mathbf{T}^2 = E_1(x) \oplus E_2(x)$ into the sum of $D_x f$-invariant subspaces $E_1(x)$ and $E_2(x)$ which satisfies

$$\|D_x f^n|_{E_1(x)}\| \, \|D_{f^n(x)} f^{-n}|_{E_2(f^n(x))}\| \leq C\lambda^n \quad (n \geq 0).$$

Clearly, if f is partially hyperbolic with contracting direction on Γ, then f has a dominated splitting on Γ.

We denote by $\Omega(f)$ the set of points $x \in \mathbf{T}^2$ such that for any neighborhood V of x there exists $n > 0$ satisfying $f^n(V) \cap V \neq \emptyset$. We say that an n-periodic point p is *hyperbolic* if the absolute values of eigenvalues of $D_p f^n$ are different from 1 and is *sink* (respectively *source*) if all the absolute values of eigenvalues of $D_p f^n$ are smaller(respectively larger) than 1. If μ has positive and negative Lyapunov exponents then the set of hyperbolic periodic points is not empty ([12]).

Lemma 3.5 ([18]). *Assume that* f *has a dominated splitting on* $\Omega(f)$ *and all the periodic points in* $\Omega(f)$ *are hyperbolic. Then* $\Omega(f)$ *is represented as a union* $\Omega(f) = \Gamma_1 \cup \Gamma_2$ *of* Γ_1 *and* Γ_2 *where* Γ_1 *is a hyperbolic set for* f *and* Γ_2 *consists of a finite union of periodic simple closed curves* $\mathscr{C}_1, \dots, \mathscr{C}_n$ *such that each* $\mathscr{C}_i$ *is normally hyperbolic and* $f^{m_i} : \mathscr{C}_i \to \mathscr{C}_i$ *is conjugated to an irrational rotation* (m_i *denotes the period of* $\mathscr{C}_i$).

In particular, in the case when $\Omega(f) = \mathbf{T}^2$, f *is an Anosov diffeomorphism.*

Remark 3.6. If f is topologically transitive, then all the periodic points are not sink nor source.

Proof of Theorem A. Assume that f is an Anosov diffeomorphism. Then f satisfies the conditions (1)–(3) of Theorem A. Indeed, the condition (1) is obtained from the definitions of Anosov and partially hyperbolic diffeomorphisms. Since any Anosov diffeomorphism of $\mathbf{T}^2$ is topologically conjugate to some hyperbolic toral automorphism ([6] Theorem 6.3) and hyperbolic toral automorphisms are topologically transitive, we have the condition (2). The condition (3) is the direct consequence of [23] as stated in Introduction. Therefore it remains only to show that the converse statement holds.

To do so we prepare the following claim.

Claim. All the periodic points of $\mathbf{T}^2$ are hyperbolic.

If we have the claim, then f satisfies the assumption of Lemma 3.5. Thus we have the conclusion.

To show the claim, we choose an arbitrary periodic point p with period n. To simplicity we replace f by f^n. Then p is a fixed point of f. Since f is uniformly contracting along E_1, we have to show that $\|D_p f|_{E_2(p)}\| > 1$. Since f is topologically transitive, by Remark 3.6 it doesn't happen that $\|D_p f|_{E_2(p)}\| < 1$. Assume that $\|D_p f|_{E_2(p)}\| = 1$. Then we lead a contradiction by using the method in the proof of Theorem A in [11].

Fix $0 < \varepsilon_1 < \min\{\varepsilon, \delta/(2\lambda_1)\}$ and define a neighborhood $\mathscr{P}$ of p by

$$\mathscr{P} = \{[y,x] \mid y \in \tilde{W}^{cu}_{\varepsilon_1}(p), x \in \tilde{W}^{s}_{\varepsilon_1}(p)\}.$$

When we identify $\tilde{W}^{cu}_{\varepsilon_1}(p)$ with an interval $I_{\varepsilon_1} \subset \mathbf{R}$ and p is a fixed point of f, the graph of $f|_{\tilde{W}^{cu}_{\varepsilon_1}(p)}$ satisfies $|f|_{\tilde{W}^{cu}_{\varepsilon_1}(p)}(y)| > |y|$ for any $y \in \tilde{W}^{cu}_{\varepsilon_1}(p)\setminus\{p\}$ (Figure 1) because f is topologically transitive. Since f^{-1} is uniformly expanding along E_1, $f^{-1}(\mathscr{P})$ intersects $\mathscr{P}$ transeversely along stable direction (see Figure 2).

Let ξ^u be a measurable partition subordinate to the W^u-foliation and $\{\mu^u_x\}$ (μ-a.e.x) denote a canonical system of conditional measures of μ w.r.t. ξ^u. Since μ is an SRB measure, we can take a measurable function $g : \mathbf{T}^2 \to \mathbf{R}$ satisfying

$$g(z) = \frac{d\mu^u_y}{dm^u_y}(z)$$

for μ-a.e.y and m^u_y-a.e.$z \in \xi^u(y)$ ([13]). Here m^u_y denotes the Lebesgue measure on $W^u(y)$. Moreover it is known ([14] Corollary 6.1.4) that for μ-a.e.y, g is strictly positive on $\xi^u(y)$ and $\log g$ is Lipshitz continuous on $\xi^u(y)$.

