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CONTINUITY OF THE LINEAR OPERATOR METHOD FOR STRICT AND WEAK MEASURE TOPOLOGIES[1]

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The continuous dependence of the solution to the Dirichlet problem on the boundary data is an interesting and useful notion [4], and motivated by this, we have already posed an analagous question for the linear operator method of L. Sario [1], [7], [10]. In this situation, since the harmonic singularity functions of $H(W')$ defined oh the boundary neighborhood W' are unbounded, only topologies applicable in such function spaces are of interest. Two examples of such topologies are realized in the so-called $\mathfrak{S}$ topologies of [2] and [11] where, on the one hand, $\mathfrak{S}$ consists of all compact sets, and on the other, $\mathfrak{S}$ consists of all finite sets. The continuity of the linear operator method was investigated for such topologies in [6], wherein the question was phrased in the framework of Rodin and Sario [7], That is, one considers the restriction $\Phi: H(W)\to H(W')$ mapping each harmonic function $h(p)$ on W to the singularity $h|_{W'}(p)$. The linear operator method, whose essence is the extension of singularity functions modulo regular singularity functions, is then considered in terms of the algebraic isomorphism $\Phi: H(W)/K\to H(W')/LC(\alpha)$, where K is the set of constant functions, and $LC(\alpha)$ is the set of regular singularities. When $H(W')$ and $H(W)$ carry the topology of compact convergence, it was shown in [7] that the mapping $\bar{\Phi}$ is in fact a topological isomorphism when its range and domain are equipped with the usual quotient topologies. In this sense, it is said there that the linear operator method is continuous.

1. Purpose of this investigation. Our purpose here is the further consideration of linear topologies defined on $H(W')$ and $H(W)$ for which the question of continuity of the operator method is meaningful. First, the weak measure and strict topologies denoted α and β by Buck [3], would seem to be interesting candidates. Second, the Mackey and strong topologies seem to be interesting candidates as well. Furthermore, the results of Rubel and Shields [8] suggest that one would obtain spaces topologically different from those encountered in [6].

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However, our interest is in spaces of unbounded harmonic functions, and in order to construct facsimilies of such topologies, it is necessary to restrict our attention to a certain subspace $H_{K'}(W')$ of the space $H(W')$ of all singularity functions as well as to the subspace $H_K(W)=H(W)\cap H_{K'}(W')$. Two natural constraints seem relevant: (i) the subspace $H_K(W')$ should be sufficiently comprehensive to include the typical applications that are made of the linear operator method [1], and (ii) the restriction $\Phi: H_K(W)\rightarrow H_K(W')$ must, when quotientized, be an algebraic isomorphism. For this last condition, it is sufficient that $LC(\alpha)\subset H_{K'}(W')$. One suggestion for the definition for such a subspace $H_{K'}(W')$ is made in the work of B. A. Taylor for the space of entire functions [13], and this is the content of Definition 4.

The topologies τ and τ' on $H_K(W)$ and $H_{K'}(W')$ will always be taken as the same kind. For example, in [6], when τ was taken as an $\mathfrak{S}$-topology on $H(W)$, then τ' was taken as the topology determined by the reduction $\mathfrak{S}'$ of $\mathfrak{S}$ to sets of $\overline{W}'$. However, such a direct relation between topologies of interest isn't always so apparent. Hence an attempt to distinguish those pairs of topologies of interest in the domain and range of $\bar{\Phi}$ is now made.

DEFINITION 1. The pair of locally convex topologies (τ, τ') on H_K and $H_{K'}$ is called compatible if $\Phi: (H_K, \tau)\rightarrow(H_{K'}, \tau')$ is continuous. When the topologies τ and τ' are compatible, the continuity of the linear operator method is defined in terms of quotient spaces $E=(H_K/K, q(\tau))$ and $F=(H_{K'}/LC(\alpha), q(\tau'))$.

DEFINITION 2. The linear operator method is said to be continuous when $\bar{\Phi}: E\rightarrow F$ is a topological isomorphism.

For compatible topologies, it suffices to consider the continuity of $\bar{\Psi}=\bar{\Phi}^{-1}$ and the following proposition exibits a certain uniqueness for such topologies. Indeed, when τ' is fixed, then $q(\tau)$ is the finest locally convex topology for which $\bar{\Psi}$ is continuous; and when τ is fixed, then $q(\tau')$ is the coarsest (least fine) topology for which $\bar{\Psi}$ is continuous.

