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MAPPING DEGREE AND EULER CHARACTERISTIC

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Abstract

Let V_δ denote a local level surface for function-germ $f:(\mathbf{R}^{n+1},0)\to(\mathbf{R},0)$. A mapping degree formula for difference of the Euler characteristics of $V_\delta\cap\{g\le 0\}$ and $V_\delta\cap\{g\ge 0\}$ is given, when level surfaces of a function $g:(\mathbf{R}^{n+1},0)\to(\mathbf{R},0)$ are parallelizable.

It is classically known that mapping degree is closely related to Euler characteristics. One of such relation is the following celebrated formula due to G. N. Khimshiashvili ([7]): Let $(x_0,x_1,\dots,x_n)$ be a coordinate system of $\mathbf{R}^{n+1}$. Let B_ε^{n+1} denote the open ball centered at $0\in\mathbf{R}^{n+1}$ with radius ε. Let $f:(\mathbf{R}^{n+1},0)\to(\mathbf{R},0)$ be an analytic function-germ and V_δ denote the local level surface of f, i.e.,

$$V_\delta=B_\varepsilon^{n+1}\cap f^{-1}(\delta)\quad\text{for }\ 0<|\delta|\ll\varepsilon\ll 1.$$

We denote its Euler characteristic by $\chi(V_\delta)$. Then the Khimshiashvili's formula asserts that, when f defines an isolated singularity at 0,

$$\deg(df)=\operatorname{sign}(-\delta)^{n+1}(1-\chi(V_\delta))$$

where df is the map-germ defined by

$$df:(\mathbf{R}^{n+1},0)\to(\mathbf{R}^{n+1},0),\quad x\mapsto(f_{x_0}(x),f_{x_1}(x),\dots,f_{x_n}(x)).$$

Here f_{x_i} denote the partial derivative of f by x_i, $i=0,1,\dots,n$.

We consider a relative version of this formula. In [3], the first author considered the mapping degree of map-germs

$$F:(\mathbf{R}^{n+1},0)\to(\mathbf{R}^{n+1},0),\quad x\mapsto(f(x),f_{x_1}(x),\dots,f_{x_n}(x))$$

and showed that, if F is finite, then

$$\deg(F)=\operatorname{sign}(-\delta)^{n+1}(\chi(V_\delta(x_0\le 0))-\chi(V_\delta(x_0\ge 0))$$

where $V_\delta(x_0\le 0)=\{x\in V_\delta: x_0\le 0\}$, and $V_\delta(x_0\ge 0)=\{x\in V_\delta: x_0\ge 0\}$.

In this paper, we consider an analytic function $g:(\mathbf{R}^{n+1},0)\to(\mathbf{R},0)$ so that there are C^∞-vector fields $v_1,\dots,v_n$ which span the tangent space of a level set

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of g at each regular point of g. We assume that ∇g, $v_1, \dots, v_n$ agree with the orientation of $(\mathbf{R}^{n+1}, 0)$ at each regular point of g where ∇g is the gradient vector of g. We define a map F by

$$F : (\mathbf{R}^{n+1}, 0) \to (\mathbf{R}^{n+1}, 0), \quad x \mapsto (f(x), v_1 f(x), \dots, v_n f(x)).$$

The purpose is to show (Theorem 4.1) that, if F is finite, and $V_\delta \cap \Sigma(g) = \emptyset$, then

$$\deg(F) = \operatorname{sign}(-\delta)^{n+1}(\chi(V_\delta(g \le 0)) - \chi(V_\delta(g \ge 0))) \tag{0.1}$$

where $V_\delta(g \le 0) = \{x \in V_\delta : g(x) \le 0\}$, and $V_\delta(g \ge 0) = \{x \in V_\delta : g(x) \ge 0\}$.

This formula will be proved in §4 applying Morse theory to the restriction of g to a level of f. In §1 we investigate the condition on the existence of such vector fields $v_1, \dots, v_n$ and discuss explicit construction of them in some special case in §2. Applying Morse theory to the restriction of f to a level of g, we also show another topological interpretation of $\deg F$ in §3. In §4 we investigate the condition that $g|_{V_\delta}$ is Morse and give a proof of (0.1) and its variant.

In the last section, we consider a kind of 'product' of dg and df and give a topological interpretation of its mapping degree. It is motivated by Remark 2.1 which is a consequence of the explicit form of F.

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1. Condition (P) and the definition of the map F

Let L denote an oriented $(n+1)$-dimensional C^∞-manifold and $g : L \to \mathbf{R}$ be a C^∞-function on L. We fix a Riemannian metric on L and denote the gradient of g by ∇g. We always consider the orientation of the set of regular points of the level set of g so that ∇g and the orientation of the level set of g agree with the orientation of L.

We consider the following condition on g.

(P): There exist C^∞-vector fields $v_1(x), \dots, v_n(x)$ on L which span the tangent space of the level set of g at a regular point x of g, and the orientation of a level of g there coincides with the orientation defined by $v_1(x), \dots, v_n(x)$.

Definition 1.1. Let $g : L \to \mathbf{R}$ be a C^∞-function with Condition (P). We define the map

$$F : L \to \mathbf{R}^{n+1}, \quad \text{by } x \mapsto (f(x), v_1 f(x), \dots, v_n f(x)),$$

where $f : L \to \mathbf{R}$ is a C^∞-function.

In later sections, we investigate several topological interpretations of the mapping degree of F. In the rest of this section, we investigate Condition (P) in general.

1.1. Existence of vector fields $v_1, \dots, v_n$ in Condition (P). We show the following

PROPOSITION 1.2. *Let $g : (\mathbf{R}^{n+1}, 0) \to (\mathbf{R}, 0)$ be a C^∞-function which defines an isolated singularity at 0. Then, the following conditions are equivalent.*

(i) *There exist C^∞-vector fields $v_1(x), \dots, v_n(x)$ near 0 which span the tangent space of the level set of g at a regular point x of g.*

(ii) *One of the following conditions holds.*
- $n = 1, 3, 7$.
- *n is odd, $n \neq 1, 3, 7$, and* $\deg(dg)$ *is even.*
- *n is even, and* $\deg(dg)$ *is zero.*

First we consider more general set-up. Let L be a manifold of dimension $n+1$, and let $g : L \to \mathbf{R}$ be a C^∞-function. We denote $L' = L - \Sigma(g)$, and assume that L' is parallelizable. Let E denote the vector bundle on L' whose fiber is the tangent space of each level of g. We investigate the following

QUESTION. When E is a trivial bundle?

If E is C^0-trivial, then this bundle is C^∞-trivial and there exist C^∞-vector fields $w_1(x), \dots, w_n(x)$ on L' which span E. Then $v_i(x) = b(x)w_i(x)$, $i = 1, \dots, n$, satisfy Condition (P) where b is a C^∞-function on L so that $\Sigma(g) = b^{-1}(0)$ and that b is flat at $\Sigma(g)$, that is, all partial derivatives of order k, $k = 0, 1, 2, \dots$, vanish at each point of $\Sigma(g)$.

Since L' is parallelizable, there is an oriented orthonormal frame $\boldsymbol{e}_0, \boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ of the tangent bundle of L', and we can define the following Gauss map:

$$\alpha : L' \to S^n, \quad x \mapsto (a_0, a_1, \dots, a_n) \quad \text{where } \frac{\nabla g}{\|\nabla g\|} = a_0\boldsymbol{e}_0 + a_1\boldsymbol{e}_1 + \cdots + a_n\boldsymbol{e}_n.$$

Let $\mathrm{SO}(n)$ denote the group of orthogonal $n \times n$ matrices with determinant 1. Let us consider the map defined by

$$p : \mathrm{SO}(n+1) \to S^n, \quad A \mapsto \text{the first column of } A.$$

PROPOSITION 1.3. *Under the above assumption, the following conditions are equivalent.*

(i) *The vector bundle E is C^0-trivial (and, thus C^∞-trivial).*

(ii) *There is a continuous map $\beta : L' \to SO(n+1)$ so that $\alpha = p \circ \beta$.*

(iii) *One of the following conditions holds.*
- $n = 1, 3, 7$.
- *n is odd, $n \neq 1, 3, 7$, and the induced map $\alpha_\# : \pi_n(L') \to \pi_n(S^n)$ is even.*
- *n is even, and the induced map $\alpha_\# : \pi_n(L') \to \pi_n(S^n)$ is zero.*

Here we say that a map $\alpha : G_1 \to G_2$ between two abelian groups G_1, G_2 is *even* if for any $g_1 \in G_1$ there is $g_2 \in G_2$ with $f(g_1) = 2g_2$.

We say that a map $p : E \to B$ is a *fibration in the sense of Serre* if the following condition holds: for a CW complex X and a homotopy $\alpha_t : X \to B$, $0 \le t \le 1$, if there is a map $\beta_0 : X \to E$ with $p \circ \beta_0 = \alpha_0$, then there is a homotopy $\beta_t : X \to E$, $0 \le t \le 1$, with $p \circ \beta_t = \alpha_t$ for $0 \le t \le 1$.

