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ON ANGULAR DERIVATIVES OF UNIVALENT FUNCTIONS

Dedicated to Professor Yûsaku Komatu on his 60th birthday

By Kôtaro Oikawa

1. Statement of Results.

1.1. Let $W=F(Z)$ be a regular univalent function on the right-half plane $\operatorname{Re} Z>0$. If

$$\lim_{\substack{Z\to\infty \\ \text{Stolz}}} F(Z)=\infty \tag{1.1}$$

$$\lim_{\substack{Z\to\infty \\ \text{Stolz}}} \frac{Z}{F(Z)}=\tau\,, \tag{1.2}$$

then τ is called the *angular derivative* of F. Here the symbol "Stolz" means that the limiting values are taken under the restriction that Z moves within the Stolz domains $\{Z \mid r<|Z|,\ |\arg Z|<\pi/2-\varepsilon\}$, $\varepsilon>0$ being arbitrary and $r>0$ indefinite.

We are interested in the problem of finding necessary conditions and sufficient conditions for F with (1.1) to have a finite non-zero angular derivative τ. The conditions are to be expressed in terms of geometric properties of the image domain. This problem has been studied by a large number of mathematicians for these fifty years. For a brief history of studies, we refer to the introductions of Warschawski's papers [13] and [14]. Among the latest contributions are Eke [6, 7], and Warschawski [15].

In the present paper, we shall apply the method of module (i. e. the method of extremal length) to the study of the above problem.

Our Theorem 1 is an improvement of Ahlfors' result [1]. It is obtained by replacing a part of Ahlfors' proof by a pair of inequalities involving module gotten by Jenkins-Oikawa [8].

Theorem 2 is an improvement of Warschawski's result [12]. Its sufficiency part is contained implicitly in Dufresnoy [2] and Warschawski [11] (see more detailed explanation in 1.5°).

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1.2. Let Ω be the image of the right-half plane under the conformal mapping $W=F(Z)$. If F satisfies (1.1), then Ω possesses an accessible boundary

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point p over $W=\infty$, which corresponds to $Z=\infty$ in the well-known manner. Conversely, if so, F satisfies (1.1).

Suppose such an F has a finite non-zero angular derivative τ. Then Ω contains any Stolz domain about the ray $\arg W=-\arg\tau$, and the accessible boundary point p is identical with the one over $W=\infty$ determined by the ray $W=-\arg\tau$. These facts are apparent from the following:

LEMMA 1. *Let $W=H(Z)$ be a regular univalent function on the domain $\Sigma=\{Z|r_0<|Z|, |\arg Z|<\pi/2-\varepsilon_0\}$, $0<r_0$, $0\leqq\varepsilon_0<\pi/2$. Suppose*

$$\lim H(Z)=\infty, \qquad \lim\frac{Z}{H(Z)}=\tau\neq 0,\ \infty$$

as $\Sigma\ni Z\to\infty$. Then, for any $\varepsilon_1(>\varepsilon_0)$, there exists $R_1(>0)$, $r_2(>r_0)$, and $\varepsilon_2(>\varepsilon_0)$ such that $\{W|R_1<|W|, |\arg W+\arg\tau|<\pi/2-\varepsilon_1\}\subset H(\{Z|r_2<|Z|, |\arg Z|\pi/2-\varepsilon_2\})$.

The fact that $\lim((\arg Z)-(\arg H(Z)+\arg\tau))=0$ uniformly as $\Sigma\ni Z\to\infty$ furnishes the proof immediately. The detail may be omitted.

According to this observation, in the present paper, we shall consider only those Ω having the above mentioned properties. Furthermore, on performing rotation and parallel displacement of the W-plane, if necessary, we may assume in advance that F and Ω satisfy the following conditions:

(1.3) Ω is a simply connected domain with more than one boundary point having the properties that $0\notin\Omega$, $\infty\notin\Omega$, and that, for any ε_1 $(0<\varepsilon_1<\pi/2)$, there exists $R_1>0$ with $\{W|R_1<|W|, |\arg W|<\pi/2-\varepsilon_1\}\subset\Omega$,

(1.4) The accessible boundary point of Ω over $W=\infty$ determined by the positive real-axis corresponds to $Z=\infty$ under F.

By the condition (1.3) there exists a real number R such that the open interval (R, ∞) on the real-axis is contained in Ω. Let R_0 be the smallest of such R; clearly $R_0\geqq 0$.

For R with $R_0\leqq R<\infty$, denote by $\Omega(R)$ the connected component of $\Omega\cap\{W|R<|W|\}$ containing the interval (R, ∞). Then the condition (1.4) is easily seen to be equivalent with the following:

(1.5) For any R with $R_0\leqq R<\infty$, there exists an r_0 such that the image of the interval (r_0, ∞) under F is contained in $\Omega(R)$.

1.3. For R with $R_0<R<\infty$, let $A(R)$ be the connected component of $\Omega\cap\{W|R=|W|\}$ intersecting with the real-axis. It is a cross-cut of the domain Ω and, therefore, $\Omega-A(R)$ consists of two connected components, which are simply connected domains. The intervals (R_0, R) and (R, ∞) belong to different components respectively. We denote by $\Omega^*(R)$ the one containing (R, ∞).

Let $\Theta(R)$ be the angular measure of the arc $A(R)$; thus $R\Theta(R)$ is the length of $A(R)$. The function Θ is known to be measurable on the interval (R_0, ∞) (cf. Ahlfors [1], pp. 5-7).

