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THE STRUCTURE OF TRIVARIATE POISSON DISTRIBUTION

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0. Summary.

In this paper we discuss the structure of trivariate Poisson distribution. In the first section usual univariate Poisson distribution and bivariate general Poisson distribution [2] are stated. It is stated in section 2 the main result of this paper; that is, the structure of trivariate Poisson distribution. The discussion is constructed by the three parts

2.1. definition of the trivariate Bernoulli distribution
2.2. definition of the trivariate binomial distribution
2.3. definition of the trivariate Poisson distribution and the relation of the trivariate Poisson distribution and the trivariate binomial distribution.

In the part (3) some characters of the trivariate Poisson distribution and the notion of the generalization to the multivariate Poisson distribution are stated.

1. Poisson distribution and bivariate Poisson distribution.

UNIVARIATE CASE. Poisson distribution is given by

$$P\{X=k\}=\frac{\lambda^k}{k!}e^{-\lambda}$$

where k is nonnegative integer and λ is nonnegative parameter.

BIVARIATE CASE. Poisson distribution is given by

$$P\{X=k,\ Y=l\}=\sum_{\delta=0}^{\min(k,l)}\frac{\lambda_{10}{}^{k-\delta}\lambda_{01}{}^{l-\delta}\lambda_{11}{}^{\delta}}{(k-\delta)!(l-\delta)!\delta!}e^{-\lambda_{10}-\lambda_{01}-\lambda_{11}}$$

where k and l are nonnegative integers and λ_{10}, λ_{01} and λ_{11} are nonnegative parameters, see Kawamura [2].

NOTE For general treatment considering multivariate (more than three dimensional) case of the distribution we may prefer the following formulation to the above formulation.

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$$P\{X=k,\ Y=l\}=\sum_{\substack{n_{10}+n_{11}=k\\ n_{01}+n_{11}=l}}\frac{\lambda_{10}^{n_{10}}\lambda_{01}^{n_{01}}\lambda_{11}^{n_{11}}}{n_{10}!\,n_{01}!\,n_{11}!}e^{-\lambda_{10}-\lambda_{01}-\lambda_{11}}$$

where n_{10}, n_{01} and n_{11} are nonnegative integers.

Marginal distribution of X is a usual univariate Poisson distribution of parameter $\lambda_{10}+\lambda_{11}$ and marginal distribution of Y is a usual Poisson distribution of parameter $\lambda_{01}+\lambda_{11}$. Therefore we have

$$E(X)=\mathrm{Var}\,(X)=\lambda_{10}+\lambda_{01}\,,$$

$$E(Y)=\mathrm{Var}\,(Y)=\lambda_{01}+\lambda_{11}\,.$$

We have seen the generating function of the bivariate Poisson distribution

$$h(s_1,\ s_2)=e^{-(\lambda_{10}+\lambda_{01}+\lambda_{11})+\lambda_{10}s_1+\lambda_{01}s_2+\lambda_{11}s_1s_2}\,.$$

The covariance of the random vector $(X,\ Y)$ is given by

$$E(XY)=(\lambda_{10}+\lambda_{11})(\lambda_{01}+\lambda_{11})+\lambda_{11}\,,$$

$$\mathrm{Cov}\,(X,\ Y)=E(XY)-E(X)E(Y)=\lambda_{11}\,,$$

then the coefficient of correlation of $(X,\ Y)$ equals to

$$R(X,\ Y)=\lambda_{11}/\sqrt{(\lambda_{10}+\lambda_{11})(\lambda_{01}+\lambda_{11})}\,.$$

All results of bivariate Poisson distribution mentioned above are the facts proved in [2].

2. Main results.

Notations

i, j and k take the values 0 or 1 respectively. $\sum_i$, $\sum_j$, $\sum_k$, $\sum_{ij}$, $\sum_{jk}$, $\sum_{ik}$ and $\sum_{ijk}$ means the sum of i, j and k which are indicated under the sigma and if we need some condition C in the sum of i, j and k we shall indecate C under the sigma additionally as followings

$$\sum_{ijk\,C},\quad \sum_{ijk(i,j,k)\neq(0,0,0)}$$

2.1. Trivariate Bernoulli distribution.

Consider a pair of random variable $(X,\ Y,\ Z)$ which has a joint discrete distribution.

$$P(X=i,\ Y=j,\ Z=k)=p_{ijk}$$

where i, j and k take the values zero or one and the sum of p_{ijk} for all i, j and k equals to unity

$$\sum_{ijk}p_{ijk}=1$$

where $\sum_{ijk}$ means the sum i, j and k varying zero or one.