We remark that the locally trivial fibration is a fibration in the sense of Serre.

Proof of Proposition 1.3. (i) $\Rightarrow$ (ii): If E is trivial, then the associated $\mathrm{SO}(n)$-bundle with E is trivial, and thus have non-zero section. This means (ii).

(ii) $\Rightarrow$ (i): If there is a continuous map $\beta : L' \to \mathrm{SO}(n+1)$ so that $\alpha = p \circ \beta$, then there is an orthonormal frame which spans E, and we thus conclude that E is trivial.

(ii) $\Rightarrow$ (iii): Since the map $p : \mathrm{SO}(n+1) \to S^n$ is a fibration with fiber $\mathrm{SO}(n)$, we have the following homotopy exact sequence:

$$(1.2) \qquad \pi_n(\mathrm{SO}(n+1)) \xrightarrow{p_\#} \pi_n(S^n) \to \pi_{n-1}(\mathrm{SO}(n)) \xrightarrow{\imath_\#} \pi_{n-1}(\mathrm{SO}(n+1)) \to 0$$

where $\imath : \mathrm{SO}(n) \to \mathrm{SO}(n+1)$ denote an inclusion. Remark that the map $\beta : L' \to \mathrm{SO}(n+1)$ induces $\beta_\# : \pi_n(L') \to \pi_n(\mathrm{SO}(n+1))$ with $p_\# \circ \beta_\# = \alpha_\#$. Then the following fact (see [6, Chapter 8, Ex. 8]) implies (iii).

$$(1.3) \qquad \text{Kernel of } \imath_\# = \begin{cases} 0, & \text{if } n = 1, 3, 7; \\ \mathbf{Z}/2\mathbf{Z}, & \text{if } n \text{ is odd and } n \neq 1, 3, 7; \\ \mathbf{Z}, & \text{if } n \text{ is even.} \end{cases}$$

(iii) $\Rightarrow$ (ii): We show this implication as an application of the obstruction theory.

Let S^k, $k = 0, 1, \ldots, n-1$, be a k-dimensional sphere in L', and set $\alpha_k = \alpha|_{S^k}$. The map $\alpha_k : S^k \to S^n$ represents the zero element of $\pi_k(S^n)$, since $\pi_k(S^n) = 0$. Take a map $\beta'_k : S^k \to \mathrm{SO}(n+1)$ which represents the zero element of $\pi_k(\mathrm{SO}(n+1))$. Since the map $p \circ \beta'_k$ also represents zero of $\pi_k(S^n)$, there is a homotopy $\phi_t : S^k \to S^n$, $0 \le t \le 1$, with $\phi_0 = p \circ \beta'_k$ and $\phi_1 = \alpha_k$. Since $p : \mathrm{SO}(n+1) \to S^n$ is a fibration in the sense of Serre, there is a map $\beta_k : S^k \to \mathrm{SO}(n+1)$ so that $p \circ \beta_k = \alpha_k$. If there is a $(k+1)$-dimensional ball B^{k+1} in L' which bounds the sphere S^k in L', then β_k can be extended to B^{k+1}, since β_k represents the zero in $\pi_k(\mathrm{SO}(n+1))$.

Let S^n be an n-dimensional sphere in L' and set $\alpha_n = \alpha|_{S^n}$. By (iii), the homotopy class of α_n is in the kernel of $\imath_\#$ in (1.2), because of (1.3). Since $p : \mathrm{SO}(n+1) \to S^n$ is a fibration in the sense of Serre, there is a map $\beta_n : S^n \to \mathrm{SO}(n+1)$ so that $p \circ \beta_n = \alpha_n$.

Since L' is not compact, L' has a homotopy type of a CW complex of dimension $\le n$, and we complete the proof. $\square$

Remark 1.4. Let $g_1 : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ be a C^∞-function. Let U be a neighborhood of 0 and assume that g_1 is defined on U. Let $\pi : \mathbf{R}^{n+1} \to \mathbf{R}^n$ be a linear projection. Setting $g = g_1 \circ \pi$ and $L = \pi^{-1}(U)$, we have $L' = L - \Sigma(g) \simeq (U - \Sigma(g_1)) \times \mathbf{R}$, that has a homotopy type of a CW complex of dimension $\le$

$n-1$. By the above proof of the implication (iii) $\Rightarrow$ (ii), we conclude that g satisfies Condition (P).

Next we present two propositions which gives sufficient conditions for Condition (P).

Proposition 1.5. *Under the same assumption as Proposition* 1.3, *the vector bundle E is trivial if there is a continuous map $\gamma : L' \to P^n(\mathbf{R})$ so that $\varphi \circ \gamma$ is homotopic to α where $\varphi : P^n(\mathbf{R}) \to S^n$ is the map defined by*

$$[x] = [x_0 : x_1 : \cdots : x_n] \mapsto \frac{1}{S}(2x_0^2 - S, 2x_1x_0, \ldots, 2x_nx_0) \quad \text{where } S = \sum_{i=0}^{n} x_i^2.$$

Let $q : S^n \to P^n(\mathbf{R})$ denote the map defined by $(x_0, x_1, \ldots, x_n) \mapsto [x_0 : x_1 : \cdots : x_n]$. For a unit vector $x = (x_0, x_1, \ldots, x_n)$ and $y = \varphi \circ q(x)$, we see that 0, $\boldsymbol{e}_0$, x, and y are in the same plane and $2\measuredangle \boldsymbol{e}_0 0x = \measuredangle \boldsymbol{e}_0 0y$. We remark that the map φ is generically one-to-one and sends the set defined by $\{x_0 = 0\}$ to a point.

Proof. Let $x = (x_0, x_1, \ldots, x_n)$ be a non-zero vector in $\mathbf{R}^{n+1}$. Let $\psi_x : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ denote the reflection sending the vector x to $-x$. We remark that the map ψ_x is represented by the matrix

$$\left(\delta_{i,j} - \frac{2x_ix_j}{S}\right)_{i,j=0,1,\ldots,n} \quad \text{where } \delta_{i,j} = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{otherwise,} \end{cases}$$

and the first column of the matrix for $\psi_{\boldsymbol{e}_0} \circ \psi_x$ represents the map $\varphi : P^n(\mathbf{R}) \to S^n$. Let h_t denote a homotopy with $h_0 = \varphi \circ \gamma$ and $h_1 = \alpha$. We remark that there is a continuous map $\gamma_1 : L' \to \mathrm{SO}(n+1)$ with $\varphi = p \circ \gamma_1$. In fact, the map $\gamma_1 = \psi \circ \gamma$ satisfy $\varphi = p \circ \gamma_1$ where $\psi : P^n(\mathbf{R}) \to \mathrm{SO}(n+1)$ is the embedding defined by $[x] \mapsto \psi_{\boldsymbol{e}_0} \circ \psi_x$. Since $p : \mathrm{SO}(n+1) \to S^n$ is a fibration in the sense of Serre, we obtain there is a continuous map $\alpha_1 : L' \to \mathrm{SO}(n+1)$ with $\alpha = p \circ \alpha_1$, and we complete the proof. □

Proposition 1.6. *Under the same assumption as Proposition* 1.3, *the vector bundle E is trivial when one of the following conditions holds.*

- *n is odd, and the induced map $\alpha^* : H^n(S^n; \mathbf{Z}) \to H^n(L'; \mathbf{Z})$ is even.*
- *n is even, and the induced map $\alpha^* : H^n(S^n; \mathbf{Z}) \to H^n(L'; \mathbf{Z})$ is zero.*

Proof. Assume first that n is even and the induced map $\alpha^* : H^n(S^n; \mathbf{Z}) \to H^n(L'; \mathbf{Z})$ is zero. Then, by Hopf's theorem (see [5, Chapter II, 8]) there is a homotopy $A : L' \times [0,1] \to S^n$, $A(x,t) = \alpha_t(x)$, with the following properties:

- $\alpha_0 = \alpha$.
- If n is odd, then there are continuous maps $a : L' \to S^n$ and $b : S^n \to S^n$ so that b is of degree two and $\alpha = b \circ a$. We may assume that b factors through the map φ.

• If n is even, then $\operatorname{Im}\alpha_1$ is a point.
Since $p : \mathrm{SO}(n+1) \to S^n$ is a fibration in the sense of Serre, we complete the proof as in the same way in the previous proposition. □

Proof of Proposition 1.2. (i) $\Rightarrow$ (ii): If (i) holds, then (iii) of Proposition 1.3 holds, and (ii) holds.