For R_1 and R_2 with $R_0<R_1<R_2$, let $\Omega^*(R_1, R_2)$ be the connected component of $\Omega-A(R_1)\cup A(R_2)$ containing the interval (R_1, R_2). It is a simply connected domain. We consider the family of the locally rectifiable curves in $\Omega^*(R_1, R_2)$ separating $A(R_1)$ from $A(R_2)$, and denote its module (i. e., the reciprocal of extremal length) by $M(R_1, R_2)$. The relation

$$\int_{R_2}^{R_2}\frac{1}{\Theta(R)}\frac{dR}{R}\leqq M(R_1, R_2)$$

is well known.

In the following, we are interested in the function

$$M(R_*, R)-\frac{1}{\pi}\log R \tag{1.6}$$

of R defined on the interval (R_*, ∞), where $R_*>R_0$.

LEMMA 2. *Whether or not the function* (1.6) *is bounded* (*above and/or below*) *is independent of the choice of* R_*.

It is apparent from the following easily verified relation: If $R_*<R'_*$ and $2R'_*<R$, then

$$M(R'_*, R)\leqq M(R_*, R)\leqq M(R'_*, R)+M(R_*, 2R'_*)+\frac{2\pi}{\log 2}.$$

1.4. In order to state our first result, we denote by $\tilde{\Omega}$ the connected component of $\Omega\cap\{W|\operatorname{Re} W>0\}$ containing (R_0, ∞). For this domain, we introduce the quantity $\tilde{M}(R_1, R_2)$ which is, by definition, the counterpart of $M(R_1, R_2)$. It is possible to do it because the domain $\tilde{\Omega}$ satisfies (1.3).

THEOREM 1. *Let* $W=F(Z)$ *be a conformal mapping of the right-half plane onto a domain* Ω *with* (1.3). *Suppose* (1.1) (*or equivalently* (1.4), *or else* (1.5)) *is satisfied.*

(a) *If a finite non-zero angular derivative* (1.2) *exists, then, for all* $R_*(>R_0)$, *the function*

$$M(R_*, R)-\frac{1}{\pi}\log R$$

of R *on* (R_*, ∞) *is bounded* (*above and below*).

(b) *If there exist* R_* $(>R_0)$ *and* $\tilde{R}_*$ $(>R_0)$ *for which*

$$M(R_*, R)-\frac{1}{\pi}\log R$$

is bounded below and

$$\tilde{M}(\tilde{R}_*, R)-\frac{1}{\pi}\log R$$

is bounded above, then a finite non-zero angular derivative (1.2) *exists.*

This is an improvement of Ahlfors' result [1; pp. 35-36, Sätze I, II]. His proof is based on his principal inequalities and Wolff-Valiron-Landau-Carathéodory's theorem. The essence of our proof (2.1°-2.7°) is the replacement of the former by Lemmas 4, 5 (2.2°), which Jenkins-Oikawa [8] have obtained. That our result contains Ahlfors' will be apparent from the inequalities (2.8) and (3.11). How much improved will be illustrated by examples (1.6° and 1.7°).

1.5. Next we consider a more restrictive case where Ω is such that

$$\Omega(R_1)=\{Re^{i\Phi}\mid R_1<R,\ -\Phi_1(R)<\Phi<\Phi_2(R)\} \tag{1.7}$$

for an $R_1(>R_0)$. Here $\Phi_k(R)$ is a function on (R_1, ∞) subject to the following conditions $(k=1, 2)$:

$$0<\Phi_k(R)\leqq 2\pi\,; \tag{1.8}$$

(1.9) there exists a finite number V_k which dominates the total variation of $\Phi_k(R)$ over any closed subinterval of (R_1, ∞).

THEOREM 2. *Let $W=F(Z)$ be a conformal mapping of the right-half plane onto a domain Ω with* (1.3), (1.7), (1.8) *and* (1.9). *Suppose* (1.1) (*or equivalently* (1.4), *or else* (1.5)) *is satisfied. A finite non-zero angular derivative* (1.2) *exists if and only if the finite limit*

$$\lim_{R\to\infty}\int_{R_*}^{R}\frac{\pi-\Theta(R)}{\Theta(R)}\frac{dR}{R}$$

exists for some (*or equivalently all*) R_* $(>R_0)$. *In this case*

$$\tau=\lim\frac{Z}{F(Z)}$$

holds for the unrestricted approach $Z\to\infty$, $\mathrm{Re}\, Z>0$ (*i.e., Z is not restricted to Stolz domains*).

Proof is given in 3.1°-3.9°. This theorem is an improvement of Warschawski's result [12: p. 530, Theorem XIII], where the smoothness and other properties of the functions Φ_1 and Φ_2 are assumed. Under the assumption of the present theorem, Dufresnoy [2] (cf. also Dufresnoy et Ferrand [3; Théorème 2]) showed that the condition is sufficient for the existence of the angular derivative $\tau\neq 0, \infty$. On the other hand, since the "Unbewalltheit" condition of Warschawski [11; Satz 7] is satisfied, the angular derivative is the unrestricted derivative. In this sense, the sufficiency part of the present theorem is regarded as being known.

1.6. We present two illustrative examples for Theorem 1.

Example 1. Given $0<\varepsilon_n<\pi/2$ with $\lim\varepsilon_n=0$ and $0<R_n$ with $R_{n+1}/R_n\geqq 1+c$ for some $c>0$, let

$$\Omega=\{W\mid \mathrm{Re}\, W>0\}-\bigcup_{n=1}^{\infty}C_n\,,$$

where $C_n=\{W \mid |W|=R_n,\ \pi/2-\varepsilon_n \leqq |\arg W| \leqq \pi/2\}$. Then the $W=F(Z)$ has $\tau \neq 0$, ∞ if any only if

$$\sum_{n=1}^{\infty} \varepsilon_n{}^2 < \infty .$$

Remark. The if-part of this conclusion has been obtained by Wolff [16] without assuming $R_{n+1}/R_n \geqq 1+c>1$. Ahlfors [1; pp. 39-40] proved the sufficiency of $\sum \varepsilon_n < \infty$ for the existence of $\tau \neq 0$, ∞, and announced the necessity of $\sum \varepsilon_n{}^2 < \infty$. We present this example, which is derived from Theorem 1 by routines in the theory of module, in order to show how much our theorem improves that of Ahlfors.