PROPOSITION. (i) *If $q(\tau')$ is fixed on $H_{K'}/LD(\alpha)$, then $q(\tau)$ is at least as fine as each locally convex topology Q for which $\bar{\Psi}$ is continuous.* (ii) *If $q(\tau)$ is fixed on H_K/K then, $q(\tau')$ is at least as coarse as each locally convex topology Q' for which $\bar{\Psi}$ is continuous.*

Proof. To establish (i), we consider V a Q-neighborhood of θ for which there is $U\in q(\tau)$ satisfying $\bar{\Psi}(U)\subset V$. But, with the compatibility of τ and τ', it follows that $\bar{\Psi}$ is open, and this means that $W\subset\bar{\Psi}(U)\subset V$ for some $q(\tau)$-neighborhood of θ. Hence it follows that $Q\leqq q(\tau)$.

The second second statement is proved in the same way, for suppose that $U'\in q(\tau')$. Since $\bar{\Psi}(U')\in q(\tau)$, it follows from the continuity of that $\bar{\Psi}(V')\subset\bar{\Psi}(U')$ for some $V'\in Q'$. But $\bar{\Psi}$ is an algebraic isomorphism, and this means that $V'\subset U'$; that is $q(\tau')\leqq Q'$.

An example of the part (i) of this proposition occurs in [5], where the topology of interest on each space is the compact open topology.

The mapping $\overline{\Psi}=\overline{\Phi}^{-1}$ is given in [7] as $\overline{\Psi}[s]=[Df]\cup[LDf+s]$ and this is realized from the linear mapping $\Psi: H_{K'}\to H_K$ defined as:

$$\text{(1)} \qquad \Psi(s)=\begin{cases} Df & \text{on } \Omega \\ LDf+s-Ls & \text{on } W'. \end{cases}$$

Here, in terms of a regular region Ω with $\partial\Omega\subset W'$, f satisfies $(I-LD)f=s-Ls$ in a subspace X of $C(\partial\Omega)$ in which $\|LD\|<1$.

Of interest here are the α and β topologies defined on the space $H_K(W)$, where they are dehoted α_K and β_K and called the weak K-Borel measure topology and the K-strict topology respectively. The Mackey topology τ_K and the strong topology γ_K are then readily defined in terms of the dual $(H_K, \alpha_K)'$. In these terms, it is shown that the linear operator method is a continuous operator in the sense of the definition already cited. However, the question about pairwise distinctness of these topologies is only partially answered.

2. Notation and definitions. In order to describe a class of harmonic singularities satisfying the constraints of (i) and (ii) of § 1, we start with an arbitrary Riemann surface W_0, from which W is formed by removing a finite point set $\{p_1, \cdots, p_K\}$. A neighborhood of the ideal boundary $W'\subset W$ is then realized by $\bigcup_1^k(\Delta(p_i)\backslash\{p_i\})\cup W_0'$, where W_0' is a boundary neighborhood in W_0. The compact boundary of W' is denoted α, and the resulting union of bordered surfaces is written $\overline{W}'$, as is customary. Since the harmonic functions are to have singular behavior near p_i, we single out the following class of continuous functions which are 0 there. This is done in terms of $C_0(W)$, the class of continuous functions vanishing at ∞.

DEFINITION 3. The set of positive functions $k\in C_0(W)$ satisfying (*) $\lim_{p\to p_i} k(p)\exp|c/z|=0$ for each complex c, is denoted $K(W)$. The set of positive functions $k'\in C_0(\overline{W}')$ satisfying (*) is denoted $K(W')$. Here the conditlon (*) is understood in teams of points p in a parameter disc at p_i, and is in fact, independent of the local variable z representing p, since the condition is to hold for all c. It is now easy to single out those harmonic functions of $H(W)$ upon which a defieition of a strict topology can be made.

DEFINITION 4. The subspace $H_K(W)$ of $H(W)$ for which $hk\in C_0(W)$ for all $k\in K(W)$ is called the space of K-harmonic functions. The subspace $H_{K'}(W')$ of K-harmonic singularities is defined analogously in terms of $C_0(\overline{W}')$, and the notations $H_K(W)$ and $H_{K'}(W')$ are abbreviated to H_K and $H_{K'}$ respectively.