(ii) $\Rightarrow$ (i): The implication (iii) $\Rightarrow$ (i) of Proposition 1.3 implies (ii) $\Rightarrow$ (i). The explicit construction of $v_1, \dots, v_n$ in the next section gives another proof when $n = 1, 3, 7$. When $n \neq 1, 3, 7$, Proposition 1.6 also gives another proof by Hopf's theorem (ibid.). □

2. Explicit construction of vector fields $v_1, \dots, v_n$ in Condition (P)

Let $g : (\mathbf{R}^{n+1}, 0) \to (\mathbf{R}, 0)$ be a polynomial (resp. analytic) function. Assume that one of the conditions in Proposition 1.3 (iii) (or in Proposition 1.2 (ii) when g defines isolated singularity at 0) holds. Then are there polynomial (resp. analytic) vector fields $v_1, \dots, v_n$ which span the tangent space of the level of g at a regular point of g? The answer is affirmative if one of the following conditions holds.

(a) $n = 1, 3, 7$.

(b) g_{x_0} is not negative.

We are going to prove this assertion to construct vector field $v_1, \dots, v_n$ explicitly. Let $L = \mathbf{R}^{n+1}$ and we denote by ∂_{x_i} the unit vector $\boldsymbol{e}_i = (0, \dots, \overset{i+1}{1}, \dots, 0)$ for $i = 0, 1, \dots, n$.

2.1. Case (a). If $n = 1, 3, 7$, our explicit construction of the vector fields $v_1, \dots, v_n$ is based on the multiplicative structure of complex, quotanion, Cayley numbers, respectively.

Case $n = 1$: We consider the complex numbers $\mathbf{C} = \mathbf{R} + \mathbf{R}\boldsymbol{i}$ where $\boldsymbol{i}^2 = -1$, and identify it with $\mathbf{R}^2$. Under this identification $\nabla g = g_{x_0} + g_{x_1}\boldsymbol{i}$. Then $\boldsymbol{i}\nabla g = -g_{x_1} + g_{x_0}\boldsymbol{i}$ span the tangent space of the level set of g at a regular point of g. In other words, the vector field v_1 in Condition (P) is given by the following:

$$v_1 = \boldsymbol{i}\nabla g = -g_{x_1}\partial_{x_0} + g_{x_0}\partial_{x_1}.$$

Case $n = 3$: We consider the quotanion numbers $Q = \mathbf{R} + \mathbf{R}\boldsymbol{i} + \mathbf{R}\boldsymbol{j} + \mathbf{R}\boldsymbol{k}$ with

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1, \quad \boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k}, \quad \boldsymbol{jk} = -\boldsymbol{kj} = \boldsymbol{i}, \quad \boldsymbol{ki} = -\boldsymbol{ik} = \boldsymbol{j}.$$

We set $\bar{x} = a_0 - a_1\boldsymbol{i} - a_2\boldsymbol{j} - a_3\boldsymbol{k}$ when $x = a_0 + a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$. Since $x\bar{x} = \sum_{i=0}^{3} a_i^2$, x has the inverse $\bar{x}/(x\bar{x})$ if $x \neq 0$. We identify Q with $\mathbf{R}^4$. We remark that $\langle x, y\rangle := \mathrm{Re}(x\bar{y})$ $(x, y \in Q)$ is the Euclidean inner product of $\mathbf{R}^4$. Under this identification we have that $\nabla g = g_{x_0} + g_{x_1}\boldsymbol{i} + g_{x_2}\boldsymbol{j} + g_{x_3}\boldsymbol{k}$. Since $(1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ forms an orthonormal frame of the tangent space of $\mathbf{R}^4$, $(\nabla g, \boldsymbol{i}\nabla g, \boldsymbol{j}\nabla g, \boldsymbol{k}\nabla g)$

forms also an orthogonal frame of the tangent space of $\mathbf{R}^4$, when $\nabla g \neq 0$. This implies that $\boldsymbol{i}\nabla g$, $\boldsymbol{j}\nabla g$, $\boldsymbol{k}\nabla g$ span the tangent space of the level set of g at a regular point of g. In other words, the vector fields v_1, v_2, v_3 in Condition (P) are given by the following:

$$\begin{aligned} v_1 &= \boldsymbol{i}\nabla g = -g_{x_1}\partial_{x_0} + g_{x_0}\partial_{x_1} - g_{x_3}\partial_{x_2} + g_{x_2}\partial_{x_3}, \\ v_2 &= \boldsymbol{j}\nabla g = -g_{x_2}\partial_{x_0} + g_{x_3}\partial_{x_1} + g_{x_0}\partial_{x_2} - g_{x_1}\partial_{x_3}, \\ v_3 &= \boldsymbol{k}\nabla g = -g_{x_3}\partial_{x_0} - g_{x_2}\partial_{x_1} + g_{x_1}\partial_{x_2} + g_{x_0}\partial_{x_3}. \end{aligned}$$

Case $n = 7$: We consider Cayley numbers $\mathfrak{C} = Q + Qe$ with

$$(q + re)(s + te) = (qs - \bar{t}r) + (tq + r\bar{s})e, \quad q, r, s, t \in Q.$$

We set $\bar{x} = \bar{q} - re$ when $x = q + re$. Since $x\bar{x} = q\bar{q} + r\bar{r}$, x has the inverse $\bar{x}/(x\bar{x})$ if $x \neq 0$. We identify $\mathfrak{C}$ with $\mathbf{R}^8$ and remark that $\langle x, y \rangle := \mathrm{Re}(x\bar{y})$ $(x, y \in \mathfrak{C})$ is the Euclidean inner product of $\mathbf{R}^8$. Under this identification we have that $\nabla g = g_{x_0} + g_{x_1}\boldsymbol{i} + g_{x_2}\boldsymbol{j} + g_{x_3}\boldsymbol{k} + (g_{x_4} + g_{x_5}\boldsymbol{i} + g_{x_6}\boldsymbol{j} + g_{x_7}\boldsymbol{k})e$. Then $\boldsymbol{i}\nabla g$, $\boldsymbol{j}\nabla g$, $\boldsymbol{k}\nabla g$, $e\nabla g$, $\boldsymbol{i}e\nabla g$, $\boldsymbol{j}e\nabla g$, $\boldsymbol{k}e\nabla g$ span the tangent space of the level set of g at a regular point of g. In other words, the vector fields $v_1, \dots, v_7$ in Condition (P) are given by the following:

$$\begin{aligned} v_1 &= \boldsymbol{i}\nabla g = -g_{x_1}\partial_{x_0} + g_{x_0}\partial_{x_1} - g_{x_3}\partial_{x_2} + g_{x_2}\partial_{x_3} - g_{x_5}\partial_{x_4} + g_{x_4}\partial_{x_5} + g_{x_7}\partial_{x_6} - g_{x_6}\partial_{x_7}, \\ v_2 &= \boldsymbol{j}\nabla g = -g_{x_2}\partial_{x_0} + g_{x_3}\partial_{x_1} + g_{x_0}\partial_{x_2} - g_{x_1}\partial_{x_3} - g_{x_6}\partial_{x_4} - g_{x_7}\partial_{x_5} + g_{x_4}\partial_{x_6} + g_{x_5}\partial_{x_7}, \\ v_3 &= \boldsymbol{k}\nabla g = -g_{x_3}\partial_{x_0} - g_{x_2}\partial_{x_1} + g_{x_1}\partial_{x_2} + g_{x_0}\partial_{x_3} - g_{x_7}\partial_{x_4} + g_{x_6}\partial_{x_5} - g_{x_5}\partial_{x_6} + g_{x_4}\partial_{x_7}, \\ v_4 &= e\nabla g = -g_{x_4}\partial_{x_0} + g_{x_5}\partial_{x_1} + g_{x_6}\partial_{x_2} + g_{x_7}\partial_{x_3} + g_{x_0}\partial_{x_4} - g_{x_1}\partial_{x_5} - g_{x_2}\partial_{x_6} - g_{x_3}\partial_{x_7}, \\ v_5 &= \boldsymbol{i}e\nabla g = -g_{x_5}\partial_{x_0} - g_{x_4}\partial_{x_1} + g_{x_7}\partial_{x_2} - g_{x_6}\partial_{x_3} + g_{x_1}\partial_{x_4} + g_{x_0}\partial_{x_5} + g_{x_3}\partial_{x_6} - g_{x_2}\partial_{x_7}, \\ v_6 &= \boldsymbol{j}e\nabla g = -g_{x_6}\partial_{x_0} - g_{x_7}\partial_{x_1} - g_{x_4}\partial_{x_2} + g_{x_5}\partial_{x_3} + g_{x_2}\partial_{x_4} - g_{x_3}\partial_{x_5} + g_{x_0}\partial_{x_6} + g_{x_1}\partial_{x_7}, \\ v_7 &= \boldsymbol{k}e\nabla g = -g_{x_7}\partial_{x_0} + g_{x_6}\partial_{x_1} - g_{x_5}\partial_{x_2} - g_{x_4}\partial_{x_3} + g_{x_3}\partial_{x_4} + g_{x_2}\partial_{x_5} - g_{x_1}\partial_{x_6} + g_{x_0}\partial_{x_7}. \end{aligned}$$

Remark 2.1. In the above construction, the map F (Definition 1.1) coincides with

$$p(\overline{\nabla g}, \nabla f) : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}, \quad x \mapsto p(\overline{\nabla g(x)}, \nabla f(x))$$

except the first component, where $p : \mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ is the product of the complex, quotanion, Cayley numbers, respectively. In fact, the $\boldsymbol{e}_i$ component, $i = 1, \dots, n$, of F is $\langle \boldsymbol{e}_i\nabla g, \nabla f \rangle = \mathrm{Re}(p(-\boldsymbol{e}_i\overline{\nabla g}, \nabla f)) = \mathrm{Re}(p(-\boldsymbol{e}_i, p(\overline{\nabla g}, \nabla f))$, which is the $\boldsymbol{e}_i$ component of $p((\overline{\nabla g}, \nabla f))$. Here we use the fact $\mathrm{Re}((ab)c) = \mathrm{Re}(a(bc))$ for any complex, quotanion, Cayley numbers a, b, c, respectively.