We shall call this distribution as trivariate Bernoulli distribution. The marginal distribution is usual univariate Bernoulli or bivariate Bernoulli. The distribution of X is given by

$$P(X=i)=\sum_{jk} p_{ijk}=p_{i00}+p_{i10}+p_{i01}+p_{i11}, \qquad (i=0, 1).$$

Therefore we have

$$E(X)=\sum_{jk} p_{1jk}=p_{100}+p_{110}+p_{101}+p_{111}=P(X=1).$$

Marginal distribution of (X, Y), (Y, Z) and (X, Z) is bivariate Bernoulli. The joint distribution of (X, Y) is given by

$$P(X=i,\ Y=j)=\sum_{k} p_{ijk}=p_{ij0}+p_{ij1},$$

where i, j and k take the value zero or one and similarly we can varify the marginal distribution of (Y, Z) and (X, Z).

Covariance of (X, Y) of bivariate Bernoulli distribution law.

$$\begin{aligned}\operatorname{Cov}(X, Y)&=E(XY)-E(X)E(Y)\\ &=P(X=1,\ Y=1)-P(X=1)P(Y=1)\\ &=P(X=1,\ Y=1)-[P(X=1,\ Y=0)+P(X=1,\ Y=1)]\\ &\qquad\qquad\cdot[P(X=0,\ Y=1)+P(X=1,\ Y=1)]\\ &=P(X=0,\ Y=0)P(X=1,\ Y=1)-P(X=1,\ Y=0)P(X=0,\ Y=1).\end{aligned}$$

If we assume the value of the covariance of the bivariate Bernoulli distribution law to be zero; $\operatorname{Cov}(X, Y)=0$ then we have the fact that X and Y are independent random variables as follows

$$P(X=0,\ Y=0)P(X=1,\ Y=1)=P(X=1,\ Y=0)P(X=0,\ Y=1)$$

this equality implies

$$P(X=i,\ Y=j)=P(X=i)P(Y=j)$$

for all i and j. Then X and Y are mutually independent random variables as to be proved.

LEMMA. *The moment generating function of the trivariate Bernoulli distribution is given by*

$$\begin{aligned}g(s_1, s_2, s_3)&=[p_{000}+p_{100}s_1+p_{010}s_2+p_{001}s_3+p_{110}s_1s_2+p_{011}s_2s_3+p_{101}s_1s_3+p_{111}s_1s_2s_3]\\ &=\sum_{ijk} p_{ijk}s_1^{i}s_2^{j}s_3^{k}.\end{aligned}$$

2.2. Trivariate binomial distribution.

We shall derive the distribution of the sum of n mutually independent random vectors (X_i, Y_i, Z_i) $(i=1, 2, \cdots, n)$ which have identical trivariate Bernoulli distribution law. We shall denote $E[n_{000}, \cdots, n_{111}]$ the event that in the first n observed random vectors the 2^3 events $(X=i, Y=j, Z=k)$ occur n_{ijk} times where i, j and k varies zero or one respectively and the sum of the numbers n_{ijk} equals to n. Then the probability of the event $E[n_{000}, \cdots, n_{111}]$ equals to

$$P\{E[n_{000}, \cdots, n_{111}]\}=\frac{n!}{\prod_{ijk} n_{ijk}!}\prod_{ijk} p_{ijk}^{n_{ijk}}$$

We shall denote $F[k_1, k_2, k_3]$ the event $\sum_{i=1}^{n} X_i=k_1$, $\sum_{i=1}^{n} Y_i=k_2$, $\sum_{i=1}^{n} Z_i=k_3$ for all integers k_1, k_2 and k_3 satisfying $0\leqq k_1, k_2, k_3\leqq n$. The event $F[k_1, k_2, k_3]$ is expressed as the union of disjoint events as followings

$$F[k_1, k_2, k_3]=\bigcup_{[\sum_{jk} n_{1jk}=k_1, \sum_{ik} n_{i1k}=k_2, \sum_{ij} n_{ij1}=k_3]} E[n_{000}, \cdots, n_{111}].$$

In the following lines we shall denote the condition of the union as $[C]$. Therefore we have

$$P\{F[k_1, k_2, k_3]\}=\sum_{[C]} P\{E[n_{000}, \cdots, n_{111}]\}.$$

LEMMA. *The distribution of the convolution of n independent identical trivariate Bernoulli distribution is given by*

$$P\{\Sigma X_i=K_1, \Sigma Y_i=K_2, \Sigma Z_i=K_3\}=\sum_{[C]} p\{E[n_{000}, n_{100}, \cdots, n_{111}]\}.$$

We shall call this trivariate Bernoulli distribution.