Proof. $M(R_*, R)-(1/\pi)\log R$ is always bounded below. Besides $\tilde{\Omega}=\Omega$. Thus, $\tau \neq 0$, ∞ exists if and only if

$$M(R_*, R)-\frac{1}{\pi}\log R \qquad \text{is bounded above.}$$

Map Ω by $w=\log W$ (principal value) conformally onto D (cf. 2.1°). Let $\Gamma(a, b)$ be the family of the curves joining $\{w \mid \operatorname{Im} w>0\} \cap \partial D$ and $\{w \mid \operatorname{Im} w<0\} \cap \partial D$ within $D(a, b)=\{w \mid a<\operatorname{Re} w<b\} \cap D$. Let $m(a, b)$ be the module of $\Gamma(a, b)$. Evidently, the above mentioned condition is equivalent to the following:

$$m(a, b)-\frac{b}{\pi} \qquad \text{is bounded above} \tag{1.10}$$

as a function of b on (a, ∞).

Suppose (1.10) is satisfied. Let $u_n=\log R_n$, $D_n=\{w \in D \mid u_n-\varepsilon_n/2<\operatorname{Re} w< u_n+\varepsilon_n/2\}$. Take n_0 so that $\bar{D}_n \cap \bar{D}_{n+1}=\emptyset$ for every $n \geqq n_0$. Put $a=u_{n_0}+\varepsilon_{n_0}/2$. For an arbitrary N, take b with $u_N+\varepsilon_N/2 \leqq b<u_{N+1}-\varepsilon_{N+1}/2$. Let $\Gamma^*(a, b)$ be the family of the curves joining $\{w \mid \operatorname{Re} w=a\} \cap D$ and $\{w \mid \operatorname{Re} w=b\} \cap D$ within $D(a, b)$, and let Γ_n^* be the family of the curves joining $\{w \mid \operatorname{Re} w=u_n-\varepsilon_n/2\} \cap D$ and $\{w \mid \operatorname{Re} w=u_n+\varepsilon_n/2\}$ within D_n. Then

$$m(a, b)=m(\Gamma^*(a, b))^{-1} \geqq \frac{1}{\pi}\Big(b-a-\sum_{n=n_0}^{N} \varepsilon_n\Big)+m(\Gamma_n^*)^{-1} .$$

The density $\rho_n(w)$ which is defined to be equal to 1 on $\{w \mid |\operatorname{Im} w|<\pi/2-\varepsilon_n/2\} \cap D_n$ and 0 elsewhere satisfies $\int_\gamma \rho_n |dw| \geqq \varepsilon_n$ for every $\gamma \in \Gamma_n^*$. Accordingly

$$m(\Gamma_n^*)^{-1} \geqq \varepsilon_n^2\Big(\iint \rho_n^2 du dv\Big)^{-1}=\frac{\varepsilon_n}{\pi-\varepsilon_n} \geqq \frac{\varepsilon_n}{\pi}\Big(1+\frac{\varepsilon_n}{\pi}\Big)$$

and, therefore,

$$m(a, b) \geqq \frac{b-a}{\pi}+\sum_{n=n_0}^{N}\Big(\frac{\varepsilon_n}{\pi}\Big)^2$$

for arbitrary N. We conclude $\sum \varepsilon_n^2<\infty$.

Conversely, suppose $\sum \varepsilon_n^2<\infty$. Take n_0 so that $4\varepsilon_n<\pi$ for every $n \geqq n_0$. Put $a=u_{n_0}-\varepsilon_{n_0}$. For an arbitrary b $(>a)$, take N with $u_N+\varepsilon_N \leqq b<u_{N+1}+\varepsilon_{N+1}$ and

consider the density $\rho(w)$ defined as follows: $\rho(w)=2$ on $\bigcup_{n=n_0}^{N}\{w \mid u_n-\varepsilon_n<\operatorname{Re} w<u_n+\varepsilon_n,\ |\operatorname{Im} w| \geqq \pi/2-2\varepsilon_n\}$, $\rho(w)=1$ on other part of $D(a, b)$, and $\rho(w)=0$ elsewhere. Since $\int_\gamma \rho|dw| \geqq \pi$ for every $\gamma \in \Gamma(a, b)$, we get

$$m(a, b) \leqq \frac{1}{\pi^2}\iint \rho^2 du dv=\frac{1}{\pi^2}\left\{\pi(b-a)+24 \sum_{n=n_0}^{\infty} \varepsilon_n^2\right\},$$

thus (1.10).

1.7. *Examrle* 2. Given R_n, R_n' with $0<R_n<R_n'<R_{n+1}$, let

$$\Omega=\{W \mid 0<|W|,\ |\arg W|<\pi\}-\bigcup_{n=1}^{\infty} E_n \cup E_n',$$

where $E_n=\{W \mid \operatorname{Re} W=0,\ R_n' \leqq|\operatorname{Im} W| \leqq R_{n+1}\}$, $E_n'=\{W \mid |W|=R_{n+1},\ \pi/2 \leqq|\arg W| \leqq \pi\}$. $W=F(Z)$ has $\tau \neq 0, \infty$ if

$$\sum_{n=1}^{\infty}\left(\log \frac{R_n'}{R_n}\right)^2<\infty .$$

Remark. This shows that the angular derivative $\tau \neq 0, \infty$ may exist eventhough the relevant boundary point is an interior point of the closure of Ω. This condition can also be derived from an already known criterion, e.g., Lelong-Ferrand [10; p. 26].