2.2. Case (b). Assume that g_{x_0} is not negative. This means that the mapping degree of dg is zero. We define vector fields v_i, $i = 1, \dots, n$, by

$$v_i = g_{x_i}\partial_{x_0} + \sum_{j=1}^{n}(g_{x_i}g_{x_j} - \delta_{i,j}T)\partial_{x_j} \quad \text{where } T = g_{x_0} + \sum_{j=1}^{n} g_{x_j}^2.$$

Then $v_1, \dots, v_n$ span the tangent space of each level of g at each regular point of g.

It is clear that these $v_1, \dots, v_n$ are polynomial (resp. analytic) vector fields when g is a polynomial (resp. analytic).

Proof. It is easy to see that $\langle \nabla g, v_i \rangle = 0$ for $i = 1, \dots, n$. So it is enough to show that ∇g, $v_1, \dots, v_n$ are linearly independent on $\mathbf{R}^n - \Sigma(g)$. The coefficient matrix of vector fields ∇g, $v_1, \dots, v_n$ is

$$M = \begin{pmatrix} g_{x_0} & g_{x_i} \\ g_{x_j} & g_{x_i}g_{x_j} - \delta_{i,j}T \end{pmatrix}_{i,j=1,\dots,n}$$

and its determinant is $T^{n-1}\sum_{i=0}^{n} g_{x_i}^2$. This implies that ∇g, $v_1, \dots, v_n$ are linearly dependent only on $\{T = 0\} \cup \Sigma(g)$. By assumption $\{T = 0\} \cup \Sigma(g) = \Sigma(g)$, and we are done. $\square$

Remark that $M\boldsymbol{e}_0 = \nabla g$, $M\nabla g = \|\nabla g\|^2 \boldsymbol{e}_0$, and $Mv = -Tv$ when $\langle v, \boldsymbol{e}_0 \rangle = \langle v, \nabla g \rangle = 0$.

Remark 2.2. The matrix appeared in the proof of Proposition 1.5 suggests another explicit construction of the vector field $v_1, \dots, v_n$ in some special case. Let us find an x with $\varphi(x) = \nabla g / \|\nabla g\|$ where ∇g denotes the gradient of g. Looking the first component, we have $\dfrac{2x_0^2}{S} - 1 = \dfrac{g_{x_0}}{\|\nabla g\|}$ and

$$\left(1 - \frac{g_{x_0}}{\|\nabla g\|}\right)x_0^2 = (x_1^2 + \dots + x_n^2)\left(1 + \frac{g_{x_0}}{\|\nabla g\|}\right).$$

We then obtain

$$\begin{aligned}\left(\frac{x_0}{1 + \dfrac{g_{x_0}}{\|\nabla g\|}}\right)^2 &= \frac{x_1^2 + \dots + x_n^2}{\left(\dfrac{g_{x_1}}{\|\nabla g\|}\right)^2 + \dots + \left(\dfrac{g_{x_n}}{\|\nabla g\|}\right)^2} \\ &= \frac{x_1^2 + \dots + x_n^2}{\left(\dfrac{2x_1x_0}{S}\right)^2 + \dots + \left(\dfrac{2x_nx_0}{S}\right)^2} = \left(\frac{S}{2x_0}\right)^2.\end{aligned}$$

We thus conclude

$$(x_0, x_1, \dots, x_n) = k(\nabla g \pm \|\nabla g\|\boldsymbol{e}_0) \quad \text{where } k = \frac{S}{2x_0\|\nabla g\|}.$$

Choosing the sign $+$, and setting $k = 1$, we have

$$v_i := \varphi(\boldsymbol{e}_i) = \frac{1}{\|\nabla g\|}\left(g_{x_0}\partial_{x_0} + \sum_{j=1}^{n}\left(\frac{g_{x_i}g_{x_j}}{\|\nabla g\| + g_{x_0}} + \delta_{i,j}\|\nabla g\|\right)\partial_{x_j}\right), \quad i = 1, \dots, n.$$

They are the desired vector fields which make sense whenever $\nabla g + \|\nabla g\|\boldsymbol{e}_0 \neq 0$. Remark that the last condition implies the mapping degree of dg is zero. But, in this construction, it is not clear that $v_1, \dots, v_n$ are polynomial (resp. analytic) vector fields when g is a polynomial (resp. analytic).

3. Restricting f to the level of g

THEOREM 3.1. *Let L be a C^∞-manifold of dimension $n+1$ and $f, g : L \to \mathbf{R}$ C^∞-functions. We assume that 0 is a regular value of $g : L \to \mathbf{R}$ and set $N = g^{-1}(0)$. We assume that g satisfies Condition (P) and the map*

$$\bar{F} : N \to S^n, \quad x \mapsto \frac{(f(x), v_1 f(x), \dots, v_n f(x))}{\|(f(x), v_1 f(x), \dots, v_n f(x))\|},$$

is well-defined and finite.

(i) *If $L_+ = \{x \in L : f(x) \geq 0\}$ is compact, then*

$$\deg \bar{F} = \chi(N(f \geq 0), N(f = 0))$$

where $N(f \geq 0)$ denotes the set $\{x \in N : f(x) \geq 0\}$, and so on.

(ii) *If $L_- = \{x \in L : f(x) \leq 0\}$ is compact, then we obtain*

$$\deg \bar{F} = (-1)^{n+1}\chi(N(f \leq 0), N(f = 0)).$$

Proof. Take the point $(1, 0, \dots, 0)$ and consider its preimage by $\bar{F}$. They are the critical points of $f : N \to \mathbf{R}$ in the region $\{f > 0\}$. If $f|_N$ is Morse (we can assume this after small perturbation of f if necessary), we obtain

$$\mathrm{Hess}(f|_N)(x) = \frac{\partial \bar{F}}{\partial y}(x),$$

where y denotes an oriented coordinate system of N. This implies the first equality.

Next take the point $(-1, 0, \dots, 0)$ and apply the similar discussion for $-f$ on the region $\{f \leq 0\}$. We then obtain the second equality. □

When F induces a finite map germ $F_0 : (L, F^{-1}(0)) \to (\mathbf{R}^{n+1}, 0)$, $\deg \bar{F} = \deg F_0$.

Remark 3.2. Assume that L is compact. If n is odd, we have that $\deg \bar{F} = \frac{1}{2}\chi(N(f = 0))$ and $\chi(N(f \geq 0)) = \chi(N(f \leq 0))$. We consider the following Gauss map

$$G : N(f = 0) \to S^{n-1}, \quad x \mapsto \frac{(v_1 f(x), \dots, v_n f(x))}{\|(v_1 f(x), \dots, v_n f(x))\|}.$$

Using the fact stated in [8, §6], we obtain that the degree of this Gauss map is equal to the sum of indices of ∇f in $N(f \geq 0)$, which is equal to $\deg \overline{F}$. So we conclude that $\deg G = \frac{1}{2}\chi(N(f = 0))$.