In order to consider the Definition 2, it is important to establish that the mapping $\Phi: H_K(W)\to H_{K'}(W')$ induces an isomorphism between $H_K(W)/K$ and

$H_{K'}(W')/LC(\alpha)$ in the manner of [7]. First, when $h\in H_K$, then $h|_{W'}\in H_{K'}$, and furthermore the inclusions $K\subset H_K$ and $LC(\alpha)\subset H_{K'}$ hold. Hence the quotient map $\bar{\Phi}: H_K/K\rightarrow H_{K'}/LC(\alpha)$ is defined. Now, if $[s]\in H_{K'}/LC(\alpha)$, then of course, $[s]\in H(W')/LC(\alpha)$ as well, and it follows from [7] that $[\Phi h]=[s]$ for some $h\in H(W)$. Since this means that $h-s\in LC(\alpha)$, it follows that $h\in H_K$, and $\bar{\Phi}$ remains an isomorphism between the quotients of K-harmonic functions and K-harmonic singularities.

3. The K-strict topology. By virtue of the Definition 4, it follows that $\sup_W |h(p)k(p)|=\|h\|_k$ is finite for each pair $h\in H_K$ and $k\in K(W)$. In terms of the notation $V(k;r)$ for $\{h;\|h\|_k\leqq r\}$, the relation $V(k_1, r_1)\cap V(k_2, r_2)\supset V(k_1\vee k_2, r_1\wedge r_2)$ implies that the collection $B=\{V(k;r); k\in K(W)$ and $r>0\}$ is a base for a filter. In fact, the base $B=\{V(k;r); k(p)\leqq 1$ and $r>0\}$ will suffice since $V(k, r)=V(k/k_0, r/k_0)$ when $k_0=\sup_W k(p)$. Furthermore, the sets $V(k, r)$ are balanced (circled), convex and absorbing (radial) and obviously satisfy $V(k, r/2)+V(k, r/2)\subset V(k, r)$. That is, B in a neighborhood base at θ for a locally convex topology β_K defined on $H_K(W)$. In an analogous manner, a K-strict topology $\beta_{K'}$ is defined on $H_{K'}(W')$.

THEOREM 1. *The mapping $\Psi: H_{K'}\rightarrow H_K$ is continuous when each space is equipped with the K-strict topology.*

Proof. If V_θ is an arbitrary β_K neighborhood of θ in $H_K(W)$, then $V(k;r)\subset V_\theta$ for some $V(k;r)$ with $k(p)\leqq 1$, and since Ψ is linear and $rV(k;1)=V(k;r)$, it suffices to take $r=1$. In terms of $a=\min\{k'(z); z\in\alpha\}$ and $b=\min\{k'(z); z\in\partial\Omega\}$, we form the $\beta_{K'}$ neighborhood $V'(k';(1/4)(a\wedge b)(1-\|LD\|)$, where $k'=k|_{\bar{W}'}$, and claim that $\Psi(V')\subset V$. To see this, we let $s\in V'$, and observe with (1) that

$$\|\Psi(s)\|_K=\sup_W |\Psi(s)k|=\sup_{\bar{\Omega}} |Df\cdot k|\vee \sup_{W'} |(LDf+s-Ls)k|. \tag{2}$$

Since f and s are related on $\partial\Omega$ by $(I-LD)f=s-Ls$, it follows that

$$\sup_{\partial\Omega}|f|\leqq(1-\|LD\|)^{-1}(\sup_{\partial\Omega}|s|+\sup_{\partial\Omega}|Ls|)\leqq(1-\|LD\|)^{-1}(\|s\|)_{\partial\Omega}+\|s\|_\alpha) \tag{3}$$

But harmonic K-singularities $s\in V'$ satisfy $|s(p)|\leqq(1/4)(1-\|LD\|)$ on $\partial\Omega\cup\alpha$ and the estimate $\sup_{\partial\Omega}|f|\leqq 1/2$ follows directly from (3).