By the definition of ξ^u, there exist $r > 0$, $x_0 \in Y_\mu$ and a closed set $A \subset Y_\mu \cap B(x_0, r)$ with $\mu(A) > 0$ such that for any $y \in A$

(a) $\xi^u(y) \supset B^u(y, 2r)$ where $B^u(y, 2r)$ denotes the ball centered at y with radius $2r$ in $W^u(y)$,

(b) $\lim_{n\to\pm\infty} \frac{1}{n} \log\|D_z f^n|_{E_2(z)}\| = \chi_2(z) > 0$ for m^u_y-a.e. $z \in \xi^u(y)$,

(c) there exists $C_0 > 0$ (independent of y) such that for $z \in B^u(y, 2r)$

$$C_0^{-1} \le g(z) \le C_0. \tag{3.3}$$

Let $\eta^u(y)$ denote the connected component of $\xi^u(y) \cap B(x_0, r)$ which contains $y \in A$ and write $B^u = \bigcup_{y\in A} \eta^u(y)$. For any $y \in A$, by (b) we have that $l(f^i(\eta^u(y))) \to \infty$ as $i \to \infty$, where $l(I)$ denotes the length of an interval I in the unstable manifold. Since f is topologically transitive, there exists $k_1 > 0$ such that $f^{k_1}(B^u)$ meets transversely one of the components of $f^{-1}(\mathscr{P})\setminus\mathscr{P}$ along E_2. This intersection is denoted by $\mathscr{Q}^{(1)}$, and clearly we have $m^u_y(\mathscr{Q}^{(1)}) > 0$ for $y \in \mathscr{Q}^{(1)}$. Furthermore we can show that $\mu(\mathscr{Q}^{(1)}) > 0$. Indeed, since μ is f-invariant and is an SRB measure,

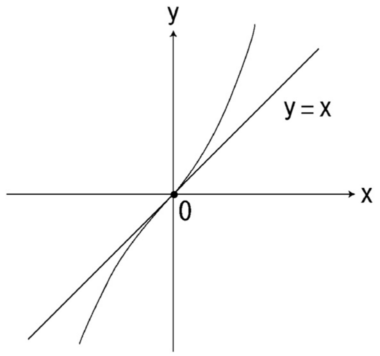

FIGURE 1. The graph of $f|_{\tilde{W}^{cu}_{\varepsilon_1}(p)}$

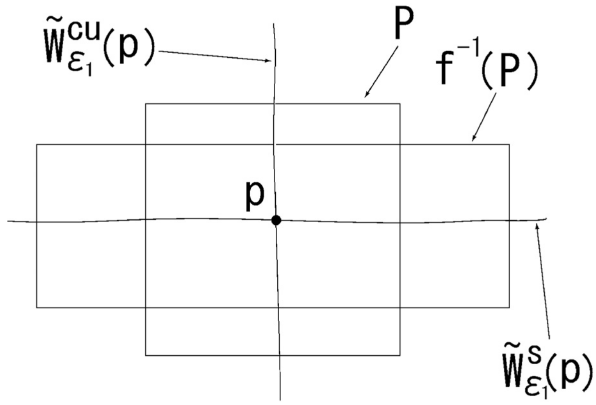

FIGURE 2. A transverse intersection of $f^{-1}(\mathscr{P}) \cap \mathscr{P}$

$$
\begin{aligned}
(3.4) \qquad \mu(\mathscr{Q}^{(1)}) &= \mu(f^{-k_1}(\mathscr{Q}^{(1)})) = \int \mu^u_x(f^{-k_1}(\mathscr{Q}^{(1)}))\, d\mu(x) \\
&= \int_{B^u} \left(\int_{f^{-k_1}(\mathscr{Q}^{(1)})} g\, dm^u_x \right) d\mu(x) \\
&= \int_{B^u} \left(\int_{\mathscr{Q}^{(1)}} g(f^{-k_1}(z)) \| D_z f^{-k_1}|_{E_2(z)} \|\, dm^u_{f^{k_1}(y)}(z) \right) d\mu(y) \\
&\geq C_0^{-1} \inf_{z \in \mathscr{Q}^{(1)}} \{ \| D_z f^{-k_1}|_{E_2(z)} \| \} \int_{B^u} m^u_{f^{k_1}(y)}(\mathscr{Q}^{(1)})\, d\mu(y) \quad (\because\ (3.3)) \\
&= C_1 \int m^u_x(\mathscr{Q}^{(1)})\, d\mu(x)
\end{aligned}
$$

where $C_1 = C_0^{-1} \inf_{z \in \mathscr{Q}^{(1)}} \{ \| D_z f^{-k_1}|_{E_2(z)} \| \}$. The last term of (3.4) is positive because $\mu(B^u) > 0$ and $m^u_{f^{k_1}(y)}(\mathscr{Q}^{(1)}) > 0$ for $y \in B^u$.

