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ON SEMI-SYMMETRIC METRIC φ-CONNECTIONS IN A SASAKIAN MANIFOLD

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§ 0. Introduction.

Let M be an n-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{U; x^h\}$ and with the fundamental metric tensor g_{ji}, where and in the sequel the indices $h, i, j, \cdots$ run over the range $\{1, 2, \cdots, n\}$. A linear connection D with components Γ^h_{ji} of M is said to be semi-symmetric if its torsion tensor $S_{ji}{}^h=\Gamma^h_{ji}-\Gamma^h_{ij}$ is of the form $S_{ji}{}^h=\delta^h_j p_i-\delta^h_i p_j$, p_i being a 1-form and is said to be metric if it satisfies $D_k g_{ji}=0$.

The components of a semi-symmetric metric connection in a Riemannian manifold are given by [3]

$$\Gamma^h_{ji}=\left\{ {h \atop j\ i} \right\}+\delta^h_j p_i-g_{ji}p^h, \tag{0.1}$$

$\left\{ {h \atop j\ i} \right\}$ being the Christoffel symbols formed with g_{ji} and $p^h=p_t g^{th}$. One of present authors [3] proved that: In order that a Riemannian manifold admits a semi-symmetric metric connection whose curvature tensor vanishes, it is necessary and sufficient that the Riemannian manifold is conformally flat.

Let M be a Kaehlerian manifold with Hermitian metric g_{ji} and almost complex structure tensor $F_i{}^h$. A linear connection D with components Γ^h_{ji} of M is called a complex conformal connection if it satisfies

$$D_k e^{2p} g_{ji}=0, \qquad D_k e^{2p} F_{ji}=0, \qquad (F_{ji}=F_j{}^t g_{ti})$$

and

$$\Gamma^h_{ji}-\Gamma^h_{ij}=-2F_{ji}q^h$$

for a certain scalar p and a vector field q^h.

The components of a complex conformal connection are given by [4]

$$\Gamma^h_{ji}=\left\{ {h \atop j\ i} \right\}+\delta^h_j p_i+\delta^h_i p_j-g_{ji}p^h+F_j{}^h q_i+F_i{}^h q_j-F_{ji}q^h, \tag{0.2}$$

where $p_i=\partial_i p$, $p^h=p_t g^{th}$, $q_i=-p_t F_i{}^t$ and $q^h=q_t g^{th}$, ∂_i denoting the partial derivation with respect to x^i. One of the present authors [4] proved that: If, in an n-dimensional Kaehlerian manifold ($n\geqq 4$), there exists a scalar function p

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such that the complex conformal connection (0, 2) is of zero curvature, then the Bochner curvature tensor of the manifold vanishes.

Let M be a Sasakian manifold with structure tensors $(\varphi_i{}^h, \xi^h, \eta_i, g_{ji})$ [2]. A linear connection D with components Γ^h_{ji} of M is called a contact conformal connection if it satisfies

$$D_k e^{2p} g_{ji} = 2e^{2p} p_k \eta_j \eta_i, \qquad D_j \varphi_i{}^h = 0, \qquad D_j \xi^h = 0$$

and

$$\Gamma^h_{ji} - \Gamma^h_{ij} = -2\varphi_{ji} u^h$$

for a certain scalar p and a vector field u^h.

The components of a contact conformal connection are given by [5]

$$\Gamma^h_{ji} = \left\{ {h \atop j\ i} \right\} + (\delta^h_j - \eta_j \xi^h) p_i + (\delta^h_i - \eta_i \xi^h) p_j - (g_{ji} - \eta_j \eta_i) p^h \tag{0.3}$$
$$+ \varphi_j{}^h (q_i - \eta_i) + \varphi_i{}^h (q_j - \eta_j) - \varphi_{ji} (q^h - \xi^h),$$

where $p^h = p_t g^{th}$, $q_i = -p_t \varphi_i{}^t$, $q^h = q_t g^{th}$ and p satisfies $p_i \xi^i = 0$. One of the present authors [5] proved that: If, in a $(2m+1)$-dimensional Sasakian manifold $(2m+1>3)$, there exists a scalar function p such that the contact conformal connection (0.3) is of zero curvature, then the contact Bochner curvature tensor of the manifold vanishes.