Proof. It suffices to verify that $M(R_*, R)-(1/\pi)\log R$ is bounded below. As before, we shall show that

$$m(a, b)-\frac{b}{\pi} \quad \text{is bounded below.} \tag{1.11}$$

Consider $\Gamma^*(a, b)$ as before. Let $u_n=\log R_n$, $u_n'=\log R_n'$, and put $a=u_1$. For an arbitrary b $(>a)$, take N with $u_N<b \leqq u_{N+1}$. The density $\rho(w)$ which is, by definition, equal to 1 on $\{w \mid |\operatorname{Im} w| \leqq \pi/2\} \cup \bigcup_{n=1}^{N+1}\{w \mid u_n \leqq \operatorname{Re} w \leqq u_n+2(u_n'-u_n),\ \pi/2 \leqq |\operatorname{Im} w| \leqq \pi/2+(u_n'-u_n)\}$, and 0 elsewhere, satisfies $\int_\gamma \rho|dw| \geqq(b-a)$ for every $\gamma \in \Gamma^*(a, b)$. Thus

$$\begin{aligned} m(a, b)=m(\Gamma^*(a, b))^{-1} & \geqq(b-a)^2\left(\iint \rho^2 du dv\right)^{-1} \\ & =(b-a)^2(\pi(b-a)+4 \sum_{n=1}^{N+1}(u_n'-u_n)^2)^{-1} \\ & \geqq \frac{b-a}{\pi}-\frac{4}{\pi^2} \sum_{n=1}^{N}(u'-u)^2 . \end{aligned}$$

Therefore, the convergence of $\sum(u_n'-u_n)^2$ implies (1.11).

2. Proof of Theorem 1.

2.1. By means of $z=\log Z$, the principal value, the right-half plane $\operatorname{Re} Z>0$ is mapped conformally onto the horizontal strip

$$S=\left\{z \mid |\operatorname{Im} z|<\frac{\pi}{2}\right\}.$$

The domain Ω with (1.3) in the W-plane is mapped onto D by

$$w=\log W, \tag{2.1}$$

where log is the branch being real-valued on the interval (R_0, ∞). The function $W=F(Z)$ is transformed to

$$w=f(z);$$

namely $f(z)=\log F(e^z)$.

In accordance with the condition (1.1),

$$\lim f(z)=\infty \qquad \text{as} \quad S_\delta \ni z \to +\infty \tag{2.2}$$

for any δ $(0<\delta<\pi/2)$, where

$$S_\delta=\left\{z \mid |\operatorname{Im} z|<\frac{\pi}{2}-\delta\right\}.$$

The relation (1.2) is

$$\lim\,(z-f(z))=\log \tau \qquad \text{as} \quad S_\delta \ni z \to +\infty \tag{2.3}$$

for every δ.

Corresponding to (1.3), D satisfies the following condition:

(2.4) For every ε $(0<\varepsilon<\pi/2)$, there exists a such that $\{w \mid a<\operatorname{Re} w,\ |\operatorname{Im} w|<\pi/2-\varepsilon\} \subset D$.

Let a_0 be the smallest value of the a with $(a, \infty) \subset D$. Clearly $a_0=\log R_0$. For every a with $a_0 \leqq a<\infty$, we denote by $D(a)$ the image of $\Omega(R)$, $a=\log R$, under the mapping (2.1). According to (1.5) we have the following:

(2.5) For every a $(a_0 \leqq a<\infty)$, there exists an x_0 such that $f((x_0, \infty)) \subset D(a)$.

Let $\sigma(a)$ and $D^*(a)$ be the images under (2.1) of $A(R)$ and $\Omega^*(R)$, respectively, where $a=\log R$. The length $\theta(a)$ $(=\Theta(e^a))$ of $\sigma(a)$ satisfies

$$0<\theta(a) \leqq 2\pi \tag{2.6}$$

for $a_0<a<\infty$ and, because of (2.4),

$$\pi \leqq \varliminf_{a\to+\infty} \theta(a). \tag{2.7}$$

For a, b with $a_0<a<b$, let $D^*(a, b)$ be the image of $\Omega^*(e^a, e^b)$ under (2.1).

Denote by $m(a, b)$ the module of $\Gamma(a, b)$ which is by definition the family of the locally rectifiable curves in $D^*(a, b)$ separating $\sigma(a)$ from $\sigma(b)$. Let $m_0(a, b)$ be the module of the family $\{\sigma(u) | a<u<b\}$. We have $m(a, b)=M(e^a, e^b)$ and

$$\int_a^b \frac{du}{\theta(u)} = m_0(a, b) \leqq m(a, b).$$

Finally, let $\tilde{D}$ be the connected component of $D \cap S$ containing (a_0, ∞), and $\tilde{m}(a, b)$ be the quantity $m(a, b)$ considered with respect to the domain $\tilde{D}$. Now, Parts (a) and (b) of Theorem 1 are equivalent respectively to the following:

PROPOSITION 1. *If the finite limiting value* (2.3) *exists, then*

$$m(a_*, b) - \frac{b}{\pi}$$

is bounded (*above and below*) *for* $a_* < b < \infty$, *where* a_* $(>a_0)$ *is arbitrary.*

PROPOSITION 2. *If there exist* a_* $(>a_0)$ *and* $\tilde{a}_*$ $(>a_0)$ *such that*

$$m(a_*, b) - \frac{b}{\pi}$$

is bounded below for $a_* < b < \infty$, *and*

$$\tilde{m}(\tilde{a}_*, b) - \frac{b}{\pi}$$

is bounded above for $\tilde{a}_* < b < \infty$, *then the finite limiting value* (2.3) *exists.*

2.2. For a greater than a_0, $\gamma_a = f^{-1}(\sigma(a))$ is a cross-cut of the domain S. Its end points are not $+\infty$ and, except possibly for one a, not $-\infty$ and, therefore,

$$x'(a) = \inf_{z \in \gamma_a} \operatorname{Re} z, \qquad x''(a) = \sup_{z \in \gamma_a} \operatorname{Re} z$$

are finite.