Define

$$\mathcal{Q}^{(i)} = \{z \in \mathcal{Q}^{(1)} \mid f^j(z) \in \mathcal{P}, (1 \le j \le i)\} \quad (i \ge 2)$$

and remark that $\mathcal{Q}^{(i)} \supset \mathcal{Q}^{(j)}$, $f^i(\mathcal{Q}^{(i)}) \cap f^j(\mathcal{Q}^{(j)}) = \emptyset$ for $1 \le i < j$ and $\bigcup_{i \ge 1} f^i(\mathcal{Q}^{(i)}) \subset \mathcal{P}$. Let $\pi : \mathcal{Q}^{(1)} \to \tilde{W}^{cu}_{\varepsilon}(p)$ be the projection sliding along local stable manifolds. Since the fixed point p satisfies the condition of Lemma 3.3, there exists $L_1 > 0$ such that

$$m^u_x(\mathcal{Q}^{(i)}) \ge L_1 m^u_p(\pi(\mathcal{Q}^{(i)})) \tag{3.5}$$

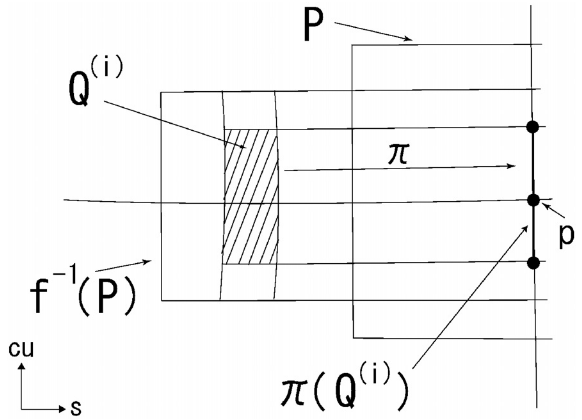

FIGURE 3. The figure of $\mathcal{Q}^{(i)}$

for any $x \in \mathcal{Q}^{(1)}$ (see Figure 3). In (3.4), replacing $\mathcal{Q}^{(1)}$ by $\mathcal{Q}^{(i)}$, we have

$$\mu(\mathcal{Q}^{(i)}) \ge C_1 \int m^u_x(\mathcal{Q}^{(i)})\, d\mu(x) \quad (i \ge 1). \tag{3.6}$$

By (3.5), (3.6) and $m^u_p(\pi(\mathcal{Q}^{(i)})) = m^u_p(f^{-i}(\pi(\mathcal{Q}^{(1)})))$,

$$\mu(\mathcal{Q}^{(i)}) \ge C_1 L_1 m^u_p(f^{-i}(\pi(\mathcal{Q}^{(1)}))).$$

Then, since μ is f-invariant,

$$\mu\left(\sum_{i=1}^{\infty} f^i(\mathcal{Q}^{(i)})\right) = \sum_{i=1}^{\infty} \mu(f^i(\mathcal{Q}^{(i)})) = \sum_{i=1}^{\infty} \mu(\mathcal{Q}^{(i)}) \ge C_1 L_1 \sum_{i=1}^{\infty} m^u_p(f^{-i}(\pi(\mathcal{Q}^{(1)}))).$$

The last expression above goes to ∞ by Lemma 3.2. This is a contradiction with $\mu(\mathbf{T}^2) = 1$. Therefore $\|D_p f|_{E_2(p)}\| > 1$, i.e., p is hyperbolic. This completes the proof. □

4. Proof of Theorem B

In this section we deal with a $C^{1+\alpha}$-diffeomorphism f of $\mathbf{T}^2$ $(0 < \alpha < 1)$ which has a non-hyperbolic fixed point p and satisfies the following three assumptions:

Assumption 1. There exist a norm $\|\cdot\|$ on $\mathbf{T}^2$, $0 < \lambda < 1$ and a $D_x f$-invariant decomposition $\mathrm{T}_x\mathbf{T}^2 = E_1(x) \oplus E_2(x)$ into subspaces $E_1(x)$ and $E_2(x)$ which satisfy

$$\|D_x f|_{E_1(x)}\| \leq \lambda, \quad \|D_x f|_{E_2(x)}\| \begin{cases} = 1 & (x = p), \\ > 1 & (x \neq p). \end{cases}$$

For any $0 < \varepsilon < 1$, if we denote the local stable and unstable manifolds $\tilde{W}_\varepsilon^s(x)$ and $\tilde{W}_\varepsilon^u(x)$ at $x \in \mathbf{T}^2$ by

$$\tilde{W}_\varepsilon^s(x) = \{y \in \mathbf{T}^2 \mid d(f^n(y), f^n(x)) \leq \varepsilon, (n \geq 0)\} \quad \text{and}$$
$$\tilde{W}_\varepsilon^u(x) = \{y \in \mathbf{T}^2 \mid d(f^{-n}(y), f^{-n}(x)) \leq \varepsilon, (n \geq 0)\}$$

respectively, then it follows from Assumption 1 that $\tilde{W}_\varepsilon^s(x)$ and $\tilde{W}_\varepsilon^u(x)$ are $C^{1+\alpha}$-manifolds with $T_x\tilde{W}_\varepsilon^s(x) = E_1(x)$ and $T_x\tilde{W}_\varepsilon^u(x) = E_2(x)$ ([9]). To obtain the Lipschitz continuity of the holonomy map along local stable manifolds (Lemma 4.3), we impose the following assumption.