On the other hand, the present authors [6] defined a semi-symmetric metric F-connection in a Kaehlerian manifold as a linear connection D which satisfies

$$D_k g_{ji} = 0, \qquad D_k F_i{}^h = 0$$

and

$$\Gamma^h_{ji} - \Gamma^h_{ij} = \delta^h_j p_i - \delta^h_i p_j - 2F_{ji} q^h$$

for a certain 1-form p_i and a vector field q^h.

The components of a semi-symmetric metric F-connection are given by [6]

$$\Gamma^h_{ji} = \left\{ {h \atop j\ i} \right\} + \delta^h_j p_i - g_{ji} p^h + F_j{}^h q_i + F_i{}^h q_j - F_{ji} q^h \tag{0.4}$$

where $p^h = p_t g^{th}$, $q_i = -p_t F_i{}^t$ and $q^h = q_t g^{th}$.

We proved [6] that: If, in an n-dimensional Kaehlerian manifold $(n \geqq 4)$, there exists a scalar function p such that the semi-symmetric metric F-connection (0.4) with $p_i = \partial_i p$ is of zero curvature, then the Bochner curvature tensor of the manifold vanishes.

We also gave another definition of a semi-symmetric metric F-connection in a Kaehlerian manifold as a linear connection which satisfies

$$D_k g_{ji} = 0, \qquad D_k F_i{}^h = 0$$

and

$$\Gamma^h_{ji} - \Gamma^h_{ji} = \delta^h_j p_i - \delta^h_i p_j + F_j{}^h q_i - F_i{}^h q_j - 2F_{ji} q^h$$

for a certain 1-forms p_i and q_i, q^h being defined to be $q^h = q_t g^{th}$.

The components of a semi-symmetric metric F-connection in this sense are given by [6]

(0.5) $$\Gamma_{ji}^h=\left\{ {h \atop j\ i} \right\}+\delta_j^h p_i-g_{ji}p^h+F_j{}^h q_i-F_{ji}q^h,$$

where p_i is a 1-form and $q_i=-p_t F_i{}^t$. We proved [6] that: If, in an n-dimensional Kaehlerian manifold ($n\geqq 4$), there exists a scalar function p such that the cuvature tensor $R_{kji}{}^h$ of a semi-symmetric metric F-connection (0.5) with $p_i=\partial_i p$ is of the form $R_{kji}{}^h=\alpha_{kj}F_i{}^h$, then the Bochner curvature tensor of the manifold vanishes, and also that; If a Kaehlerian manifold of dimension $n\geqq 4$ admits a semi-symmetric metric F-connection (0.5) with $p_i=\partial_i p$ in the latter sense such that the torsion tensor $S_{ji}{}^h$ satisfies $D_k S_{ji}{}^h=0$ and the curvature tensor $R_{kji}{}^h$ is of the form $R_{kji}{}^h=\alpha_{kj}F_i{}^h$, then the manifold is of constant holomorphic sectional curvature.

The main purpose of the present paper is to obtain results similar to those for semi-symmetric metric F-connections in a Kaehlerian manifold, in the case of a Sasakian manifold, the Bochner curvature tensor in a Kaehlerian manifold being replaced by the so-called contact Bochner curvature tensor.

§ 1. Preliminaries.

Let M be a $(2m+1)$-dimensional differentiable manifold covered by a system of coordinate neighborhoods $\{U: x^h\}$ in which there are given a tensor field $\varphi_i{}^h$ of type (1.1), a vector field ξ^h and a 1-form η_i satisfying

(1.1) $$\varphi_j{}^i\varphi_i{}^h=-\delta_j^h+\eta_j\xi^h,\qquad \varphi_i{}^h\xi^i=0,\qquad \eta_i\varphi_j{}^i=0,\qquad \eta_i\xi^i=1,$$

where and in the sequel the indices $h, i, j, \cdots$ run over the range $\{1, 2, \cdots, 2m+1\}$. Such a set of φ, ξ and η is called an almost contact manifold. If the set (φ, ξ, η) satisfies