LEMMA. *The moment generating function of the trivariate Bernoulli distribution is given by*

$$g(s_1, s_2, s_3)=[\sum_{ijk} p_{ijk}s_1^i s_2^j s_3^k]^n.$$

2.3. Trivariate Poisson distribution.

2.3.1. Definition of the trivariate Poisson distribution.

In the preceding section we have defined the trivariate binomial distribution by the method of convolution of the n independent identical trivariate Bernoulli distribution. Then the expected numbers of events $(0, 0, 0), \cdots, (1, 1, 1)$ in the n independent observations of (X, Y, Z) equal to $np_{000}, \cdots, np_{111}$ respectively. As given in the method of famous Poisson's theorem in one dimensional case we assume the conditions $[D]$

$$[D] \quad np_{100}=\lambda_{100}, \cdots, np_{111}=\lambda_{111}$$

where $\lambda_{100}, \cdots, \lambda_{111}$ are fixed nonnegative parameters and we assume $n\to\infty$, then we have

$$P\{E[n_{000}, \cdots, n_{111}]\}=\frac{n!}{n_{000}!\cdots n_{111}!}\,p_{000}{}^{n_{000}}\cdots p_{111}{}^{n_{111}}$$

$$=\frac{n!}{n_{000}!}\,\frac{\lambda_{100}{}^{n_{100}}\lambda_{010}{}^{n_{010}}\cdots\lambda_{111}{}^{n_{111}}}{n_{100}!\,n_{010}!\cdots n_{111}!}\,\frac{[1-\sum_{ijk\,(i,j,k)\neq(0,0,0)} p_{ijk}]^{n_{000}}}{n^{n_{100}}n^{n_{010}}\cdots n^{n_{111}}}$$

$$=\frac{n!}{n^{n_{100}}n^{n_{010}}\cdots n^{n_{111}}n_{000}!}\,\frac{\lambda_{100}{}^{n_{100}}\lambda_{010}{}^{n_{010}}\cdots\lambda_{111}{}^{n_{111}}}{n_{100}!\,n_{010}!\cdots n_{111}!}$$

$$\cdot\left[1-\frac{\sum_{ijk\,(i,j,k)\neq(000)}\lambda_{ijk}{}^{n_{000}}}{n}\right]^{n-\sum_{ijk\,(i,j,k)\neq(0,0,0)} n_{ijk}}$$

$$\longrightarrow\frac{\lambda_{100}{}^{n_{100}}\lambda_{010}{}^{n_{010}}\cdots\lambda_{111}{}^{n_{111}}}{n_{100}!\,n_{010}!\cdots n_{111}!}\,e^{-\lambda_{100}-\lambda_{010}-\cdots-\lambda_{111}}$$

see Kendall and Stuart [3] in the bivariate case.

Under the condition $[D]$ we have the probability of the event $F[k_1, k_2, k_3]$ equals

$$P\{F[k_1, k_2, k_3]\}\longrightarrow\sum_{[C]}\frac{\lambda_{100}{}^{n_{100}}\lambda_{010}{}^{n_{010}}\cdots\lambda_{111}{}^{n_{111}}}{n_{100}!\,n_{010}!\cdots n_{111}!}\,e^{-\lambda_{100}-\lambda_{010}-\cdots-\lambda_{111}}.$$

We shall call this limitting distribution as trivariate Poisson distribution.

THEOREM 3.1. *The sum vector of n independent identical trivariate Bernoulli random vectors has a limitting trivariate distribution*

$$P\{X_1=K_1, X_2=K_2, X_3=K_3\}=\sum_{[C]}\frac{\lambda_{100}{}^{n_{100}}\lambda_{010}{}^{n_{010}}\cdots\lambda_{111}{}^{n_{111}}}{n_{100}!\,n_{010}!\,n_{111}!}\,e^{-\lambda_{100}-\lambda_{010}-\cdots-\lambda_{111}}$$

where we assumed the condition $[D]$.