Assumption 2. (1) For any $x \in \mathbf{T}^2$ and $0 < \varepsilon < 1$, each local unstable manifold $\tilde{W}_\varepsilon^u(x)$ is a C^2-embedding and (2) W^u-foliation $\{\tilde{W}_\varepsilon^u(x) \mid x \in \mathbf{T}^2\}$ is C^2-continuous, i.e. the correspondence $x \mapsto \tilde{W}_\varepsilon^u(x)$ is C^2-continuous.

Assumption 3. If we identify $\tilde{W}_\varepsilon^u(p)$ with $I_\varepsilon = [-\varepsilon, \varepsilon]$, then the graph of $f|_{\tilde{W}_\varepsilon^u(p)}$ can be represented as

$$f|_{\tilde{W}_\varepsilon^u(p)}(x) = \begin{cases} x + x^{1+\alpha} + o(x^2) & (x \geq 0), \\ x - |x|^{1+\alpha} - o(x^2) & (x < 0). \end{cases}$$

Assumption 3 implies that f is of $C^{1+\alpha}$ on $\tilde{W}_\varepsilon^u(p)$ and it is crucial in proving the existence of an SRB measure.

Remark 4.1 ([19] Chapter VIII 8.8). We can construct a $C^{1+\alpha}$-diffeomorphism f above as follows: Let f_0 be a hyperbolic toral automorphism of $\mathbf{T}^2$ with two different eigenvalues $0 < \lambda_1 < 1 < \lambda_2$. We slowly deform f_0 near the origin along only unstable direction until it satisfies Assumptions 1 and 3.

By Assumption 1, f is partially hyperbolic with contracting direction but not an Anosov diffeomorphism and there exists $\delta > 0$ such that if $d(x, y) < \delta$, then $\tilde{W}_\varepsilon^s(x)$ and $\tilde{W}_\varepsilon^u(y)$ have a single transverse intersection point $[x, y] = \tilde{W}_\varepsilon^s(x) \cap \tilde{W}_\varepsilon^u(y)$ ([9] Theorem 5.5). Thus f is *expansive*, i.e. there exists $\eta > 0$ such that if $x, y \in \mathbf{T}^2$ and $d(f^i(x), f^i(y)) < \eta$ $(i \in \mathbf{Z})$ then $x = y$.

Moreover we can check that f satisfies the uniformly shadowing property ([2] Theorem 5.4, [3] Theorem 2.2.17).

A sequence $\{x_i\}_{i \in \mathbf{Z}} \subset \mathbf{T}^2$ is called a *β-pseudo orbit* for f if $d(f(x_i), x_{i+1}) < \beta$ for all $i \in \mathbf{Z}$. A point $x \in \mathbf{T}^2$ is called an *α-shadowing point* for a β-pseudo orbit $\{x_i\}_{i \in \mathbf{Z}}$ if $d(f^i(x), x_i) < \alpha$ $(i \in \mathbf{Z})$. We call that f satisfies *uniformly shadowing* *property* if for any $\alpha > 0$ there exists $\beta > 0$ such that for a δ-pseudo orbit there exists at least one α-shadowing point. Since f is expansive and satisfies the uniformly shadowing property, f is topologically conjugate to some hyperbolic toral automorphism ([8] Theorem) and is topologically transitive.

To conclude Theorem B it is enough to show the following:

PROPOSITION 4.2. *If $0 < \alpha < 1$, then f admits an SRB measure.*

In [11] Hu and Young gave a direct proof of the Lipschitz continuity of the holonomy map along the stable leaves for C^2-almost Anosov diffeomorphisms. We emphasize that in their proof they use only Assumptions 1 and 2. So the Lipschitz continuity also holds for our case:

LEMMA 4.3 ([1] Proposition 2.5). *There exists $L > 0$ such that for any $y \in \mathbf{T}^2$, interval $J \subset \tilde{W}^u_\varepsilon(y)$ and $q \in \tilde{W}^s_\varepsilon(y)$ with $d(y,q) < \delta$,*

$$L^{-1}\ell(J) \leq \ell([J,q]) \leq L\ell(J).$$

By Assumption 3, we have the following ([24] p. 180):

LEMMA 4.4 ([11], [24]). *For any interval $J \subset \tilde{W}^u_\varepsilon(p)$*

$$\sum_{i=0}^{\infty} \ell(f^{-i}(J)) < \infty.$$

For any $z \in \mathbf{T}^2$ we denote by

$$\mathscr{R}_{\varepsilon'}(z) = \{[y,x] \mid y \in \tilde{W}^u_{\varepsilon'}(z), x \in \tilde{W}^s_{\varepsilon'}(z)\} \quad (0 < \varepsilon' \leq \varepsilon)$$

a *rectangle* of z. Combining Lemma 4.3 with the proofs of Proposition 3.1 in [11] and Lemma 5 in [24], we have the following:

LEMMA 4.5 ([11], [24]). *For any small rectangle $\mathscr{P}$ of p, there exist $\delta_1 > 0$ and $K > 0$ such that for any $x \in \mathbf{T}^2$ and any interval $J \subset \tilde{W}^u_\varepsilon(x)$ with $l(J) \leq \delta_1$ and $J \cap \mathscr{P} = \emptyset$*

$$\frac{1}{K} \leq \frac{|\det(D_y f^{-n}|_{E_2(y)})|}{|\det(D_z f^{-n}|_{E_2(z)})|} \leq K \quad (y, z \in \gamma, n \geq 1)$$

where $\det(D_y f|_{E_2(y)})$ denotes the Jacobian at y of f restricted to $E_2(y)$.