(1.2) $$N_{ji}{}^h+(\partial_j\eta_i-\partial_i\eta_j)\xi^h=0,$$

where $N_{ji}{}^h$ is the Nijenhuis tensor formed with $\varphi_i{}^h$ and $\partial_j=\partial/\partial x^i$, then the almost contact structure is said to be normal and the manifold is called a normal almost contact manifold. If, in an almost contact manifold, there is given a Riemannian metric g_{ji} such that

(1.3) $$g_{ts}\varphi_j{}^t\varphi_i{}^s=g_{ji}-\eta_j\eta_i,\qquad \eta_i=g_{ih}\xi^h,$$

then the almost contact structure is said to be metric and the manifold is called an almost contact metric manifold. In this case, $\varphi_{ji}=\varphi_j{}^t g_{ti}$ is skew-symmetric. Since $\eta_i=g_{ih}\xi^h$, we shall write η^h in stead of ξ^h in the sequel. If an almost contact metric manifold satisfies $\varphi_{ji}=\frac{1}{2}(\partial_j\eta_i-\partial_i\eta_j)$, then the almost contact metric structure is called a contact manifold. A manifold with a normal contact structure is called a Sasakian manifold.

It is a well known fact that in a Sasakian manifold we have

(1.4) $$\nabla_i\eta^h=\varphi_i{}^h,$$

(1.5) $$\nabla_j \varphi_i{}^h = -g_{ji}\eta^h + \delta_j^h \eta_i ,$$

∇_j denoting the operator of covariant differentiation with respect the Levi-Civita connection. Since we have $\mathcal{L} g_{ji} = \nabla_j \eta_i + \nabla_i \eta_j = \varphi_{ji} + \varphi_{ij} = 0$, $\mathcal{L}$ denoting the Lie derivative with respect to η^h, η^h is a Killing vector.

Now the contact Bochner curvature tensor in a Sasakian manifold is given by [1], [5]

(1.6) $$\begin{aligned} B_{kji}{}^h = & K_{kji}{}^h + (\delta_k^h - \eta_k \eta^h) L_{ji} - (\delta_j^h - \eta_j \eta^h) L_{ki} + L_k{}^h (g_{ji} - \eta_j \eta_i) \\ & - L_j{}^h (g_{ki} - \eta_k \eta_i) + \varphi_k{}^h M_{ji} - \varphi_j{}^h M_{ki} + M_k{}^h \varphi_{ji} - M_j{}^h \varphi_{ki} \\ & - 2(M_{kj} \varphi_i{}^h + \varphi_{kj} M_i{}^h) + (\varphi_k{}^h \varphi_{ji} - \varphi_j{}^h \varphi_{ki} - 2\varphi_{kj} \varphi_i{}^h) , \end{aligned}$$

$K_{kji}{}^h$ being the Riemann-Christoffel curvature tensor of the manifold, where

(1.7) $$L_{ji} = -(1/2(m+1))[K_{ji} + (L+3) g_{ji} - (L-1) \eta_j \eta_i] , \qquad L_k{}^h = L_{kt} g^{th} ,$$

(1.8) $$M_{ji} = -L_{jt} \varphi_i{}^t , \qquad M_k{}^h = M_{kt} g^{th} ,$$

and consequently

(1 9) $$M_{ji} = (1/2(m+1))[K_{jt} \varphi_i{}^t - (L+3) \varphi_{ji}] ,$$

and

(1.10) $$L = g^{ji} L_{ji} ,$$

K_{ji} being the Ricci tensor of the manifold. The L_{ji} is symmetric and M_{ji} is skew-symmetric. From (1.7) and (1.10), we find

(1.11) $$L = -\{K + 2(3m+2)\} / 4(m+1) ,$$

K being the scalar curvature of the manifold.

§ 2. Semi-symmetric metric φ-connections.