The first term on the right side of (2) is now easily bounded by 1/2, since $\sup_{\bar{\Omega}}|Df\cdot k|\leqq\sup_{\Omega}|Df|=\sup_{\partial\Omega}|f|$. The second term on the right side of (2) is estimated in much the same manner. To start, we observe that $\sup_{W'}|kLDf|\leqq\sup_{W'}|LDf|=\sup_{\alpha}|Df|\leqq\sup_{\partial\Omega}|f|\leqq 1/2$, and that $\sup_{W'}|s\cdot k'|\leqq 1/4$ for each $s\in V'$. Finally, the inequality $1-\|LD\|<1$ implies that $\sup_{W'}|k'Ls|\leqq\sup_{W'}|Ls|=\sup_{\alpha}|s|<1/4$. The proof is complete because $\|\Psi(s)\|_k$ is no larger than 1.

Since $(\beta_K, \beta_{K'})$ is easily seen to be a compatible pair, it follows from Theorem 1 and the proposition that the quotient topologies are unique topologies of interest in the sense of Definition 1.

COROLLARY. (i) *When $H_{K'}/LC(\alpha)$ is given the $q(\beta_{K'})$ topology, then $q(\beta_K)$ is the finest locally convex topology on H_K/K for which Ψ is continuous.* (ii) *When H_K/K is given the $q(\beta_K)$ topology, then $q(\beta_{K'})$ is the coarsest locally convex topology on $H_{K'}/LC(\alpha)$ for which Ψ is continuous.*

4. The Weak K-Borel Measure Topology. To start, we let $M_K(W)$ be the set of all measures of the form $kd\mu$, where $k\in K(W)$ and μ is a finite signed Borel measure. Now, with writing

$$k_1d\mu_1+k_2d\mu_2=k_1\vee k_2\Big(\frac{k_1}{k_1\vee k_2}d\mu_1+\frac{k_2}{k_1\vee k_2}d\mu_2\Big),$$

it follows that $M_K(W)$ is a vector space since $k_1\vee k_2$ belongs to $K(W)$ when k_1 and k_2 belong. Certainly for each $\nu\in M_K(W)$, $\langle h, \nu\rangle=\int hd\nu$ is finite, and this bilinear form places $H_K(W)$ and $M_K(W)$ in duality provided that ν_1 is taken equivalent to ν_2 when $\langle \nu_1, h\rangle=\langle \nu_2, h\rangle$ for all $h\in H_K$. Without changing the notation, we assume that such is understood for M_K.

DEFINITION 5. The weak K-Borel measure topology on $H_K(W)$, denoted α_K, is the weakest (locally convex) topology for which each of the functionals $\phi_\nu: h\to\langle h, \nu\rangle$ is continuous. When W is replaced by $\overline{W}'$, the resulting weak topology on $H_{K'}$ is denoted by $\alpha_{K'}$.

It is a standard matter that each linear functional ϕ on H_K, which is continuous for α_K, is of the form ϕ_ν for some $\nu\in M_K$[11; p. 124]. Hence the continuity of Ψ for the weak K-Borel measure topology will follow directly from standard theorems about weakened topologies provided that $(H_K, \beta_K)'=M_{K'}$, and this is now established in the manner of Rubel and Shields [8], with appropriate accomodations for the unbounded functions of H_K.

THEOREM 2. *In terms of the usual embeddings of M_K into the algebraic dual H_K^* of H_K, $(H_K, \beta_K)'=M_K=(H_K, \alpha_K)'$.*

Proof. It is easily checked that $\alpha_K\leqq\beta_K$, because each basic weak neighborhood V_θ is a finite intersection of sets $\{h; |\langle h, \nu_i\rangle|\leqq 1\}$. Now for each i, $d\nu_i=k_id\mu_i$, and it follows that

$$\{h; \|hk_0\|\leqq c\}\subset\Big\{h; \|hk_i\|\leqq\frac{1}{\|\mu_i\|}\Big\}\subset\{h; |\langle h, \nu_i\rangle|\leqq 1\},$$

wheae k_0 is $k_1\vee\cdots\vee k_n$ and $c=(\max\{\|\mu_1\|, \cdots, \|\mu_n\|\})^{-1}$.