Now we introduce here the notion of a Markov partition. For any rectangle $\mathscr{R}$ and $x \in \mathscr{R}$, let $\gamma^\sigma(x)$ be *the stable* ($\sigma = s$) and *unstable* ($\sigma = u$) *leaf* which is the connected component of $\tilde{W}^\sigma_\varepsilon(x) \cap \mathscr{R}$ containing x. A rectangle $\mathscr{R} \subset \mathbf{T}^2$ is said to be proper if $\mathrm{Cl}(\mathrm{int}(\mathscr{R})) = \mathscr{R}$ where $\mathrm{Cl}(A)$ and $\mathrm{int}(A)$ denote the closure and interior of a set A respectively. We say that $\{\mathscr{R}_i\}_{i=0}^{r-1}$ is a *Markov partition* if (i) each $\mathscr{R}_i$ is proper, (ii) $\{\mathscr{R}_i\}_{i=0}^{r-1}$ is a cover of $\mathbf{T}^2$, (iii)

$\mathrm{int}(\mathscr{R}_i) \cap \mathrm{int}(\mathscr{R}_j) = \emptyset$ for $i \neq j$ and (iv) for any $x \in \mathrm{int}(\mathscr{R}_i) \cap f^{-1}(\mathrm{int}(\mathscr{R}_j))$, $f(\gamma^s(x)) \subset \gamma^s(f(x))$ and $f(\gamma^u(x)) \supset \gamma^u(f(x))$.

Since f is expansive and satisfies uniformly shadowing property, f has a Markov partition $\{\mathscr{R}_i\}_{i=0}^{r-1}$ with arbitrary diameter (see [3]). So we can assume that the diameter of $\{\mathscr{R}_i\}_{i=0}^{r-1}$ is less than $0 < \varepsilon_2 < \delta/2$. Moreover, since f is topologically conjugate to some hyperbolic toral automorphism, each element of $\{\mathscr{R}_i\}_{i=0}^{r-1}$ is homeomorphic to a parallelogram ([22] Theorem 4.1) and its boundary consists of two stable leaves and two unstable leaves.

We consider elements of $\{\mathscr{R}_i\}_{i=0}^{r-1}$ containing the fixed point p. Then p is contained in the interior of some $\mathscr{R}_i$ or, in the boundaries of some $\mathscr{R}_i$s. If the latter happens, one of the boudary leaves of $\mathscr{R}_i$ is an unstable or stable leaf with p. This implies that the cardinarity of the set of all $\mathscr{R}_i$ containing p is less than 4. Since we can assume that all the eigenvalues of $D_p f$ are positive (by replacing f by f^2, if necessary), we have

(a) $f(\mathrm{int}(\mathscr{R}_i)) \cap \mathrm{int}(\mathscr{R}_i) \neq \emptyset$ whenever $p \in \mathscr{R}_i$, and

(b) $f(\mathrm{int}(\mathscr{R}_i)) \cap \mathrm{int}(\mathscr{R}_j) = \emptyset$ whenever $p \in \mathscr{R}_i \cap \mathscr{R}_j$ $(i \neq j)$.

If we take a neighborhood $\mathscr{P}$ of p where

$$\mathscr{P} = \mathrm{int}\left(\bigcup_{p \in \mathscr{R}_i} \mathscr{R}_i\right),$$

then f is uniformly hyperbolic outside $\mathscr{P}$ (by Assumption 1). By (a) and (b) we have $f(\mathscr{P}) \cap \mathscr{P} = \mathrm{int}(\bigcup_{p \in \mathscr{R}_i} f(\mathscr{R}_i) \cap \mathscr{R}_i)$.

Let $R(x)$ be the smallest positive integer such that $(f^R)(x) = f^{R(x)}(x) \in \mathbf{T}^2 \backslash \mathscr{P}$ for $x \in \mathbf{T}^2 \backslash \mathscr{P}$. By Assumption 1, *the first return map* f^R is defined for m-a.e.$x \in \mathbf{T}^2 \backslash \mathscr{P}$ where m denotes the Lebesgue measure on $\mathbf{T}^2$. We set

$$\Gamma_i = \{y \in \mathbf{T}^2 \backslash \mathscr{P} \mid R(y) = i\} \quad (i \geq 1),$$

then $f^R(x) = f^i(x)$ for $x \in \Gamma_i$. We define

$$\mathscr{Q}^{(i)} = \{z \in f^{-1}(\mathscr{P}) \backslash \mathscr{P} \mid f^j(z) \in \mathscr{P}, (1 \leq j \leq i)\} \quad (i \geq 1).$$

Then $f^i(\mathscr{Q}^{(i)}) \cap f^j(\mathscr{Q}^{(j)}) = \emptyset$ for $i \neq j$, $\mathscr{P} = \bigcup_{i \geq 1} f^i(\mathscr{Q}^{(i)})$ and $\mathscr{Q}^{(i)} = \bigcup_{j \geq i+1} \Gamma_j$.