We consider a linear connection D with torsion in a Sasakian manifold the components of which are Γ_{ji}^h. If D satisfies

(2 1) $$D_k g_{ji} = 0 ,$$

D is called a metric connection. If we put

(2.2) $$\Gamma_{ji}^h = \left\{ {h \atop j \ i} \right\} + U_{ji}{}^h ,$$

then $U_{ji}{}^h$ are components of a tensor and the torsion tensor of D is given by

(2 3) $$S_{ji}{}^h = U_{ji}{}^h - U_{ij}{}^h .$$

If the connection D is metric, i. e. (2.1) holds, then from (2.2) we have $U_{kji} + U_{kij} = 0$, where $U_{kji} = U_{kj}{}^t g_{ti}$. From (2.3) and this, we find

(2.4) $$U_{ji}{}^{h}=\frac{1}{2}(S_{ji}{}^{h}+S^{h}{}_{ji}+S^{h}{}_{ij}),\ \text{where}\ S^{h}{}_{ji}=S_{tj}{}^{s}g^{th}g_{si}$$

If D satisfies

(2.5) $$D_{j}\varphi_{i}{}^{h}=0\,,\qquad D_{j}\eta^{h}=0\,,$$

D is called a φ-connection. In this case, we have, from (2.2) and (2.5),

(2.6) $$U_{ji}{}^{t}\varphi_{t}{}^{h}-U_{jt}{}^{h}\varphi_{i}{}^{t}=-g_{ji}\eta^{h}+\delta_{j}^{h}\eta_{i}$$

(2.7) $$U_{ji}{}^{h}\eta^{i}=-\varphi_{j}{}^{h}\,.$$

Assume that the torsion tensor $S_{ji}{}^{h}$ of the linear connection D is of the form

(2.8) $$S_{ji}{}^{h}=(\delta_{j}^{h}-\eta_{j}\eta^{h})p_{i}-(\delta_{i}^{h}-\eta_{i}\eta^{h})p_{j}-2\varphi_{ji}u^{h},$$

where p_{i} is a 1-form and u^{h} a vector field. We call a semi-symmetric connection a linear connection whose torsion tensor has the form (2.8).

Now we suppose that a linear connection D is semi-symmetric and metric. Then substituting (2.8) into (2.4), we find

(2.9) $$U_{ji}{}^{h}=(\delta_{j}^{h}-\eta_{j}\eta^{h})p_{i}-(g_{ji}-\eta_{j}\eta_{i})p^{h}+\varphi_{j}{}^{h}u_{i}+\varphi_{i}{}^{h}u_{j}-\varphi_{ji}u^{h}\,,$$

where $p^{h}=p_{t}g^{th}$ and $u_{i}=g_{it}u^{t}$.

Next suppose that a linear connection D is skew-symmetric, metric and moreover is a φ-connection. Then substituting (2.9) into (2.6) and contracting, we find

(2.10) $$u_{i}=-p_{t}\varphi_{i}{}^{t}-\eta_{i}$$

and substituting (2.9) into (2.7), we find

(2.11) $$p_{i}\eta^{i}=0\,.$$

Thus we have

PROPOSITION 2.1. *In a Sasakian manifold with structure tensors* $(\varphi_{i}{}^{h},\ \eta_{i},\ g_{ji})$. *A semi-symmetric metric* φ*-connection is given by*

(2.12) $$\Gamma_{ji}^{h}=\left\{{h \atop j\ \ i}\right\}+(\delta_{j}^{h}-\eta_{j}\eta^{h})p_{i}-(g_{ji}-\eta_{j}\eta_{i})p^{h}+\varphi_{j}{}^{h}u_{i}+\varphi_{i}{}^{h}u_{j}-\varphi_{ji}u^{h}\,,$$

where p_{i} *is a* 1*-form satisfying* (2.11) *and* u_{i} *is given by* (2.10).

If p_{i} in (2.12) is the gradient of a scalar function p, we call the connection a special semi-symmetric metric φ-connection.

§ 3. Curvature tensor of a semi-symmetric metric φ-connection.

We consider a special semi-symmetric metric φ-connection (2.12) in a Sasakian manifold and compute the curvature tensor $R_{kji}{}^{h}$ of the connection D. By a long but straightforward computation, we find

$$R_{kji}{}^h = K_{kji}{}^h - (\delta_k^h - \eta_k\eta^h)p_{ji} + (\delta_j^h - \eta_j\eta^h)p_{ki} - p_k{}^h(g_{ji} - \eta_j\eta_i) + p_j{}^h(g_{ki} - \eta_k\eta_i) - \varphi_k{}^h q_{ji} + \varphi_j{}^h q_{ki} - q_k{}^h\varphi_{ji} + q_j{}^h\varphi_{ki} + (\nabla_k q_j - \nabla_j q_k)\varphi_i{}^h + 2\varphi_{kj}(q_i p^h - p_i q^h) + (\varphi_k{}^h\varphi_{ji} - \varphi_j{}^h\varphi_{ki} - 2\varphi_{kj}\varphi_i{}^h), \tag{3.1}$$