Hence, to finish the proof, we need only establish that each linear functional ϕ, continuous for the K-strict topology, is representable by means of a measure ν in M_K, and the argument required is that of [8]. In particular, since ϕ is

β_K-continuous, it induces a bounded linear functional $\tilde{\phi}(kh)=\phi(h)$ on the vector space $kH_K(W)$ of $C_0(W)$. Now because $\tilde{\phi}$ has an extension to $C_0(W)$, it is representable as $\tilde{\phi}(kh)=\langle kh, \mu\rangle$ for some finite signed Borel measure μ. That is, $\phi(h)=\langle h, \nu\rangle$, where $d\nu=kd\mu$.

COROLLARY 1. *The function $\Psi: H_{K'}\to H_K$ defined in (1) by $\Psi(s)=Df\cup LDf+s-Ls$ is continuous when the domain and range are equipped with the weak K-Borel measure topology.*

Proof. According to the Theorem 1, Ψ is continuous for the K-strict topology and continuity for the weakened topologies $\sigma(H_{K'}, H'_{K'})$ and $\sigma(H_K, H'_K)$ follows from standard theorems of functional analysis when each dual is taken with respect to the K-strict topology. But Theorem 2 implies that H'_K is in fact $M_K(W)$ and the weak topology $\sigma(H_K, H'_K)$ is the topology α_K.

COROLLARY 2. *When the topology $q(\alpha_{K'})$ $(q(\alpha_K))$ is fixed on $H_{K'}/LC(\alpha)$ (H_K/K), then $q(\alpha_K)$ $(q(\alpha_{K'}))$ is the finest (coarsest) locally convex topology for which Ψ is continuous.*

5. Comparison of the α_K and β_K topologies. If the topologies α_K and β_K are equal, then of course the Corollary 1 of Theorem 2 is of no interest. So the purpose of this section is to establish that β_K is properly finer than α_K by exibiting a topological property which these fail to share. Some properties that these topologies do share are examined as well.

THEOREM 3. *The space H_K is β_K-complete and fails to be α_K-complete.*

Proof. The idea of each of these assertions is again given by Rubel and Shields [8], and only a verification that their proof is valid for the unbounded functions of H_K need be given.

To demonstrate the first assertion, we let F be a β_K-Cauchy filter on H_K. With [8], we observe that for an arbitrary $k_0\in K(W)$, k_0F is a base for a Cauchy filter in the norm topology on $C_0(W)$, and in this topology, $k_0F\to f_0\in C_0(W)$. Hence it follows that F converges to $h_0=f_0/k_0$ uniformly on compact sets, and h_0 is then harmonic.

The demonstration is complete with establishing (i) $h_0\in H_K(W)$ and (ii) F converges to h_0 in the β_K topology. Obviously $h_0k_0\in C_0(W)$, and the same is true for h_0k for each $k\in K(W)$; because, kF donverges to, say $f\in C_0(W)$, and with the uniqueness of limits, we conclude that $f/k=h_0$. Hence it follows that $h_0k\in C_0(W)$ and $h_0\in H_K(W)$. As for (ii), since k_0F converges to f_0 in the norm topology, for $\varepsilon>0$, there is $F\in F$ for which

$$F\subset h_0+\left\{\frac{f}{k_0}; f\in C_0(W) \text{ and } \|f\|\leqq\varepsilon\right\}.$$

But $h_0\in H_K(W)$ and $F\subset H_K(W)$ as well, so it follows that $F\subset h_0+\{h\in H_K(W);$

$\|hk_0\| \leqq \varepsilon\}$. Hence (ii) is established.

To show that (H_K, α_K) fails to be complete, one need only construct a Cauchy filter F on H_K which converges to an element f of $C_K(W)$ and fails to be harmonic. Here, $C_K(W)$ is the set of continuous functions f on W for which $kf \in C_0(W)$ for each $k \in K(W)$.