For any rectangle $\mathscr{R}$, any unstable leaf γ^u and any $\rho > 0$, we say that $\mathscr{V}_\rho \subset \mathscr{R}$ is a *u-subset* of γ^u with radius ρ if $\mathscr{V}_\rho = \bigcup_{y \in \tilde{B}^s(x,\rho)} \gamma^u(y)$ for $x \in \gamma^u$, where $\tilde{B}^s(x,\rho)$ denotes the closed ball in $\tilde{W}^s_\varepsilon(x)$ centered at x with radius ρ. For any interval $\omega \subset \gamma^u$, $\mathscr{S}_\omega$ is an *s-subset* corresponding to ω if $\mathscr{S}_\omega = \bigcup_{y \in \omega} \gamma^s(y)$. We denote $\partial^s(\mathscr{R})$ the two stable leaves which contain the different extreme points of any unstable leaf $\gamma^u \subset \mathscr{R}$. $\partial^u(\mathscr{R})$ is also defined. The boudary of $\mathscr{R}$, $\partial(\mathscr{R})$, is represented as $\partial(\mathscr{R}) = \partial^s(\mathscr{R}) \cup \partial^u(\mathscr{R})$.

Let ξ^u be the measurable partition subordinate to W^u-foliation. For an f^R-invariant probability measure ν on $\mathbf{T}^2 \backslash \mathscr{P}$, let $\{\nu^u_x\}$ (ν-a.e.x) denote the canonical system of conditional measures of ν w.r.t. ξ^u and m^u_x denote the Lebesgue measure on $\gamma^u(x)$.

LEMMA 4.6. *There exists an f^R-invariant Borel probability measure μ such that $\mu_x^u \ll m_x^u$ (μ-a.e.x).*

Proof. Let γ_0 be an unstable leaf which intersects one of the components of $f^{-1}(\mathscr{P})\backslash\mathscr{P}$ and m_0^u be the Lebesgue measure on γ_0. To simplicity we assume that $m_0^u(\gamma_0) = 1$, and so define a probability measure of $\mathbf{T}^2\backslash\mathscr{P}$ by

$$\mu_n = \frac{1}{n}\sum_{j=0}^{n-1}\sum_{i=1}^{\infty}(f^R)_*^j(m_0^u|_{\Gamma_i}) \quad (n \geq 1).$$

Here $(f^R)_*^j(m_0^u|_{\Gamma_i})$ is the push-forward of $m_0^u|_{\Gamma_i}$ by $(f^R)^j$. Then there exist a probability measure μ on $\mathbf{T}^2\backslash\mathscr{P}$ and a subsequence $\{\mu_{n_j}\}_{j\geq 1} \subset \{\mu_n\}_{n\geq 1}$ such that $\mu_{n_j} \to \mu$ $(j \to \infty)$. Clearly μ is f^R-invariant.

To obtain the conclusion it is enough to show that there exists $K_1 > 0$ such that for μ-a.e.x and any interval $\omega \subset \gamma^u(x)$

$$\frac{1}{K_1}m_x^u(\omega) \leq \mu_x^u(\omega) \leq K_1 m_x^u(\omega). \tag{4.1}$$

For any $x \in \mathbf{T}^2$, to simplicity set $\gamma = \gamma^u(x)$ and choose any interval $\omega \subset \gamma$. $\mathscr{S}_\omega$ denotes the s-subset corresponding to ω and $\mathscr{V}_\rho$ the u-subset of γ with radius $0 < \rho < \delta/2$. Firstly we prove that there exists $K_0 > 0$ such that

$$\frac{1}{K_0}\frac{m_x^u(\omega)}{m_x^u(\gamma)} \leq \frac{\mu_n(\mathscr{V}_\rho \cap \mathscr{S}_\omega)}{\mu_n(\mathscr{V}_\rho)} \leq K_0\frac{m_x^u(\omega)}{m_x^u(\gamma)}. \tag{4.2}$$

To see this, we use the arguments in the proof of Lemma 5.2 in [11] and Theorem 1 in [25]. We set $\gamma_i^j = (f^R)^j(\gamma_0 \cap \Gamma_i) \cap \mathscr{V}_\rho$. Since f is topologically transitive, for $n > 0$ large enough there exist $i \geq 1$ and $0 \leq j \leq n-1$ such that $\gamma_i^j \neq \emptyset$. By Lemma 4.3 there exists $L > 0$ such that for $0 \leq j \leq n-1$ and $i \geq 1$ with $\gamma_i^j \neq \emptyset$,

$$\frac{1}{L^2}\frac{m_x^u(\omega)}{m_x^u(\gamma)} \leq \frac{m_{\gamma_i^j}^u(\gamma_i^j \cap S_\omega)}{m_{\gamma_i^j}^u(\gamma_i^j)} \leq L^2\frac{m_x^u(\omega)}{m_x^u(\gamma)}. \tag{4.3}$$