where

$$p_{ji} = \nabla_j p_i - p_j p_i + (q_j - \eta_j)(q_i - \eta_i) + \frac{1}{2} p_t p^t (g_{ji} - \eta_j\eta_i), \tag{3.2}$$

$$q_{ji} = \nabla_j q_i - p_j(q_i - \eta_i) - p_i(q_j - \eta_j) + \frac{1}{2} p_t p^t \varphi_{ji}, \tag{3.3}$$

$$q_i = -p_t\varphi_i{}^t, \qquad p_k{}^h = p_{kt}g^{th}, \qquad q_k{}^h = q_{kt}g^{th}$$

and consequently

$$q_{ji} = -p_{jt}\varphi_i{}^t, \qquad p_{ji} = q_{jt}\varphi_i{}^t + \eta^j\eta^i.$$

If we assume that $R_{kji}{}^h = 0$, then we have from (3.1)

$$K_{kji}{}^h = (\delta_k^h - \eta_k\eta^h)p_{ji} - (\delta_j^h - \eta_j\eta^h)p_{ki} + p_k{}^h(g_{ji} - \eta_j\eta_i) - p_j{}^h(g_{ki} - \eta_k\eta_i) + \varphi_k{}^h q_{ji} - \varphi_j{}^h q_{ki} + q_k{}^h\varphi_{ji} - q_j{}^h\varphi_{ki} + \alpha_{kj}\varphi_i{}^h + \varphi_{kj}\beta_i{}^h - (\varphi_k{}^h\varphi_{ji} - \varphi_j{}^h\varphi_{ki} - 2\varphi_{kj}\varphi_i{}^h), \tag{3.4}$$

where we have put $\alpha_{kj} = -(\nabla_k q_j - \nabla_j q_k)$ and $\beta_i{}^h = 2(p_i q^h - q_i p^h)$. We can write (3.4) in the covariant form

$$K_{kjih} = (g_{kh} - \eta_k\eta_h)p_{ji} - (g_{jh} - \eta_j\eta_h)p_{ki} + p_{kh}(g_{ji} - \eta_j\eta_i) - p_{jh}(g_{ki} - \eta_k\eta_i) + \varphi_{kh}q_{ji} - \varphi_{jh}q_{ki} + q_{kh}\varphi_{ji} - q_{jh}\varphi_{ki} + \alpha_{kj}\varphi_{ih} + \varphi_{kj}\beta_{ih} - (\varphi_{kh}\varphi_{ji} - \varphi_{jh}\varphi_{ki} - 2\varphi_{kj}\varphi_{ih}), \tag{3.5}$$

where $\beta_{ih} = 2(p_i q_h - p_h q_i)$.

Using the identity $K_{kjih} = K_{ihkj}$ and $p_{ji} = p_{ij}$, we find from (3.5)

$$q_{ji} + q_{ij} = 0. \tag{3.6}$$

Using $p_{ji} = p_{ij}$ and $q_{ji} = -q_{ij}$, we can find, from (3.5), p_{ji}, q_{ji}, α_{kj} and β_{ih} as follows.

$$p_{ji} = -L_{ji}, \qquad q_{ji} = -M_{ji}, \tag{3.7}$$

$$\alpha_{kj} = 2M_{kj} + \{(L+1)/(m+2)\}\varphi_{kj}, \tag{3.8}$$

$$\beta_{ih} = 2M_{ih} - \{(L+1)/(m+2)\}\varphi_{ih}, \tag{3.9}$$

L_{ji} and M_{ji} being given by (1.7) and (1.9) respectively.

Substituting (3.7), (3.8) and (3.9) into (3.4), we find $B_{kji}{}^h = 0$. Thus we have

THEOREM 3.1. *If, in a* $(2m+1)$*-dimensional Sasakian manifold* $(m\geqq 2)$, *there exists a scalar function* p *such that the special semi-symmetric metric* φ*-connection* (2.12) *with* $p_i=\partial_i p$ *is of zero curvature, then the contact Bochner curvature tensor of the manifold vanishes.*

§4. Another definition of semi-symmetry.