4. Restricting g to the level of f

THEOREM 4.1. *Let $f, g : B_\varepsilon^{n+1} \to \mathbf{R}$ be analytic functions with $f(0) = g(0) = 0$. We assume that the singular set of (f, g), which is defined by*

$$X = \left\{ x \in B_\varepsilon^{n+1} : \operatorname{rank}\begin{pmatrix} f_{x_0}(x) & f_{x_1}(x) & \cdots & f_{x_n}(x) \\ g_{x_0}(x) & g_{x_1}(x) & \cdots & g_{x_n}(x) \end{pmatrix} < 2 \right\},$$

is of dimension 1. *We choose $\varepsilon > 0$ small enough so that*

- *the number of connected components of $(X - \{0\}) \cap B_{\varepsilon'}^{n+1}$ does not change if $0 < \varepsilon' \leq \varepsilon$, and*
- *the functions f and g do not change the sign on each connected component of $X - \{0\}$.*

We choose δ, a regular value of f, which is close enough to 0, *and set $V_\delta = \{x \in B_\varepsilon^{n+1} : f(x) = \delta\}$. We assume that g satisfies Condition (P). If $V_\delta \cap \Sigma(g) = \emptyset$, $g|_{V_\delta}$ is a Morse function, and the map-germ*

$$F : (\mathbf{R}^{n+1}, 0) \to (\mathbf{R}^{n+1}, 0), \quad x \mapsto (f(x), v_1 f(x), \dots, v_n f(x)), \tag{4.1}$$

is finite, then we have the following:

$$\deg(F) = \operatorname{sign}(-\delta)^{n+1}(\chi(V_\delta(g \leq 0)) - \chi(V_\delta(g \geq 0))) \tag{4.2}$$

$$= \operatorname{sign}(-\delta)^{n+1}(\chi(\overline{V}_{\operatorname{sign}(\delta)-}) - \chi(\overline{V}_{\operatorname{sign}(\delta)+})) \tag{4.3}$$

Here we denote by $V_\delta(g \leq 0)$ the set $\{x \in V_\delta : g(x) \leq 0\}$, and so on. We also denote by $\overline{V}_{\operatorname{sign}(\delta)\pm}$ the set $\{x \in S_\varepsilon^n : \operatorname{sign}(\delta) f(x) \geq 0, \pm g(x) \geq 0\}$ for $0 < \varepsilon \ll 1$.

Remark 4.2. Consider the jet space $J = J^1(\mathbf{R}^{n+1}, \mathbf{R}^2)$ with coordinates

$$(x_0, x_1 \dots, x_n, y, z, p_0, p_1, \dots, p_n, q_0, q_1, \dots, q_n),$$

so that the jet section of a map $(f, g) : \mathbf{R}^{n+1} \to \mathbf{R}^2$ is defined by

$$y = f(x), \quad p_i = f_{x_i}(x), \quad z = g(x), \quad q_i = g_{x_i}(x), \quad i = 0, 1, \dots, n.$$

Let Σ_i, $i = 0, 1, 2$, be the submanifolds of the jet space J defined by

$$\operatorname{rank}\begin{pmatrix} p_0 & p_1 & \cdots & p_n \\ q_0 & q_1 & \cdots & q_n \end{pmatrix} = i.$$

If the map $(f, g) : (\mathbf{R}^{n+1}, 0) \to (\mathbf{R}^2, 0)$ is transverse to Σ_0, Σ_1 and Σ_2 on $(\mathbf{R}^{n+1} - 0, 0)$, then the singular set X of (f, g) is of dimension 1, and the number of connected components of $(X - \{0\}) \cap B^{n+1}_{\varepsilon'}$ does not change if $0 < \varepsilon' \ll 1$. This means that the condition on X is a generic condition, if $\varepsilon > 0$ is small enough.

Lemma 4.3. *Let $\gamma : (\mathbf{R}, 0) \to (X, 0)$ be a C^∞-map with $f \circ \gamma(t) \neq 0$ when $0 < |t| \ll 1$. We then obtain $\nabla f(\gamma(t))$ is not identically zero. Since $\gamma(t) \in X$, there are real numbers $\lambda(t)$ so that $\nabla g(\gamma(t)) = \lambda(t) \nabla f(\gamma(t))$, when $0 < |t| \ll 1$.*

(i) *If $g \circ \gamma(t)$ is identically zero, then $\lambda(t)$ is also identically zero.*

(ii) *If $g \circ \gamma(t)$ is not identically zero, then* $\operatorname{sign} \lambda = \operatorname{sign}(g/f)$ *along $\gamma(t)$, $0 < |t| \ll 1$.*

Proof. Assume that $\nabla f(\gamma(t))$ is identically zero. We then have

$$\frac{d}{dt} f \circ \gamma(t) = \sum_{i=0}^{n} \frac{\partial f}{\partial x_i}(\gamma(t)) \frac{d}{dt}(x_i \circ \gamma(t)) \equiv 0,$$

which implies $f \circ \gamma(t)$ is constant. This shows the first assertion. If $g \circ \gamma(t)$ is identically zero, then

$$\begin{aligned} 0 = \frac{d}{dt}(f \circ \gamma(t) \cdot g \circ \gamma(t)) &= g \circ \gamma(t) \frac{d}{dt} f \circ \gamma(t) + f \circ \gamma(t) \frac{d}{dt} g \circ \gamma(t) \\ &= \lambda(t) f \circ \gamma(t) \frac{d}{dt} f \circ \gamma(t) \end{aligned}$$

and we conclude $\lambda(t)$ is identically zero. This completes the proof of (i). The assertion (ii) is a consequence of Cauchy's mean value theorem. □

Take a point $x \in X - \{0\}$.

• If $\delta = f(x)$ is a regular value of f, then x is a critical point of $g|_{\{f=\delta\}}$.

• If $\delta' = g(x)$ is a regular value of g, then x is a critical point of $f|_{\{g=\delta'\}}$.

The following lemma clarifies when $g|_{V_\delta}$ is a Morse function.

Lemma 4.4. *Let δ be a regular value of f. For $x \in X \cap V_\delta$ there exists a real number λ so that $\nabla g(x) = \lambda \nabla f(x)$. Then $g|_{V_\delta}$ is Morse at x, if and only if*

$$\left| \begin{matrix} 0 & f_{x_j} \\ f_{x_i} & g_{x_i x_j} - \lambda f_{x_i x_j} \end{matrix} \right|_{i,j=0,1,\dots,n} \neq 0 \quad \text{at } x.$$

Proof. It is enough to prove the lemma assuming $f_{x_0}(x) \neq 0$. Then there is a function $\varphi(x_1, \dots, x_n)$ with

$$f(\varphi(x_1, \dots, x_n), x_1, \dots, x_n) \equiv \delta. \tag{4.4}$$

Differentiating (4.4) by x_i, $i = 1, \dots, n$, we obtain

$$f_{x_0} \varphi_{x_i} + f_{x_i} \equiv 0, \tag{4.5}$$

and $\varphi_{x_i} = -f_{x_0}^{-1} f_{x_i}$. Differentiating (4.5) by x_j, $j = 1, \dots, n$, we obtain that

$$f_{x_0 x_0}\varphi_{x_i}\varphi_{x_j} + f_{x_0 x_j}\varphi_{x_i} + f_{x_0 x_i}\varphi_{x_j} + f_{x_i x_j} + f_{x_0}\varphi_{x_i x_j} \equiv 0. \tag{4.6}$$

We consider the Hessian of the function $G(x_1, \dots, x_n) := g(\varphi(x_1, \dots, x_n), x_1, \dots, x_n)$ at its critical point x. Similar computation shows that $G_{x_i} = g_{x_0}\varphi_{x_i} + g_{x_i}$, $i = 1, \dots, n$, and $\lambda = g_{x_0} f_{x_0}^{-1}$ at $x \in X$. We also obtain that

$$\begin{aligned} G_{x_i x_j} &= g_{x_0 x_0}\varphi_{x_i}\varphi_{x_j} + g_{x_0 x_j}\varphi_{x_i} + g_{x_0 x_i}\varphi_{x_j} + g_{x_i x_j} + g_{x_0}\varphi_{x_i, x_j} \\ &= (g_{x_0 x_0} - \lambda f_{x_0 x_0})\varphi_{x_i}\varphi_{x_j} + (g_{x_0 x_j} - \lambda f_{x_0 x_j})\varphi_{x_i} + (g_{x_0 x_i} - \lambda f_{x_0 x_i})\varphi_{x_j} + (g_{x_i x_j} - \lambda f_{x_i x_j}) \end{aligned}$$

at x by (4.6). Therefore we conclude that

$$\begin{aligned} \det(G_{x_i x_j})_{i,j=1,\dots,n} &= \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & G_{x_i x_j} \end{vmatrix}_{i,j=1,\dots,n} \\ &= \begin{vmatrix} -1 & 0 & 0 & (g_{x_0 x_0} - \lambda f_{x_0 x_0})\varphi_{x_j} \\ 0 & 0 & -1 & \varphi_{x_j} \\ 0 & -1 & 0 & g_{x_0 x_j} - \lambda f_{x_0 x_j} \\ \varphi_{x_i} & \varphi_{x_i} & g_{x_0 x_i} - \lambda f_{x_0 x_i} & g_{x_i x_j} - \lambda f_{x_i x_j} \end{vmatrix}_{i,j=1,\dots,n} \\ &= \begin{vmatrix} -1 & 0 & g_{x_0 x_0} - \lambda f_{x_0 x_0} & 0 \\ 0 & 0 & -1 & \varphi_{x_j} \\ 1 & -1 & 0 & g_{x_0 x_j} - \lambda f_{x_0 x_j} \\ 0 & \varphi_{x_i} & g_{x_0 x_i} - \lambda f_{x_0 x_i} & g_{x_i x_j} - \lambda f_{x_i x_j} \end{vmatrix}_{i,j=1,\dots,n} \\ &= \begin{vmatrix} -1 & 0 & g_{x_0 x_0} - \lambda f_{x_0 x_0} & 0 \\ 0 & 0 & -1 & \varphi_{x_j} \\ 0 & -1 & g_{x_0 x_0} - \lambda f_{x_0 x_0} & g_{x_0 x_j} - \lambda f_{x_0 x_j} \\ 0 & \varphi_{x_i} & g_{x_0 x_i} - \lambda f_{x_0 x_i} & g_{x_i x_j} - \lambda f_{x_i x_j} \end{vmatrix}_{i,j=1,\dots,n} \\ &= -f_{x_0}^{-2} \begin{vmatrix} 0 & f_{x_j} \\ f_{x_i} & g_{x_i x_j} - \lambda f_{x_i x_j} \end{vmatrix}_{i,j=0,1,\dots,n}, \end{aligned}$$