In this theorem if we denote

$$P(k\,;\lambda)=\frac{\lambda^k}{k!}e^{-\lambda}$$

for all nonnegative integer k, then we can verify the limitting distribution as

$$\sum_{[C]}\prod_{ijk\,(i,j,k)\neq(0,0,0)} p(n_{ijk}\,;\lambda_{ijk})$$

If X_{ijk} $(i, j, k)\neq(0, 0, 0)$ has one dimensional Poisson distribution law $P(k\,;\lambda_{ijk})$ and $X_{100}, X_{010}, \cdots, X_{111}$ are independently distributed then we have the next theorem.

THEOREM 3.2. *If the trivariate random vector* (X_1, X_2, X_3) *has the trivariate Poisson distribution law then we have uniquely*

$$\begin{cases} X_1=X_{100}+X_{110}+X_{101}+X_{111} \\ X_2=X_{010}+X_{110}+X_{011}+X_{111} \\ X_3=X_{001}+X_{101}+X_{011}+X_{111} \end{cases}$$

where $X_{100}, X_{010}, \cdots, X_{111}$ are independent 2^3-1 Poisson random variables of parameters $\lambda_{100}, \lambda_{010}, \cdots, \lambda_{111}$ respectively.

Proof. If X_{ijk} is a Poisson random variable with parameter λ_{ijk} then

$$P(X_{ijk}=n_{ijk})=\frac{\lambda_{ijk}{}^{n_{ijk}}}{n_{ijk}!}e^{-\lambda_{ijk}}=P(n_{ijk};\lambda_{ijk}).$$

Therefore we have

$$P(X_1=k_1, X_2=k_2, X_3=k_3)=\sum_{[C]}\frac{\lambda_{100}{}^{n_{100}}\lambda_{010}{}^{n_{010}}\cdots\lambda_{111}{}^{n_{111}}}{n_{100}!\,n_{010}!\cdots n_{111}!}e^{-\lambda_{100}-\lambda_{010}-\cdots-\lambda_{111}}$$

$$=\sum_{[C]}\prod_{ijk\ (i,j,k)\neq(0,0,0)}P(n_{ijk};\lambda_{ijk})$$

then the sum will be expressed

$$=\sum_{[C]}P(X_{100}=n_{100}, X_{010}=n_{010}, \cdots, X_{111}=n_{111}).$$

Therefore we have

$$=P(X_{100}+X_{110}+X_{101}+X_{111}=k_1,\ X_{010}+X_{110}+X_{011}+X_{111}=k_2,$$
$$X_{001}+X_{101}+X_{011}+X_{111}=k_3),$$

this proves the theorem.

2.3.2. Characters of trivariate Poisson distribution.
Next lemma is a fundamental character of bivariate Poisson distribution.

LEMMA 3.1. *If a random vector (X_1, X_2) has a bivariate Poisson law then we have uniquely*

$$X_1=X_{10}+X_{11}, \qquad X_2=X_{01}+X_{11}$$

where X_{10}, X_{01} and X_{11} are independent 2^2-1 Poisson random variables of parameter λ_{10}, λ_{01} and λ_{11} respectively.

Proof. The joint distribution of bivariate Poisson distribution is given by

$$P(X_1=k_1, X_2=k_2)=\sum_{\substack{n_{10}+n_{11}=k_1\\ n_{01}+n_{11}=k_2}}\frac{\lambda_{10}{}^{n_{10}}\lambda_{01}{}^{n_{01}}\lambda_{11}{}^{n_{11}}}{n_{10}!\,n_{01}!\,n_{11}!}e^{-\lambda_{10}-\lambda_{01}-\lambda_{11}}.$$

If we assume X_{10}, X_{01} and X_{11} are Poisson random variables with parameter λ_{10}, λ_{01} and λ_{11} then the right side of the equation becomes

$$=\sum_{\substack{n_{10}+n_{11}=k_1\\ n_{01}+n_{11}=k_2}}P(X_{10}=n_{10}, X_{01}=n_{01}, X_{11}=n_{11})$$

$$=P(X_{10}+X_{11}=k_1, X_{01}+X_{11}=k_2).$$

It is easily shown the fact by the preceding Theorem 3.2 which is given in the next lemma.