In §3, we assumed that the torsion tensor of a linear connection D is of the form (2.8) and called it a semi-symmetric connection.

In this section, we assume that the torsion tensor of a linear connection D is of the form

$$S_{ji}{}^{h}=(\delta_j^h-\eta_j\eta^h)p_i-(\delta_i^h-\eta_i\eta^h)p_j+\varphi_j{}^h u_i-\varphi_i{}^h u_j-2\varphi_{ji}u^h\,, \tag{4.1}$$

where p_i and u_i are 1-forms and $u^h=u_t g^{th}$ and call a semi-symmetric connection a linear connection whose torsion tensor has the form (4.1).

Now we suppose that a linear connection D is semi-symmetric in this sense and metric. Then substituting (4.1) into (2.4), we find

$$U_{ji}{}^{h}=(\delta_j^h-\eta_j\eta^h)p_i-(g_{ji}-\eta_j\eta_i)p^h+\varphi_j{}^h u_i-\varphi_{ji}u^h\,, \tag{4.2}$$

where $p^h=p_t g^{th}$ and $u^h=u_t g^{th}$.

Next suppose that a linear connection D is semi-symmetric in the sense of this section, metric and moreover is a φ-connection. Then substituting (4.2) into (2.6) and contracting, we find (2.10) and substituting (4.2) into (2.7) we find (2.11). Thus we have

PROPOSITION 4.1. *In a Sasakian manifold with structure tensors* $(\varphi_i{}^h, \eta_i, g_{ji})$, *a semi-symmetric, in the sense of this section, metric* φ*-connection is given by*

$$\Gamma_{ji}^h=\left\{ {h \atop j\ i} \right\}+(\delta_j^h-\eta_j\eta^h)p_i-(g_{ji}-\eta_j\eta_i)p^h+\varphi_j{}^h u_i-\varphi_{ji}u^h\,, \tag{4.3}$$

where p_i *is a* 1*-form satisfing* (2.11) *and* u_i *is given by* (2.10).

If p_i in (4.3) is the gradient of a scalar function p, we call the connection a special semi-symmetric metric φ-connection.

We consider a special semi-symmetric metric φ-connection (4.3) in the above sense in a Sasakian manifold and compute the curvature tensor $R_{kji}{}^h$ of the connection.

Then by a straightforward computation similar to that done in the previous section, we find

$$\begin{aligned} R_{kji}{}^{h}=&K_{kji}{}^{h}-(\delta_k^h-\eta_k\eta^h)p_{ji}+(\delta_j^h-\eta_j\eta^h)p_{ki}-p_k{}^h(g_{ji}-\eta_j\eta_i)\\ &+p_j{}^h(g_{ki}-\eta_k\eta_i)-\varphi_k{}^h q_{ji}+\varphi_j{}^h q_{ki}-q_k{}^h\varphi_{ji}+q_j{}^h\varphi_{ki}\\ &+2\varphi_{kj}(p_i q^h-q_j p^h)+(\varphi_k{}^h\varphi_{ji}-\varphi_j{}^h\varphi_{ki}-2\varphi_{kj}\varphi_i{}^h)\,, \end{aligned} \tag{4.4}$$

where p_{ji} and q_{ji} are respectively given by (3.2) and (3.3).

If we assume that the curvature tensor $R_{kji}{}^h$ of a special semi-symmetric metric φ-connection in the sense of the present section is of the form $R_{kji}{}^h = \alpha_{kj}\varphi_i{}^h$ for a certain skew-symmetric tensor α_{kj}, then we have, from (4.4),

$$\begin{aligned} K_{kji}{}^h = & (\delta_k^h - \eta_k\eta^h)p_{ji} - (\delta_j^h - \eta_j\eta^h)p_{ki} + p_k{}^h(g_{ji} - \eta_j\eta_i) - p_j{}^h(g_{ki} - \eta_k\eta_i) \\ & + \varphi_k{}^h q_{ji} - \varphi_j{}^h q_{ki} + q_k{}^h\varphi_{ji} - q_j{}^h\varphi_{ki} + \alpha_{kj}\varphi_i{}^h + \varphi_{kj}\beta_i{}^h \\ & - (\varphi_k{}^h\varphi_{ji} - \varphi_j{}^h\varphi_{ki} - 2\varphi_{kj}\varphi_i{}^h) . \end{aligned}$$

where $\beta_i{}^h = 2(p_i q^h - q_i p^h)$, from which, eliminating p_{ji}, q_{ji}, α_{kj} and $\beta_i{}^h$, we find that the contact Bochner curvature tensor vanishes. Thus we have.