LEMMA 3. *There exists an* a_1 $(>a_0)$ *such that, for every* $a \geqq a_1$, *one end point of* γ_a *is on the upper edge* $\{z | \operatorname{Im} z = \pi/2\}$ *of* S *and the other is on the lower edge* $\{z | \operatorname{Im} z = -\pi/2\}$. *Furthermore*

$$\lim_{u \to \infty} x'(u) = \infty. \tag{2.9}$$

It is trivial from the boundary correspondence (1.4), (1.5).

LEMMA 4. *If* $a_1 \leqq a < b$,

$$m(a, b) \leqq \frac{1}{\pi}(x''(b) - x'(a)). \tag{2.10}$$

If further $x''(a) \leqq x'(b)$,

$$m(a, b) \leqq \frac{1}{\pi}(x'(b) - x''(a)) + 2. \tag{2.11}$$

Proof. The former is trivial. The latter is nothing but the inequality (2) of Jenkins-Oikawa [8; p. 665].

LEMMA 5. *If* a, b, *and* c *satisfy* $a_1<a<b$, $x''(a_1)\leqq x'(a)$, $a_1+2c\leqq a$, $\{w \mid a-2c<\operatorname{Re} w<b+2c, |\operatorname{Im} w|<c\}\subset D^*(a-2c, b+2c)$, *then*

$$\frac{1}{\pi}(x''(b)-x'(a))\leqq m(a, b)+\frac{4\pi}{c}. \tag{2.12}$$

Proof. The following lines are generalization of the argument in Jenkins-Oikawa [8; pp. 668-669]. Let Γ be the family of the f-images of curves joining the upper edge and lower edge of S within $\{z \mid x'(a)<\operatorname{Re} z<x''(b)\}\cap S$. Put $\Gamma_0=\Gamma-\Gamma(a, b)$. Clearly their modules $m(\Gamma)$ and $m(\Gamma_0)$ satisfy

$$\frac{1}{\pi}(x''(b)-x'(a))=m(\Gamma)\leqq m(a, b)+m(\Gamma_0).$$

In order to estimate $m(\Gamma_0)$, observe that, for every $\gamma\in\Gamma_0$, the following five cases can occur:

(i) $\gamma\subset D^*(a-2c, a+2c)$; then a subarc of γ joins $\{w \mid \operatorname{Im} w=c\}$ and $\{w \mid \operatorname{Im} w=-c\}$ within $\{w \mid a-2c<\operatorname{Re} w<a+2c, |\operatorname{Im} w|<c\}$.

(ii) $\gamma\subset D^*(b-2c, b+2c)$; similarly within $\{w \mid b-2c<\operatorname{Re} w<b+2c, |\operatorname{Im} w|<c\}$.

(iii) neither (i) nor (ii), and $\gamma\cap\overline{D^*(a+2c, b-2c)}\neq\emptyset$; then there exists a subarc of γ either joining $\sigma(a)$ and $\sigma(a+2c)$ within $D^*(a, a+2c)$, or joining $\sigma(b-2c)$ and $\sigma(b)$ within $D^*(b-2c, b)$.

(iv) neither (i) nor (ii), and $\gamma\cap\overline{D^*(b+2c)}\neq\emptyset$; then a subarc of γ joins $\sigma(b)$ and $\sigma(b+2c)$ within $D^*(b, b+2c)$.

(v) neither (i) nor (ii), and $\gamma\cap(D-D^*(a-2c))\neq\emptyset$; then a subarc of γ joins $\sigma(a-2c)$ and $\sigma(a)$ within $D^*(a-2c, a)$.

Now, consider the density $\rho(w)$ defined to be equal to 1 on $D\cap[\{w \mid a-2c<\operatorname{Re} w<a+2c\}\cup\{w \mid b-2c<\operatorname{Re} w<b+2c\}]$ and 0 elsewhere. We have $\int_\gamma \rho|dw|\geqq 2c$ for every $\gamma\in\Gamma_0$. Therefore

$$m(\Gamma_0)\leqq\frac{1}{4c^2}\iint\rho^2 du dv\leqq\frac{4\pi}{c}.$$

2.3. If $a_1\leqq a_2<b$ and $x''(a_2)\leqq x'(b)$, then (2.11) implies

$$\left(m(a_2, b)-\frac{b}{\pi}\right)+\left(-2+\frac{x''(a_2)}{\pi}\right)\leqq\frac{1}{\pi}(x'(b)-b).$$

If, further, a_2 is taken so large that $x''(a_1)\leqq x'(a_2)$, $a_1+2\pi/3\leqq a_2$, and $\{w \mid a_2-2\pi/3<\operatorname{Re} w, |\operatorname{Im} w|\leqq\pi/3\}\subset D^*(a_2-2\pi/3)$, then the (2.12) for $c=\pi/3$ implies

$$\frac{1}{\pi}(x''(b)-b)\leqq\left(m(a_2, b)-\frac{b}{\pi}\right)+\left(12+\frac{x'(a_2)}{\pi}\right).$$

Consequently, if a_2 is sufficiently large and $x''(a_2)\leqq x'(b)$ (which is satisfied if b is sufficiently large), we obtain

$$\left(m(a_2, b)-\frac{b}{\pi}\right)+\left(-2+\frac{x''(a_2)}{\pi}\right) \leqq \frac{1}{\pi}\mathrm{Re}\,(z-f(z))\leqq\left(m(a_2, b)-\frac{b}{\pi}\right)+\left(12+\frac{x'(a_2)}{\pi}\right) \tag{2.13}$$

for every $z\in\gamma_b$.