at x, which completes the proof. □

If x is a regular point of f and g, then we have

LEMMA 4.5. sign $\mathrm{Hess}(f|_{\{g=\delta'\}}) = \mathrm{sign}((-\lambda)^n \,\mathrm{Hess}(g|_{\{f=\delta\}}))$ *at* $x \in X$ *near* 0.

Proof. Since x is a regular point of g, there exists a coordinate system $(x_0, x_1, \dots, x_n)$ centered at x so that $x_0 = g(x)$. We consider f as a functions of $(x_0, x_1, \dots, x_n)$ and write $f = f(x_0, x_1, \dots, x_n)$. By implicit function theorem, there exists a C^∞-function $\psi(x_1, \dots, x_n)$ so that $f(\psi(x_1, \dots, x_n), x_1, \dots, x_n) = \delta$. Then we obtain that

$$f_{x_0}\psi_{x_i} + f_{x_i} = 0, \quad \text{for } i = 1, \dots, n, \quad \text{and}$$
$$f_{x_0}\psi_{x_i x_j} + f_{x_i x_j} \equiv 0 \bmod \psi_{x_i}, \quad \text{for } i, j = 1, \dots, n.$$

This means $f_{x_i x_j}(x) = -f_{x_0}\psi_{x_i x_j}(x)$, which implies the lemma. □

Proof of Theorem 4.1. We choose a non-zero number δ close enough to 0 so that the numbers of connected components of $\{x \in X : 0 < \operatorname{sign}(\delta) f(x) < \varepsilon\}$ do not depend on ε with $0 < \varepsilon < |\delta|$. Let $\alpha(\varepsilon)$, $0 < \operatorname{sign}(\delta)\varepsilon < |\delta|$, denote the half-branch of X which contains x. We assume that $f(\alpha(\varepsilon)) = \varepsilon$. We extend the function ε to a neighborhood of X near x and denote it by the same letter ε. We consider functions $g_1, \dots, g_n$ so that $X = \{g_1 = \cdots = g_n = 0\}$ near x and so that $\nabla g_i(x) = v_i(x)$ for $i = 1, \dots, n$. We see that $(\varepsilon, g_1, \dots, g_n)$ and $(g, g_1, \dots, g_n)$ are systems of coordinates near x. Then we obtain that

$$\frac{\partial F}{\partial(g, g_1, \dots, g_n)} = \frac{\partial(\varepsilon, g_1, \dots, g_n)}{\partial(g, g_1, \dots, g_n)} \frac{\partial F}{\partial(\varepsilon, g_1, \dots, g_n)} = \lambda^{-1} \begin{vmatrix} 1 & * \\ 0 & v_i v_j f \end{vmatrix} \quad \text{at } x,$$

since $v_i f(\alpha(\varepsilon)) = 0$ and $\langle \dot{\alpha}, \nabla f \rangle = 1$. By Lemma 4.5, we conclude that

$$\operatorname{sign} \frac{\partial F}{\partial(g, g_1, \dots, g_n)} = \operatorname{sign}((-\lambda)^{n+1} \operatorname{Hess}(g|_{\{f=\delta\}})) \quad \text{at } x. \tag{4.7}$$

Applying Morse theory to g on $\{f = \delta, g \geq 0\}$, we obtain that

$$\chi(V_\delta(g \geq 0), V_\delta(g = 0)) = \sum_{x \in X \cap V_\delta : g(x) > 0} \operatorname{sign} \operatorname{Hess}(g|_{\{f=\delta\}})(x).$$

Applying Morse theory to $-g$ on $\{f = \delta, g \leq 0\}$, we also obtain that

$$\chi(V_\delta(g \leq 0), V_\delta(g = 0)) = (-1)^n \sum_{x \in X \cap V_\delta : g(x) < 0} \operatorname{sign} \operatorname{Hess}(g|_{\{f=\delta\}})(x).$$

Taking the difference, we thus conclude that

$$\chi(V_\delta(g \geq 0)) - \chi(V_\delta(g \leq 0)) = \sum_{x \in X \cap V_\delta, g(x) \neq 0} \operatorname{sign}(g(x))^{n+1} \operatorname{Hess}(g|_{\{f=\delta\}})(x). \tag{4.8}$$

By Lemma 4.3 (i), the condition $V_\delta \cap \Sigma(g) = \emptyset$ implies that 0 is a regular value of $g|_{V_\delta}$, and we have

$$(4.8) = \sum_{x \in X \cap V_\delta} \operatorname{sign}(g(x))^{n+1} \operatorname{Hess}(g|_{\{f=\delta\}})(x). \tag{4.9}$$

By Lemma 4.3 (ii), $\operatorname{sign}(f\lambda) = \operatorname{sign}(g)$ along each connected component of $X - \{0\}$, and we obtain that

$$\begin{aligned}
(4.9) &= \sum_{x \in X \cap V_\delta} \operatorname{sign}(f(x)\lambda)^{n+1} \operatorname{Hess}(g|_{\{f=\delta\}})(x) \\
&= \operatorname{sign}(-\delta)^{n+1} \sum_{x \in X \cap V_\delta} \operatorname{sign}(-\lambda)^{n+1} \operatorname{Hess}(g|_{\{f=\delta\}})(x) \\
&= \operatorname{sign}(-\delta)^{n+1} \sum_{x \in X \cap V_\delta} \operatorname{sign} \frac{\partial F}{\partial(g, g_1, \dots, g_n)}(x) \quad (\text{by } (4.7)) \\
&= \operatorname{sign}(-\delta)^{n+1} \deg F,
\end{aligned}$$

which implies the formula (4.2). The equality (4.3) follows from the deformation argument due to [9, §11]. □

Corollary 4.6. *Let V be an analytic set of dimension $n+1$ defined near 0 in $\mathbf{R}^{m+n+1}$. Let L be the nonsingular locus of $V \cap B_\varepsilon^{m+n+1}$ for small $\varepsilon > 0$ and assume that L is oriented. Let $g : (\mathbf{R}^{m+n+1}, 0) \to (\mathbf{R}, 0)$ be an analytic function-germ. We assume that there are C^∞-vector fields $v_1(x), \dots, v_n(x)$ on B_ε^{m+n+1} so that $v_1(x), \dots, v_n(x)$ span the tangent space of $g|_L$ at each $x \in L$ and the orientation of the level of $g|_L$ there coincides with the orientation defined by $v_1(x), \dots, v_n(x)$. Let $f : (\mathbf{R}^{m+n+1}, 0) \to (\mathbf{R}, 0)$ be an analytic function-germ. We assume that*

$$V_\delta = \{x \in V \cap B_\varepsilon : f(x) = \delta\}$$

is nonsingular for a non-zero number δ which is sufficiently close to 0. If $V_\delta \cap \Sigma(g) = \emptyset$, the map-germ

$$F : (L, 0) \to (\mathbf{R}^{n+1}, 0), \quad x \mapsto (f(x), v_1 f(x), \dots, v_n f(x)).$$

is finite and $g|_{V_\delta}$ is Morse, then

$$\begin{aligned}
(4.10) \qquad \deg(F) &= \operatorname{sign}(-\delta)^{n+1}(\chi(V_\delta(g \le 0)) - \chi(V_\delta(g \ge 0))) \\
&= \operatorname{sign}(-\delta)^{n+1}(\chi(\overline{V}_{\operatorname{sign}(\delta)-}) - \chi(\overline{V}_{\operatorname{sign}(\delta)+}))
\end{aligned}$$

where $\overline{V}_{\operatorname{sign}(\delta)\pm} = \{x \in V \cap S_\varepsilon^n : \operatorname{sign}(\delta) f(x) \ge 0, \pm g(x) \ge 0\}$ for $0 < \varepsilon \ll 1$.