By Lemma 4.5, (4.3) and by using the fact that $(\sum_{i=1}^\infty a_i)/(\sum_{i=1}^\infty b_i) \leq \sup_{1\leq i}\{a_i/b_i\}$ for $a_i, b_i > 0$ $(i \geq 1)$ and $\sum_{i=1}^\infty a_i < \infty$, $\sum_{i=1}^\infty b_i < \infty$, we can find $K > 0$ such that

$$\frac{1}{KL^2}\frac{m_x^u(\omega)}{m_x^u(\gamma)} \leq \frac{(f^R)_*^j(m_0^u)(\mathscr{V}_\rho \cap \mathscr{S}_\omega)}{(f^R)_*^j(m_0^u)(\mathscr{V}_\rho)} \leq KL^2\frac{m_x^u(\omega)}{m_x^u(\gamma)}. \tag{4.4}$$

By (4.4) and the fact above again, we have (4.2).

We can choose $\mathscr{V}_\rho$ such that their boundary $\partial(\mathscr{V}_\rho)$ has μ-zero measure. Indeed, it is enough to show that $\mu(\gamma^s) = 0$ for any unstable leaf γ^s. Since there exist at most countable $\rho > 0$ such that $\mu(\partial^u(\mathscr{V}_\rho)) > 0$, except for such $\rho > 0$, we have that $\mu(\partial^u(\mathscr{V}_\rho)) = 0$. If we have the claim above, then $\mu(\partial^s(\mathscr{V}_\rho)) = 0$. Therefore $\mu(\partial(\mathscr{V}_\rho)) = 0$.

For any stable leaf γ^s and $0 < \eta < \varepsilon_2$ we set

$$U(\gamma^s, \eta) = \{[y, z] \in \mathbf{T}^2 \backslash \mathscr{P} \mid z \in \gamma^s, d(z, y) < \eta\}.$$

Then

$$U(\gamma^s, \eta) = \bigcup_{k=1}^{l} (U(\gamma^s, \eta) \cap \mathscr{R}_{m_k})$$

for $\mathscr{R}_{m_k} \neq \mathscr{R}_0$ $(k = 1, \dots, l)$ with $U(\gamma^s, \eta) \cap \mathscr{R}_{m_k} \neq \emptyset$. For any $k = 1, \dots, l$, there exist $y_k \in \mathscr{R}_{m_k}$, an interval $\omega_\eta(y_k) \subset \gamma^u(y_k)$ and $\rho(y_k) > 0$ such that (a) the u-subset $\mathscr{V}_{\rho(y_k)}$ of $\gamma^u(y_k)$ and the s-subset $\mathscr{S}_{\omega_\eta(y_k)}$ corresponding to $\omega_\eta(y_k)$ satisfy $U(\gamma^s, \eta) \cap \mathscr{R}_{m_k} = \mathscr{V}_{\rho(y_k)} \cap \mathscr{S}_{\omega_\eta(y_k)}$, (b) $m^u_{y_k}(\omega_\eta(y_k)) \to 0$ as $\eta \to 0$. By (4.2), for any $n \geq 1$,

$$\mu_n(U(\gamma^s, \eta) \cap \mathscr{R}_{m_k}) \leq K_0 \frac{m^u_{y_k}(\omega_\eta(y_k))}{m^u_{y_k}(\gamma^u(y_k))} \to 0 \quad (\eta \to 0). \tag{4.5}$$

Since $\bigcup_{k=1}^{l} U(\gamma^s, \eta) \cap \mathscr{R}_{m_k}$ contains γ^s and is open in $\mathbf{T}^2 \backslash \mathscr{P}$ w.r.t. the relative topology, we have

$$\begin{aligned} \mu(\gamma^s) &\leq \mu\left(\bigcup_{k=1}^{l} U(\gamma^s, \eta) \cap \mathscr{R}_{m_k}\right) \\ &\leq \limsup_{j \to \infty} \mu_{n_j}\left(\bigcup_{k=1}^{l} U(\gamma^s, \eta) \cap \mathscr{R}_{m_k}\right). \end{aligned} \tag{4.6}$$

By (4.5) and (4.6), we have $\mu(\gamma^s) = 0$.

Thus we can choose the finite measurable partition ξ_1 which consists of $\mathscr{V}_\rho \cap \mathscr{S}_{\omega_j}$ with $\mu(\partial(\mathscr{V}_\rho \cap \mathscr{S}_{\omega_j})) = 0$ $(1 \leq i \leq q)$ and set $\xi^u = \{\gamma^u(x) \cap \mathscr{V}_\rho \cap \mathscr{S}_{\omega_j} \mid x \in \mathscr{R}_i, \mathrm{int}(\mathscr{R}_i) \cap \mathscr{P} = \emptyset, \mathscr{V}_\rho \cap \mathscr{S}_{\omega_j} \in \xi_1\}$. Then we can find the sequence $\{\xi_\ell\}_{\ell \geq 1}$ of finite measurable partition such that

$$\xi_1 \leq \xi_2 \leq \cdots \leq \bigvee_{\ell \geq 1} \xi_\ell = \xi^u.$$

By Doob's theorem we have $\mu^\ell_x \to \mu^u_x$ (μ-a.e.x) as $\ell \to \infty$, where $\{\mu^\ell_x\}$ (μ-a.e.x) denotes the canonical system of conditional measures w.r.t. ξ_ℓ. Here we remark that $\mu^\ell_x(A) = \mu(\mathscr{V}_{\rho_\ell} \cap \mathscr{S}_{\omega_j} \cap A)/\mu(\mathscr{V}_{\rho_\ell} \cap \mathscr{S}_{\omega_j})$ for any $\mathscr{V}_{\rho_\ell} \cap \mathscr{S}_{\omega_j} \in \xi_\ell$, any Borel set A and $x \in \mathscr{V}_{\rho_\ell} \cap \mathscr{S}_{\omega_j} \cap A$.