2.4. Proof of Theorem 1, Part (a). We prove Proposition 1. Take a δ with $0<\delta<\pi/2$. Since the finite limit (2.3) exists, we find K such that $-K\leqq \mathrm{Re}\,(z-f(z))\leqq K$ for every $z\in S_\delta$ with $\mathrm{Re}\,z\geqq a_1$. Take an a_2 of (2.13) and then, for every b with $x''(a_2)\leqq x'(b)$, take $z\in\gamma_b\cap S_\delta$. We have

$$\left(m(a_2, b)-\frac{b}{\pi}\right)+\left(-2+\frac{x''(a_2)}{\pi}\right)\leqq K$$

$$\left(m(a_2, b)-\frac{b}{\pi}\right)+\left(12+\frac{x'(a_2)}{\pi}\right)\geqq -K\,.$$

Consequently, if $a_*=a_2$, $m(a_*, b)-b/\pi$ is bounded for $a_*<b<\infty$. By Lemma 2 this a_* may be replaced by any one greater than a_0.

2.5. Proof of Part (b) needs the following (Ahlfors [1; pp. 29-31]):

THEOREM OF WOLFF-VALIRON-LANDAU-CARATHÉODORY. *If a regular function $G(z)$ on the right-half plane* $\mathrm{Re}\,Z>0$ *satisfies* $\mathrm{Re}\,G(Z)>0$, *then the limiting value*

$$\lim_{\substack{z\to\infty\\ \text{Stolz}}}\frac{Z}{G(Z)}=\beta\,,\qquad 0<\beta\leqq\infty$$

exists.

2.6. The domain $\tilde{\Omega}$ of Theorem 1 possesses the accessible boundary point $\tilde{p}$ over $W=\infty$ determined by the positive real axis. Let $\tilde{F}$ be the conformal mapping of $\mathrm{Re}\,Z>0$ onto $\tilde{\Omega}$ under which $Z=\infty$ corresponds to $\tilde{p}$. By Theorem of Wolff-Valiron-Landau-Carathéodory the limiting value

$$\lim_{\substack{z\to\infty\\ \text{Stolz}}}\frac{Z}{\tilde{F}(Z)}=\tilde{\tau}\,,\qquad 0<\tilde{\tau}\leqq\infty$$

exists.

As in 2.1°, we transform $W=\tilde{F}(Z)$ to $w=\tilde{f}(z)$, which satisfies $\log\tilde{\tau}=\lim\,(z-\tilde{f}(z))$ as $S_\delta\ni z\to+\infty$, $-\infty<\log\tilde{\tau}\leqq\infty$. Now, it is not difficult to find a sequence $\{x_n\}$ of real numbers such that $\lim x_n=+\infty$ and $x_n\in\gamma_{b_n}$ for some $b_n\to+\infty$. We have

$$\log\tilde{\tau}=\lim_{n\to\infty}\mathrm{Re}\,(x_n-\tilde{f}(x_n))\,.$$

By the relation (2.13) with respect to $\tilde{f}$,

$$\mathrm{Re}\,(x_n-\tilde{f}(x_n))\leqq\left(\tilde{m}(\tilde{a}_2, b_n)-\frac{b_n}{\pi}\right)+\left(12+\frac{\tilde{x}'(\tilde{a}_2)}{\pi}\right).$$

Because of the hypothesis of Theorem 1, Part (b), namely that of Proposition 2, there exists an $\tilde{a}_*$ such that $\tilde{m}(\tilde{a}_*, b)-b/\pi$ is bounded above for $b \geqq \tilde{a}_*$. On taking $\tilde{a}_2 \geqq \tilde{a}_*$, we get $\tilde{m}(\tilde{a}_2, b)-b/\pi \leqq \tilde{m}(\tilde{a}_*, b)-b/\pi$ for $b>\tilde{a}_2$. We conclude that $\operatorname{Re}(x_n-\tilde{f}(x_n))$ is bounded above and, therefore,

$$\tilde{\tau} \neq 0,\ \infty .$$

Next, apply Theorem of Wolff-Valiron-Laudau-Carathéodory to $\hat{F}=F^{-1}\circ\tilde{F}$; the limiting value

$$\lim_{\substack{z\to\infty \\ \text{Stolz}}} \frac{Z}{\hat{F}(Z)}=\hat{\tau}\,, \qquad 0<\hat{\tau}\leqq\infty$$

exists. Observe that $W/F^{-1}(W)=(Z/\hat{F}(Z))(\tilde{F}(Z)/Z)$ if $W=\tilde{F}(Z)$. If $W\to\infty$ from a Stolz domain contained in Ω, then $Z=\tilde{F}^{-1}(W)\to\infty$ from a Stolz domain; this is verified on applying Lemma 1 to $\tilde{F}$. Therefore, on putting $\tau'=\hat{\tau}/\tilde{\tau}$, we obtain

$$\lim_{\substack{W\to\infty \\ \text{Stolz}}} \frac{W}{F^{-1}(W)}=\tau'\,, \qquad 0<\tau'\leqq\infty\,.$$

Now, $\log\tau'=\lim \operatorname{Re}(w-f^{-1}(w))$ as $S_\delta \ni w\to+\infty$. For real numbers $b_n\to+\infty$, put $z_n=f^{-1}(b_n)$. We have by (2.13)

$$\begin{aligned}\operatorname{Re}(b_n-f^{-1}(b_n)) &= \operatorname{Re}(f(z_n)-z_n)\\ &\leqq -\pi\Big(m(a_2, b_n)-\frac{b_n}{\pi}\Big)+2\pi-x''(a_2)\end{aligned}$$

for sufficiently large n. The hypothesis of Proposition 2 (cf. also Lemma 2) implies that $-(m(a_2, b_n)-b_n/\pi)$ is bounded above if a_2 is taken sufficiently large. Accordingly $\log\tau'=\lim \operatorname{Re}(b_n-f^{-1}(b_n))<\infty$, namely

$$\tau'\neq 0,\ \infty\,.$$

2.7. Proof of Theorem 1, Part (b). On applying Lemma 1 to F^{-1}, we see that, if $Z\to\infty$ from a Stolz domain, $W=F(Z)$ tends to ∞ from a Stolz domain and, therefore, $Z/F(Z)=F^{-1}(W)/W\to 1/\tau'\neq 0,\ \infty$.