Remark 4.7. We sketch how to find the formula (Theorem 4.3) in [2]. Let $(x_0, x_1, \dots, x_{m+n+q})$ denote a coordinate system of $\mathbf{R}^{m+n+q+1}$ at the origin. Let $n = 1, 3, 7$, and let m, q be non-negative integers. Let $f, g : (\mathbf{R}^{m+n+q+1}) \to (\mathbf{R}, 0)$ denote two analytic functions, and $h = (h_1, \dots, h_m) : (\mathbf{R}^{m+n+q+1}, 0) \to (\mathbf{R}^m, 0)$ a C^∞-map. We assume that g and h do not depend on the last q variables $x_{m+n+1}, \dots, x_{m+n+q}$. Set $V = h^{-1}(0)$ and L is the set of regular points of V (i.e., $L = V - \Sigma(h)$). Since L is orientable, we fix an orientation of L. Define vector fields $v_1, \dots, v_{n+q}$ by

$$(v_1,\dots,v_n)=\begin{cases}(v_{0,1}) & n=1,\\ (v_{0,1}+v_{2,3},v_{0,2}-v_{1,3},v_{0,3}+v_{1,2}) & n=3,\\ (v_{0,1}+v_{2,3}+v_{4,5}+v_{6,7},v_{0,2}-v_{1,3}-v_{4,6}+v_{5,7}, & n=7,\\ v_{0,3}+v_{1,2}+v_{4,7}+v_{5,6},v_{0,4}-v_{1,5}+v_{2,6}-v_{3,7}, & \\ v_{0,5}+v_{1,4}-v_{2,7}-v_{3,6},v_{0,6}-v_{1,7}-v_{2,4}+v_{3,5}, & \\ v_{0,7}+v_{1,6}+v_{2,5}+v_{3,4}) & \end{cases}$$

$$\text{where } v_{i,j}=\begin{vmatrix}\partial_{x_i} & \partial_{x_j} & \partial_{x_{n+1}} & \cdots & \partial_{x_{n+m}}\\ g_{x_i} & g_{x_j} & g_{x_{n+1}} & \cdots & g_{x_{n+m}}\\ (h_1)_{x_i} & (h_1)_{x_j} & (h_1)_{x_{n+1}} & \cdots & (h_1)_{x_{n+m}}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ (h_m)_{x_i} & (h_m)_{x_j} & (h_m)_{x_{n+1}} & \cdots & (h_m)_{x_{n+m}}\end{vmatrix},\quad 0\le i<j\le n,$$

and $v_{n+1}=\partial_{x_{m+n+1}},\dots,v_{n+q}=\partial_{x_{m+n+q}}$. We remark that these vectors are the same as the vectors defined in subsection 2.1 when $(m,q)=(0,0)$. Consider the map

$$F:(L,0)\to\mathbf{R}^{n+q+1},\quad x\mapsto(f,v_1f,\dots,v_{n+q}f).$$

By (4.10), we obtain that

$$\deg F=\pm(\chi(V_\delta(g\le 0))-\chi(V_\delta(g\ge 0))).$$

By the discussion in [3, §3], we obtain that $\deg(F)=\deg\{(F',h):(\mathbf{R}^{m+n+q+1},0)\to(\mathbf{R}^{m+n+q+1},0)\}$ where F' is an extension of F to $(\mathbf{R}^{m+n+q+1},0)$, and find Theorem 4.3 in [2].

Remark 4.8. Let $g:(\mathbf{R}^{n+1},0)\to(\mathbf{R},0)$ be a C^∞-function and let $v_1,\dots,v_n$ be vector fields on $(\mathbf{R}^{n+1},0)$ so that $\langle\nabla g,v_i\rangle=0$, $i=1,\dots,n$. We denote by Σ_v the set of points where $v_1,\dots,v_n$ are linearly dependent. Let $f:(\mathbf{R}^{n+1},0)\to(\mathbf{R},0)$ be a C^∞-function so that $V_\delta\cap\Sigma(f)=\emptyset$, $V_\delta\cap\Sigma(g)=\emptyset$ and $V_\delta\cap\Sigma_v=\emptyset$, where $V_\delta=\{x\in(\mathbf{R}^{n+1},0):f(x)=\delta\}$. If the map F defined by (4.1) is finite and $g|_{V_\delta}$ is Morse, then the same proof works and we obtain the formulas (4.2), (4.3). This observation is sometimes useful if we know Σ_v explicitly.

Here is an example that Σ_v can be expressed explicitly. Set $p=1,3,8$. Define

$$v_i=\begin{cases}\text{the same as in subsection 2.1 replacing } n \text{ by } p \text{ there} & i=1,\dots,p\\ g_{x_i}\nabla g-\|\nabla g\|\partial_{x_i} & i=p+1,\dots,n\end{cases}$$

Then we obtain $\Sigma_v=\{g_{x_0}=\cdots=g_{x_p}=0\}$. Suppose that $g(x)=\sum_{i=0}^n x_i^2$. Then $\Sigma_v=\{x_0=\cdots=x_p=0\}$. If $f:(\mathbf{R}^{n+1},0)\to(\mathbf{R},0)$ defines an isolated singularity with $f(\Sigma_v)=0$, and the map F defined by (4.1) is finite, then we obtain that

$$\deg F=(-1)^n\chi\{x\in S_\varepsilon^n:f(x)\ge 0\}.$$

To state a global consequence of our theorem, we introduce the following

Definition 4.9. Let M be a C^∞-manifold and let $\varphi : M \to \mathbf{R}$ be a C^∞-function. We say that *Morse theory is applicable to φ on the closed interval $[a, b]$* if the following two conditions hold.

(1) φ has at most finitely many critical points in $\varphi^{-1}[a, b]$, and all critical points are Morse singularities, that is, the Hessian determinant $\mathrm{Hess}(\varphi)(x)$ of φ is non zero at each critical point x.

(2) there is "no surgery at infinity" on $[a, b]$, which means that $\{x \in M : \varphi(x) \le c - \varepsilon\}$ and $\{x \in M : \varphi(x) \le c + \varepsilon\}$ are diffeomorphic each other for sufficiently small $\varepsilon > 0$ when c is not a critical value of φ with $c \in [a, b]$.

Theorem 4.10. *Let L be a real analytic manifold of dimension $n+1$ and let $f, g : L \to \mathbf{R}$ be analytic functions. We assume that $V_\delta = \{x \in L : f(x) = \delta\}$ is nonsingular for a non-zero number δ with $0 < |\delta| \ll 1$ and Morse theory is applicable for $g|_{V_\delta}$ on $[b_0, b_k]$. We assume that g satisfies Condition (P), and that the map*

$$F : L \to \mathbf{R}^{n+1}, \quad x \mapsto (f(x), v_1 f(x), \dots, v_n f(x)),$$

is finite. We set $F^{-1}(0) = \{P_1, \dots, P_k\}$ and $c_i = g(P_i)$ for $i = 1, \dots, k$, and assume that $b_0 < c_1 < c_2 < \cdots < c_k < b_k$. Taking b_i with $c_i < b_i < c_{i+1}$ for $i = 1, \dots, k-1$, we have

$$\deg(F) = \mathrm{sign}(-\delta)^{n+1} \sum_{i=1}^{k} (\chi(V_\delta(b_{i-1} \le g \le c_i)) - \chi(V_\delta(c_i \le g \le b_i))). \tag{4.11}$$

Moreover, if n is odd, we have

$$\deg(F) = \chi(V_\delta(b_0 \le g \le b_k), V_\delta(g = b_0)). \tag{4.12}$$

Proof. By Theorem 4.1, we obtain that

$$\deg(F) \text{ at } P_i = \mathrm{sign}(-\delta)^{n+1} (\chi(V_\delta(b_{i-1} \le g \le c_i)) - \chi(V_\delta(c_i \le g \le b_i))).$$

This implies (4.11). When n is odd, the proof of Theorem 4.1 implies

$$\deg(F) = \sum_{x \in X \cap V_\delta} \mathrm{Hess}(g|_{V_\delta})(x)$$

and the right hand side is equals to

$$\chi(V_\delta(b_0 \ge g \ge b_k), V_\delta(g = b_0)),$$

which completes the proof of (4.12). $\square$

5. Mapping degree of $\bar{p}([dg], [df])$

We denote by $\pi : \mathbf{R}^{n+1} - \{0\} \to S^n$ the projection defined by $x \mapsto x/\|x\|$. Let $f : (\mathbf{R}^{n+1}, 0) \to (\mathbf{R}, 0)$ be a C^∞-function-germs. We define a map $[df]$:

$S^n_\varepsilon \to S^n$ by $x \mapsto \pi \circ df(x)$ where S^n_ε denotes the n-sphere centered at 0 with radius ε and S^n denotes the unit sphere centered at 0. Suggested by Remark 2.1, we are interesting in the following: Let $f, g : (\mathbf{R}^{n+1}, 0) \to (\mathbf{R}, 0)$ be two C^∞-function-germs. We consider a smooth map $p : \mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ and set $\tilde{Z} = p^{-1}(0)$. We investigate the mapping degree of the map

$$\bar{p}([dg],[df]) : S^n_\varepsilon \to S^n, \quad x \mapsto \pi \circ p([dg](x),[df](x))$$

when $Z := \tilde{Z} \cap (S^n \times S^n)$ is empty.