THEOREM 4.2. *If, in a $(2m+1)$-dimensional Sasakian manifold $(m \geqq 2)$, there exists a scalar function p such that the curvature tensor $R_{kji}{}^h$ of a semi-symmetric metric φ-connection (4.3) with $p_i = \partial_i p$ in the sense of the present section is of the form $R_{kji}{}^h = \alpha_{kj}\varphi_i{}^h$, then the contact Bochner curvature tensor of the manifold vanishes.*

We now assume that a special semi-symmetric φ-connection (4.3) in the sense of the present section satisfies.

$$D_k S_{ji}{}^h = \eta_k(\delta_j^h u_i - \delta_i^h u_j) . \tag{4.5}$$

Substituting (4.1) and (4.3) into (4.5), and taking account of $D_k g_{ji} = 0$ and of $D_k \varphi_j{}^h = 0$, we find

$$\begin{aligned} & (\delta_j^h - \eta_j\eta^h)D_k p_i - (\delta_i^h - \eta_i\eta^h)D_k p_j - \varphi_j{}^h\varphi_{it}D_k p^t + \varphi_i{}^h\varphi_{jt}D_k p^t \\ & - 2\varphi_{ji}\varphi_t{}^h D_k p^t = \eta_k(\delta_j^h u_i - \delta_i^h u_j) , \end{aligned}$$

from which, contracting with respect to h and j, we find $D_k p_i = \eta_k u_i$, that is,

$$\nabla_k p_i - p_k p_i + u_k u_i + p_t p^t(g_{ki} - \eta_k\eta_i) = 0 . \tag{4.6}$$

Thus, from (?.2) and (3.3), we find

$$p_{ji} = -\frac{1}{2} p_t p^t (g_{ji} - \eta_j\eta_i) , \tag{4.7}$$

$$q_{ji} = -\frac{1}{2} p_t p^t \varphi_{ji} \tag{4.8}$$

respectively. From (?.7), (4.7) and (4.5), we have

$$L_{ji} = (L/2m)(g_{ji} - \eta_j\eta_i) , \tag{4.9}$$

$$M_{ji} = (L/2m)\varphi_{ji} , \tag{4.10}$$

Now if we assume that the curvature tensor $R_{kji}{}^h$ of the connection is of the form $R_{kji}{}^h = \alpha_{kj}\varphi_i{}^h$, then we have $B_{kji}{}^h = 0$. Thus, substituting (4.9) and (4.10) into $B_{kji}{}^h = 0$, we have

$$K_{kji}{}^h=-(L/m)\{(\delta_k^h-\eta_k\eta^h)(g_{ji}-\eta_j\eta_i)-(\delta_j^h-\eta_j\eta^h)(g_{ki}-\eta_k\eta_i)\}$$
$$+\frac{-L+m}{m}(\varphi_k{}^h\varphi_{ji}-\varphi_j{}^h\varphi_{ki}-2\varphi_{kj}\varphi_i{}^h). \tag{4.11}$$

Thus we have

THEOREM 4.3. *If a* $(2m+1)$*-dimensional Sasakian manifold* $(m\geqq 2)$ *admits a special semi-symmetric metric* φ*-connection* D *with* $p_i=\partial_i p$ *in the sense of the present section such that the torsion tensor* $S_{ji}{}^h$ *satisfies* $D_kS_{ji}{}^h=\eta_k(\delta_j^h u_i-\delta_i^h u_j)$, u_i *being given by* (2.10) *and the curvature tensor* $R_{kji}{}^h$ *is of the form* $R_{kji}{}^h=\alpha_{kj}\varphi_i{}^h$ *for a certain skew-symmetric tensor* α_{kj}, *then the manifold is of curvature of the form* (4.11).

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