3. Proof of Theorem 2.

3.1. Corresponding to the conditions (1.7)-(1.9), the domain D, the image of Ω under the transformation (2.1), satisfies the following conditions:

(3.1) $$D(a_3)=D^*(a_3)=\{u+iv \mid a_3<u,\ -\theta_1(u)<v<\theta_2(u)\}$$

for some $a_3\geqq a_1$. Here $\theta_k(u)$ is a function on (a_3, ∞) such that

(3.2) $$0<\theta_k(u)\leqq 2\pi$$

(3.3) The total variation $V_k(a, b)$ of $\theta_k(u)$ over any $[a, b]\subset(a_3, \infty)$ does not exceed a fixed number $V_k<\infty$,

$k=1, 2$. These assumptions imply the existence of the following finite limiting values, which satisfy by (2.4) the following inequalities:

$$\lim_{u\to\infty}\theta_k(u)\geqq\frac{\pi}{2}, \qquad k=1, 2. \tag{3.4}$$

From now on, we replace a_3 by a greater one, so that

$$\theta_k(u)\leqq\frac{\pi}{3}, \qquad k=1, 2 \tag{3.5}$$

for $u\geqq a_3$.

Theorem 2 is equivalent to the pair of the following propositions:

PROPOSITION 3. *If the finite limiting values*

$$\lim(z-f(z)) \qquad as \quad S_\delta\ni z\to+\infty \tag{3.6}$$

exists for every δ $(0<\delta<\pi/2)$, *then the finite*

$$\lim_{b\to\infty}\left(\int_{a_*}^{b}\frac{du}{\theta(u)}-\frac{b}{\pi}\right) \tag{3.7}$$

exists for all $a_*>a_0$.

PROPOSITION 4. *If there exists an* $a^*>a_0$ *for which the finite limiting value* (3.7) *exists, then the finite*

$$\lim_{S\ni z\to+\infty}(z-f(z)) \tag{3.8}$$

exists.

3.2. We have $\theta(u)=\theta_1(u)+\theta_2(u)$. By (2.8), (2.10), and (2.11), if $a_3\leqq a<b$,

$$\int_a^b\frac{du}{\theta(u)}\leqq\frac{1}{\pi}(x''(b)-x'(a)) \tag{3.9}$$

and, if further $x''(a)<x'(b)$,

$$\int_a^b\frac{du}{\theta(u)}\leqq\frac{1}{\pi}(x'(b)-x''(a))+2. \tag{3.10}$$

Next, by Theorem 2 af Jenkins-Oikawa [8; p. 666],

$$m(a, b)\leqq\int_a^b\frac{du}{\theta(u)}+\frac{9}{\pi}(V_1(a, b)+V_2(a, b)), \tag{3.11}$$

if $a_3\leqq a<b$. Hence, on applying Lemma 5 with respect to $c=\pi/3$ (cf. also (3.5)), we have

$$\frac{1}{\pi}(x''(b)-x'(a))\leqq\int_a^b\frac{du}{\theta(u)}+12+\frac{9}{\pi}(V_1(a, b)+V_2(a, b)) \tag{3.12}$$

for a and b with $a_4\leqq a<b$, where $a_4=a_3+2\pi/3$.

This inequality implies

$$\varliminf_{b\to\infty}\left(\int_{a_4}^{b}\frac{du}{\theta(u)}-\frac{x'(b)}{\pi}\right)>-\infty,$$

which permits us to apply some results of Jenkins-Oikawa [9] (cf. also Eke [4]). First by Lemma 4 of [9; p. 47]

$$\lim_{u\to\infty}(x''(u)-x'(u))=0; \tag{3.13}$$

secondly, the existence of the limiting value (21) of [9; p. 44] (cf. also the last line of [9; p. 46]): The finite limiting value

$$\lim\left(\int_{a}^{\operatorname{Re} f(z)}\frac{du}{\theta(u)}-\frac{1}{\pi}\operatorname{Re} z\right) \quad \text{as} \quad S_\delta\ni z\to+\infty \tag{3.14}$$

exists for all δ $(0<\delta<\pi/2)$ and a $(\geqq a_4)$.

3.3. Proof of Proposition 3. Consider

$$\frac{1}{\pi}\operatorname{Re}(z-f(z))$$
$$=\left(\frac{1}{\pi}\operatorname{Re} z-\int_{a_4}^{\operatorname{Re} f(z)}\frac{du}{\theta(u)}\right)+\left(\int_{a_4}^{\operatorname{Re} f(z)}\frac{du}{\theta(u)}-\frac{1}{\pi}\operatorname{Re} f(z)\right),$$

and let $z\in S_\delta$ tend to $+\infty$. In the right-hand side, the limit of the first term exists and is finite. Accordingly, the existence of the finite limit of the second term is equivalent to that of (3.6). The latter is, as is easily verified, equivalent to the existence of the finite limit (3.7) for $a_*=a_4$. Evidently a_* may be replaced by any one greater than a_0. The proof of Proposition 3 is hereby complete.