LEMMA 3.2. *Any marginal distribution of the trivariate Poisson distribution* *is usual Piosson or bivariate Poisson distribution.*

In the trivariate case if (X_1, X_2, X_3) has a Poisson law then by the Theorem 3.2 we have uniuely 2^3-1 Poisson random variables $X_{100}, \cdots, X_{111}$ with parameter $\lambda_{100}, \cdots, \lambda_{111}$ which are mutually independent. If we put $S(X_1)$ the set of Poisson random variables construction the random variable X_1 then we can express

$$S(X_1)=\{X_{100}, X_{110}, X_{101}, X_{111}\}$$

and similarly

$$S(X_2)=\{X_{010}, X_{110}, X_{011}, X_{111}\}$$

$$S(X_3)=\{X_{001}, X_{101}, X_{011}, X_{111}\}\,.$$

And we denote 0 which means empty set or random variable zero with probability one.

LEMMA 3.3. *If (X_1, X_2, X_3) has a trivariate Poisson distribution law then we have the fact that generally $S(X_1)\frown S(X_2)=0$ and $S(X_2)\frown S(X_3)=0$ do not imply $S(X_1)\frown S(X_3)=0$.*

Proof. If we assume $S(X_1)\frown S(X_2)=0$ then by the Theorem 3.2 we have $X_1=X_{100}+X_{101}$ and $X_2=X_{010}+X_{011}$. This means that (X_1, X_2) has a independent type bivariate Poisson distribution; that is, X_1 and X_2 has a independent Poisson distribution law. Additionary we assume $S(X_2)\frown S(X_3)=0$ then X_2 and X_3 has a independent Poisson distribution law. But we can not conclude $S(X_1)\frown S(X_3)=0$ generally under the two assumptions. Because under the two assumptions we have

$$S(X_1)\frown S(X_3)=X_{101}\,.$$

Which means $X_1=X_{100}+X_{101}$, $X_3=X_{001}+X_{101}$ has a bivariate Poisson law if we assume $X_{101}\neq 0$ with probability one.

LEMMA 3.3.1. *If we assume (X_1, X_2, X_3) has a trivariate Poisson distribution law with parameter $\lambda_{110}=\lambda_{011}=\lambda_{111}=0$ and $\lambda_{101}\neq 0$ then X_1, X_2 are mutually independent and X_2, X_3 are also but X_1, X_3 are not mutually independent.*

LEMMA 3.4. *The moment generating function of the trivariate Poisson distribution is given by*

$$h(s_1, s_2, s_3)=\exp\{-\sum_{\substack{ijk\\(i,j,k)\neq(0,0,0)}}\lambda_{ijk}+\sum_{\substack{ijk\\(i,j,k)\neq(0,0,0)}}\lambda_{ijk}s_1{}^i s_2{}^j s_3{}^k\}\,.$$

Proof.

$$h(s_1, s_2, s_3)=\lim_{n\to\infty} g(s_1, s_2, s_3)^n$$

$$=\lim_{n\to\infty}[\sum_{i,j,k} p_{ijk}s_1{}^i s_2{}^j s_3{}^k]^n$$

$$=\lim_{n\to\infty}\Big[1-\frac{1}{n}\sum_{\substack{ijk\\(i,j,k)\neq(0,0,0)}}\lambda_{ijk}+\frac{1}{n}\sum_{\substack{ijk\\(i,j,k)\neq(0,0,0)}}\lambda_{ijk}s_1{}^i s_2{}^j s_3{}^k\Big]^n$$

$$= \exp\{-\sum_{\substack{ijk \\ (i,j,k)\neq(0,0,0)}} \lambda_{ijk} + \sum_{\substack{ijk \\ (i,j,k)\neq(0,0,0)}} \lambda_{ijk} s_1{}^i s_2{}^j s_3{}^k\}\ .$$

See Feller [1] or Kawamura [2]. The moment generating function of the bivariate case.

LEMMA 3.5. *If* $\lambda_{101}=\lambda_{111}=0$ *then* X_1, X_2 *are not independent and* X_2, X_3 *also but* X_1, X_3 *are independent random variable.*

LEMMA 3.5.1. *We assume* X_1, X_2, X_3 *has a trivariate Poisson distribution law where* X_1, X_2 *and* X_2, X_3 *has a bivariate Poisson distribution law of dependent type and if we assume* $\lambda_{101}=\lambda_{111}=0$ *then* X_1, X_3 *has a independent Poisson distribution law.*

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