Lemma 5.1. *Let M be an oriented manifold of dimension $\geq n$ and let ω be the volume form of the sphere S^n so that $\int_{S^n} \omega = 1$. We consider a C^∞-map $f : M \to S^n$. Then $\deg(f|_X) = \int_X f^*\omega$ for any oriented n-cycle X of M so that $f|_X$ is proper and finite.*

The proof is similar to the proof of Theorem 12 in [10, Chapter 8].

Proof. Let y be a regular value of $f|_X$ and let U be an open neighborhood of y. Let ω' be an n-form of S^n which is cohomologous to ω and $\operatorname{supp}(\omega') \subset U$. Let $\{x_1, \dots, x_k\}$ be the preimage of y. Choosing U small we may assume that $(f|_X)^{-1}(U) = U_1 \cup \cdots \cup U_k$ where each U_i is an open neighborhood of x_i in X and each U_i is diffeomorphic to U. Then we have $\int_{U_i} (f|_X)^*\omega' = \pm \int_U \omega' = \pm 1$ where the sign is $+$ (resp. $-$) when $f|_{U_i}$ is orientation preserving (resp. reversing). Thus we have

$$\deg(f|_X) = \sum_{i=1}^k \int_{U_i} (f|_X)^*\omega' = \int_X (f|_X)^*\omega' = \int_X (f|_X)^*\omega = \int_X f^*\omega,$$

and this completes the proof. □

Let $\boldsymbol{e}_i$, $i = 0, 1, \dots, n$, denote the unit vector $(0, \dots, \overset{i+1}{1}, \dots, 0)$ in $\mathbf{R}^{n+1}$. We investigate when Z is empty. When $Z = \emptyset$, we can consider the following map:

$$\bar{p} : S^n \times S^n \to S^n, \quad (x, y) \mapsto \pi \circ p(x, y).$$

We define the *class* of $\bar{p}$, denoted by $h(\bar{p})$, the image of the fundamental class of S^n by the map

$$H^n(S^n; \mathbf{Z}) \xrightarrow{\bar{p}^*} H^n(S^n \times S^n; \mathbf{Z}) = \mathbf{Z}^2,$$

where the last equality presents the natural identification between the cohomology group $H^n(S^n \times S^n; \mathbf{Z})$ and the free $\mathbf{Z}$-module generated by the cohomology classes corresponding to $S^n \times \boldsymbol{e}_0$ and $\boldsymbol{e}_0 \times S^n$.

Proposition 5.2. *There is a C^∞-map $\bar{p} : S^n \times S^n \to S^n$ so that $h(\bar{p}) = (k_1, k_2)$ if and only if one of the following conditions holds.*

- $n = 1, 3, 7$.
- *n is odd, $n \neq 1, 3, 7$, and* $k_1 k_2 \equiv 0 \pmod 2$.
- *n is even, and* $k_1 k_2 = 0$.

Proof. Assume first that n is even. Let ω denote the volume form of S^n. Let $p_i : S^n \times S^n \to S^n$, $i = 1, 2$, denote the i-th projection. We remark that $\bar{p}^*\omega$ is cohomologous to $k_1(p_1)^*\omega + k_2(p_2)^*\omega$. The assertion comes from the following:

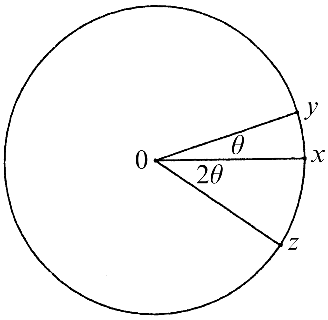
$$0 = (\bar{p}^*\omega) \wedge (\bar{p}^*\omega) = (k_1(p_1)^*\omega + k_2(p_2)^*\omega) \wedge (k_1(p_1)^*\omega + k_2(p_2)^*\omega)$$
$$= 2k_1k_2(p_1)^*\omega \wedge (p_2)^*\omega.$$

We next consider the case that n is odd. Let $f_i : S^n \to S^n$ be a C^∞-map of degree k_i. We remark that their homotopy classes is $k_i\iota_n$ where ι_n is the identity map of S^n. It is enough to determine all (k_1, k_2) so that the Whitehead product $[k_1\iota_n, k_2\iota_n] = k_1k_2[\iota_n, \iota_n]$ vanishes. By the theorem of J. Adams [1, Theorem 1.1.1], $[\iota_n, \iota_n] = 0$ if and only if $n = 1, 3, 7$. This implies the second assertion. Since $[\iota_n, \iota_n]$ is of order 2 when $n \neq 1, 3, 7$, we obtain the last assertion. $\square$

When $n = 1, 3, 7$, and a map $\bar{p}$ with $(k_1, k_2) = (1, 1)$, is induced by the product of complex, quotanion, Cayley numbers respectively.

When $n = 1$, we identify $\mathbf{R}^2$ with $\mathbf{C}$ by $(x, y) \mapsto z = x + y\boldsymbol{i}$. The map $p_{k_1,k_2} : \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}^2$ defined by $(z_1, z_2) \mapsto z_1^{k_1}z_2^{k_2}$ represents a map which class is (k_1, k_2). Remarking $z^{-1} = \bar{z}$ on S^1, we see all the classes (k_1, k_2) are represented by polynomial maps.

When n is odd, a map $S^n \times S^n \to S^n$ with $(k_1, k_2) = (1 - k, k)$ is represented by the following way: Take $x, y \in S^n$, and consider the great circle containing x, y. The image of (x, y) is z in the great circle defined by $\measuredangle x0z = k\measuredangle x0y$ described in the following picture in the case $k = -2$.



An explicit formula for this map is described by the following: For $x, y \in S^n$, we set $z = p_k(u, v)x + q_k(u, v)(y - ux)$ where $u = \langle x, y \rangle$, $v = |y - ux|$. Here $p_k(u, v)$ and $q_k(u, v)$ denote real polynomials defined by $(u + v\boldsymbol{i})^k = p_k(u, v) + q_k(u, v)v\boldsymbol{i}$.

Proposition 5.3. *Let $\bar{p} : S^n \times S^n \to S^n$ be a C^∞-map with $h(\bar{p}) = (k_1, k_2)$, and let $f_i : S^n \to S^n$, $i = 1, 2$, be two C^∞-maps. We define a map by*

$$f := \bar{p}(f_1, f_2) : S^n \to S^n, \quad x \mapsto \bar{p}(f_1(x), f_2(x)).$$

Then we have $\deg(f) = k_1 \deg(f_1) + k_2 \deg(f_2)$.

Proof. Let ω denote the volume form of S^n with $\int_{S^n} \omega = 1$. Let $p_i : S^n \times S^n \to S^n$, $i = 1, 2$, denote the i-th projection. We remark that $\bar{p}^*\omega$ is cohomologous to $k_1(p_1)^*\omega + k_2(p_2)^*\omega$. Then we have $\deg(f) = \int_{S^n} f^*\omega = \int_{S^n}(k_1(p_1)^*\omega + k_2(p_2)^*\omega) = k_1 \deg(f_1) + k_2 \deg(f_2)$. □

References

[1] J. F. Adams, On the non-existence of elements of Hopf invariant one, Ann. of Math. **72** (1960), 20–104.

[2] N. Dutertre, Degree formula for a topological invariant of bifurcations of function-germs, Kodai Math. J. **23** (2001), 442–461.

[3] T. Fukui, An algebraic formula for a topological invariant of bifurcation of 1-parameter family of function-germ, Stratifications, singularities and differential equations **II**, Stratifications and topology of singular spaces (D. Trotman & L. Wilson eds.), Travaux en cours **55**, Hermann, 1997, 45–54.

[4] T. Fukui, Mapping degree formula for 2-parameter bifurcation of function-germs, Topology **32** (1993), 567–571.

[5] S. Hu, Homotopy theory, Pure and applied mathematics **VIII**, Academic Press, 1959.

[6] D. Husemoller, Fiber bundles, 3rd ed., Graduate texts in mathematics **20**, Springer-Verleg, 1993.

[7] G. N. Khimshiashvili, On the local degree of a smooth mapping (in Russian), Bull. Acad. Sci. Gerogian SSR **85** (1977), 309–312.

[8] J. Milnor, Topology from the differentiable viewpoint, The University Press of Virginia, Charlottesville, 1965.

[9] J. Milnor, Singular points of complex hypersurfaces, Annals of Mathematical Studies **61**, Princiton University Press, 1968.

[10] M. Spivak, A comprehensive introduction to differential geometry, 2nd ed., **I**, Publish or Perish, Inc., Wilmington, Del., 1979.

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