The filter is constructed in the manner of the net of [8]. That is, for each finite set $\{\nu_1, \cdots, \nu_n\}$ of linearly independent measures of $M_K(W)$, we define

$$F_{\nu_1, \cdots, \nu_n} = \{h \in H_K ; \langle h, \nu_i \rangle = \langle f, \nu_i \rangle, \ i=1, \cdots, n\} .$$

The collection B of all such sets is obviously a base for a filter on H_K provided that each fails to be empty. On the space $V = sp(\nu_1, \cdots, \nu_n)$, equipped with the topology induced by $\sigma(M_K, H_K)$, there is a continuous linear functional ϕ for which $\phi(\nu_i) = \langle f, \nu_i \rangle$. Now, since $(M_K, \sigma(M_K, H_K))$ is locally convex, ϕ has a continuous $(\sigma(M_K, H_K))$ linear extension ϕ_0 to M_K([11], p. 45). But M_K and H_K are a dual pair, and this means that $\phi_0(\nu) = \langle h, \nu \rangle$ for $h \in H_K$, that is, $\langle f, \nu_i \rangle = \langle h, \nu_i \rangle$. Of course $h \in F_{\nu_1, \cdots, \nu_n}$, which fails to be empty.

The proof is complete with the observation that the filter F on H_K, when considered as a filter on $C_K(W)$ converges to f for α_K.

The next theorem and corollary show some topological properties that these topologies share.

THEOREM 4. *The space H_K fails to be barreled when equipped with either the α_K or the β_K topology.*

Proof. Since $\alpha_K < \beta_K$, there is a β_K neighborhood V of θ which fails to be an α_K neighborhood. Certainly V may be taken as convex, balanced and β_K-closed, and of course it is absorbing. But according to Theorem 2, (H_K, α_K) and (H_K, β_K) nave the same dual, and therefore have the same closed convex sets. Hence V is barrel of (H_K, α_K) which fails to be a neighborhood.

To show that (H_K, β_K) fails to be barreled, we need only recall that a locally convex space E is barreled exactly when each weakly bounded set of E' is equicontinuous [11, p. 141]. Because (H_K, α_K) fails to be barreled, this means that in $(H_K, \alpha_K)'$ there is such a weakly bounded set failing to be equicontinuous. But the equality of $(H_K, \alpha_K)'$ and $(H_K, \beta_K)'$ was established in Theooem 2, and the existence then of such a set establishes that (H_K, β_K) fails to be barreled.

COROLLARY. *Neither of the spaces (H_K, α_K) or (H_K, β_K) is metrizable.*

Proof. If H_K were metrizable in the topology β_K, then (H_K, β_K) would be a Fréchet space, and sinch Fréchet spaces are barreled, the assumption of metrizability would contradict Theorem 4. Hence (H_K, β_K) isn't metrizable.

The assumption of metrizability for (H_K, α_K) would mean that the weak topology $\sigma(H_K, M_K)$ is metrizable. But according to the proposition of [6; p. 400], this would imply that $M_K(W)$ has a countable Hamel basis, and there are already uncountably many lineary independent point masses in M_K. Hence

$\sigma(H_K, M_K)$ cannot be metrizable.

6. The Mackey and Strong Topologies. Since each of the topologies α_K and β_K is consistent with the duality M_K, it would seem worth noting that there is a Mackey topology τ_K; that is, τ_K is the finest locally convex topology on H_K whose dual is M_K. It is a standard theorem of, say [11], that, since Ψ is continuous for $\alpha_{K'}$ and α_K, it is also continuous for $\tau_{K'}$ and τ_K. As was the case in [8], the question of whether or not τ_K is properly finer than β_K seems quite difficult.

Among locally convex topologies which may not be consistent with the duality $\langle H_K, M_K\rangle$ is the so-called K-strong topology, denoted γ_K. Starting with the duality $\langle H_K, M_K\rangle$, this is formed in the usual way. That is, a neighborhood base at θ in H_K consists of the set (B^0) of polars of sets $B\subset M_K$ which are $\sigma(M_K, H_K)$ bounded. Now, according to the Corollary 1 of Theorem 2, the linear mapping $\Psi: H_{K'}\to H_K$ is continuous when the range and domain are given the weak topologies. Hence there is an adjoint Ψ^* of Ψ [11, p. 128]. Now it follows cirectly that Ψ is strongly continuous; because when B is a $\sigma(M_K, H_K)$ bounded set of M_K, and B^0 is its polar, then Ψ^*B is a $\sigma(M_{K'}, H_{K'})$ bounded set of $M_{K'}$ and $\Psi(\Psi^*B)^0\subset B^0$. Hence it has been proved that

THEOREM 5. *The mapping $\Psi: H_{K'}\to H_K$ is continuous when each space is equipped with the strong topology.*