Since (4.2) holds for $\mathscr{V}_{\rho_\ell} \cap \mathscr{S}_{\omega_j} \in \xi_l$ $(1 \leq j \leq q)$ instead of $\mathscr{V}_\rho$ and any interval $\omega \subset \omega_j$ with $\mu(\partial(\mathscr{V}_{\rho_\ell} \cap \mathscr{S}_\omega) = 0$, by taking $n \to \infty$ in (4.2), we have

$$\frac{1}{K_0} \frac{m^u_x(\omega)}{m^u_x(\gamma \cap \mathscr{S}_{\omega_j})} \leq \frac{\mu((\mathscr{V}_{\rho_\ell} \cap \mathscr{S}_{\omega_j}) \cap \mathscr{S}_\omega)}{\mu(\mathscr{V}_{\rho_\ell} \cap \mathscr{S}_{\omega_j})} \leq K_0 \frac{m^u_x(\omega)}{m^u_x(\gamma \cap \mathscr{S}_{\omega_j})}. \tag{4.7}$$

Doob's theorem ensures that we have

$$\frac{1}{K_0} \frac{m^u_x(\omega)}{m^u_x(\gamma^u(x) \cap \mathscr{S}_{\omega_j})} \leq \mu^u_x(\omega) \leq K_0 \frac{m^u_x(\omega)}{m^u_x(\gamma^u(x) \cap \mathscr{S}_{\omega_j})} \quad (\mu\text{-a.e.}x) \tag{4.8}$$

as $\ell \to \infty$ in (4.7). Since $\mathscr{R}_i \cap \mathscr{S}_{\omega_j}$ is a rectangle $(0 \le i \le r-1, 1 \le j \le q)$, by Lemma 4.3 there exists $K_1 > 0$ such that for any $x \in \mathscr{R}_i$,

$$\frac{1}{K_1} < m_x^u(\gamma^u(x) \cap \mathscr{S}_{\omega_j}) < K_1. \tag{4.9}$$

Therefore we have (4.1) by (4.8) and (4.9). □

Proof of Proposition 4.2. Let μ be the f^R-invariant Borel probability measure on $\mathbf{T}^2 \backslash \mathscr{P}$ in Lemma 4.6. Then

$$\bar{\mu} = \mu + \sum_{i=1}^{\infty} f_*^i(\mu|_{\mathscr{Q}^{(i)}})$$

is a finite measure. To see this, it is enough to prove that $\bar{\mu}(\mathscr{P}) < \infty$.

Let $\pi : \mathscr{Q}^{(1)} \to \gamma^u(p)$ be the projection sliding along local stable manifolds. As in proof of Theorem A, by Lemma 4.3 there exists $L > 0$ such that for any $x \in f^{-1}(\mathscr{P}) \backslash \mathscr{P}$,

$$m_x^u(\mathscr{Q}^{(i)}) \le L m_p^u(\pi(\mathscr{Q}^{(i)})) \quad (i \ge 1). \tag{4.10}$$

By (4.1) there exists $K > 0$ such that

$$\mu(\mathscr{Q}^{(i)}) = \int \mu_x^u(\mathscr{Q}^{(i)})\, d\mu(x) \le K \int m_x^u(\mathscr{Q}^{(i)})\, d\mu(x) \quad (i \ge 1). \tag{4.11}$$

By (4.11), (4.10) and $m_p^u(\pi(\mathscr{Q}^{(i)})) = m_p^u(f^{-i}(\pi(\mathscr{Q}^{(1)})))$,

$$\mu(\mathscr{Q}^{(i)}) \le 2KL \cdot m_p^u(f^{-i}(\pi(\mathscr{Q}^{(1)}))) \quad (i \ge 1). \tag{4.12}$$

By $\mathscr{P} = \bigcup_{i=1}^{\infty} f^i(\mathscr{Q}^{(i)})$, $f^i(\mathscr{Q}^{(i)}) \cap f^j(\mathscr{Q}^{(j)}) = \emptyset$ $(i \ne j)$, $\bar{\mu}(f^i(\mathscr{Q}^{(i)})) = \mu(\mathscr{Q}^{(i)})$ and (4.12), we have

$$\bar{\mu}(\mathscr{P}) = \sum_{i=1}^{\infty} \mu(\mathscr{Q}^{(i)}) \le 2KL \sum_{i=1}^{\infty} m_p^u(f^{-i}(\pi(\mathscr{Q}^{(1)}))).$$

Lemma 4.4 ensures that the last term above converges.

By Lemma 4.6 the normalized measure of $\bar{\mu}$ is an SRB measure. This concludes the proposition. □

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