In fact, the image $\Psi^*\nu$, a measure supported in $\overline{W'}$, is realizable in a rather special way; namely, it can be written as the sum of a measure ν_2 and a measure λ supported on $\alpha\cup\partial\Omega$, where $\nu_2(E)=\nu(E\cap\overline{W'})$. To see this, we first write $\nu=\nu_1+\nu_2$ where $\nu_1(E)=\nu(E\cap W\setminus\overline{W'})$ and $\nu_2(E)=\nu(E\cap\overline{W'})$. Of course it follows that

$$\langle\Psi s, \nu\rangle=\langle\Psi s, \nu_1\rangle+\langle\Psi s, \nu_2\rangle=\langle Df, \nu_1\rangle+\langle LDf+s-Ls, \nu_2\rangle$$

Now the linear functional $f\to\langle Df, \nu_1\rangle$ is bounded on $C(\partial\Omega)$ and can, with the Riesz Representation Theorem, be written as $f\to\langle f, \lambda_1\rangle$ for some finite signed Borel measure supported on $\partial\Omega$. But $\langle f, \lambda_1\rangle=\langle(I-LD)^{-1}(s-Ls), \lambda_1\rangle$ where the functional $\phi: g\to\langle(I-LD)^{-1}g, \lambda_1\rangle$ is defined only on X, a closed subspace of $C(\partial\Omega)$. However, with the Hahn-Banach Theorem, the functional ϕ has an extension ϕ_0 to which the Riesz Theorem again applies; that is $\langle(I-LD)^{-1}(s-Ls), \lambda_1\rangle=\langle s-Ls, \lambda_2\rangle$ on $\partial\Omega$. One more application of the representation theorem to $C(\alpha)$ yields $\langle Ls, \lambda_2\rangle=\langle s, \lambda_3\rangle$, where λ_3 is supported on α. Hence it follows that $\langle Df, \nu_1\rangle=\langle s, \lambda_2-\lambda_3\rangle$.

Except for $\langle s, \nu_2\rangle$, $\langle\Psi s, \nu_2\rangle$ is handled in exactly the same way; and $\langle\Psi s, \nu\rangle=\langle s, \lambda+\nu_2\rangle$. Hence the mapping Ψ^* is only $\nu\to\lambda+\nu_2$, with λ supported in $\alpha\cup\partial\Omega$.

As was the case with α_K and β_K topologies, we conclude that $\Phi: h\to h|_{W'}$ induces a topological isomorphism in the quotients of the strong topologies. Evidently the same can be said of the Mackey topology. But criteria for establishing that $\beta_K < \tau_K < \gamma_K$ remain to be found.

REFERENCES

[1] AHLFORS, L. AND L. SARIO, Riemann surfaces, Princeton University Press, Princeton, N. J. (1960).
[2] BOURBAKI, N., Espaces vectoriels topologiques; ch. 1, 2, Hermann and Co., Paris (1955).
[3] BUCK, R. C., Bounded continuous functions on a locally compact space, Michigan Math. J. **5** (1958).
[4] GARABEDIAN, P. R., Partial differential equations, John Wiley and Sons, New York (1964).
[5] LI, T. F., On topologies for continuity of Sario's normal operator in elliptic partial differential equations, Ph. D. Thesis, North Carolina State University (1972).
[6] NICKEL, P. A., The linear operator method and linear $\mathfrak{S}$ topologies, Kōdai Math. Sem. Rep. **24** (1972).
[7] RODIN, B. AND L. SARIO, Convergence of normal operators, Kōdai Math. Sem. Rep. **19** (1967).
[8] RUBEL, L. A., AND A. L. SHIELDS, The space of bounded analytic functions on a region, Ann. de linstitut Fourier **16** (1966).
[9] RUDIN, W., Real and complex analysis, McGraw-Hill Book Company, New York (1966).
[10] SARIO, L. A., A linear operator method on arbitrary Riemann surfaces, Trans. Amer. Math. Soc. **72** (1952).
[11] SCHAEFER, H. H., Topological vector spaces, The MacMillan Company, New York (1966).
[12] TAYLOR, A. E., Introduction to functional analysis, John Wiley and Sons, Inc., New York (1958).
[13] TAYLOR, B. A., Duality and entire functions, Ph. D. thesis, University of Illinois, (1965).

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