Remark 1. Same reasoning is found in Eke [5]. He considered more general domain Ω, and did not rule out the case where the limiting value of (3.7) is $-\infty$.

Remark 2. The above argument shows that the existence of finite limit (3.7) conversely implies that of $\lim\operatorname{Re}(z-f(z))$ as $S_\delta\ni z\to+\infty$. But this conclusion is of no use for the proof of Proposition 4.

3.4. Suppose that the hypothesis of Proposition 4 is satisfied. From (3.4), we have

$$\lim_{u\to\infty}\theta_k(u)=\frac{\pi}{2}, \qquad k=1, 2. \tag{3.15}$$

Next, for every x greater than $x''(a_4)$, let $s_x=\{z\,|\operatorname{Re} z=x, |\operatorname{Im} z|<\pi/2\}$. Clearly $f(s_x)\subset D(a_4)$. Put

$$u'(x)=\inf_{z\in s_x}\operatorname{Re} f(z), \qquad u''(x)=\sup_{z\in s_x}\operatorname{Re} f(z).$$

LEMMA 6.

$$\lim_{x\to\infty}u'(x)=\infty, \tag{3.16}$$

(3.17) $$\lim_{x\to\infty}(u''(x)-u'(x))=0\,.$$

Proof. For any b, we have $u'(x)>b$ whenever $x>x''(b)$. This means (3.16). Next,

$$\frac{1}{4\pi}(u''(x)-u'(x))\leqq\int_{u'(x)}^{u''(x)}\frac{du}{\theta(u)}\leqq m(u'(x),\,u''(x))$$
$$\leqq\frac{1}{\pi}(x''(u''(x))-x'(u'(x)))$$
$$\leqq\frac{1}{\pi}(x''(u''(x))-x'(u''(x)))+\frac{1}{\pi}(x''(u'(x))-x'(u'(x)))$$

From (3.16) and (3.13), we obtain (3.17).

LEMMA 7.
$$x'(b)-x'(a)=b-a+o(1)$$
as $b>a\to+\infty$.

Proof. For the sake of simplicity, write a' and b'' for $u'(x'(a))$ and $u''(x''(b))$, respectively. If a is sufficiently large,

$$\frac{1}{\pi}(x''(b)-x'(a))\leqq m(a',\,b'')\leqq\int_{a'}^{b''}\frac{du}{\theta(u)}+\frac{9}{\pi}(V_1(a',\,b'')+V_2(a',\,b''))\,.$$

Observe $a'<b''$ if $a<b$, and $a'\to\infty$ as $a\to\infty$. We have

$$\int_{a'}^{b''}\frac{du}{\theta(u)}-\int_a^b\frac{du}{\theta(u)}\longrightarrow 0$$
$$V_1(a',\,b'')+V_2(a',\,b'')\longrightarrow 0$$

as $b>a\to+\infty$ and, therefore,

$$\frac{1}{\pi}(x''(b)-x'(a))\leqq\int_a^b\frac{du}{\theta(u)}+o(1)\,.$$

Together with (3.9), we obtain

$$\frac{1}{\pi}(x''(b)-x'(a))=\int_a^b\frac{du}{\theta(u)}+o(1)\,.$$

On the other hand, the hypothesis of Proposition 4 implies

$$\frac{1}{\pi}(b-a)=\int_a^b\frac{du}{\theta(u)}+o(1)$$

as $b>a\to+\infty$. On combining these, we obtain the desired relation.

3.5. As the first step of the Proof of Proposition 4, let us verify the existence of the finite limit $\lim_{S\ni z\to\infty}\mathrm{Re}\,(z-f(z))$. We have, by Lemma 7, the finite limit $\lim_{a\to\infty}(x'(a)-a)=\beta$ and, by (3.13), $\lim_{a\to\infty}(x''(a)-a)=\beta$. Therefore, for any $\varepsilon>0$, it is possible to find a_5 such that

$$|x'(a)-a-\beta|<\varepsilon\,, \qquad |x''(a)-a-\beta|<\varepsilon$$

whenever $a \geqq a_5$. For all $z \in S$ with $\operatorname{Re} z > x''(a_5)$, we have $\operatorname{Re} f(z) > a_5$ and, therefore, $-\varepsilon < \operatorname{Re}(z-f(z))-\beta<\varepsilon$. Consequently

$$\lim_{S \ni z \to \infty} \operatorname{Re}(z-f(z))=\beta\,.$$

3.6. Proof of the existence of $\lim \operatorname{Im}(z-f(z))$ needs some preparation.[1)] For an arbitrary $c>0$, there exists an $a_6(c)$ such that

$$x''(a)-x'(a)<\frac{c}{4} \quad \text{and} \quad |(x'(b)-x'(a))-(b-a)|<\frac{c}{4}$$

whenever $a_6(c) \leqq a < b$; this is a consequence of (3.13) and Lemma 7. It is not difficult to prove that, far a $(>a_6(c))$, b $(\geqq a+c)$, and η $(|\eta|<\pi/2)$, the connected component of $\{z \,|\, \operatorname{Im} z=\eta\} \cap f^{-1}(D^*(a, b))$ which joins γ_a and γ_b is determined uniquely.

Apply this for $c=1/2$, and put $a_6=a_6(1/2)$. For a $(>a_6)$ and η $(|\eta|<\pi/2)$, we denote by

$$\xi(a, \eta)$$

the uniquely determined connected component of $\{z \,|\, \operatorname{Im} z=\eta\} \cap f^{-1}(D^*(a, a+1))$ joining γ_a and γ_{a+1}. Notice the following readily verified relation:

$$\{z \,|\, x''(a) \leqq \operatorname{Re} z,\ \operatorname{Im} z=\eta\} \subset \bigcup_{u \geqq a} \xi(u, \eta) \tag{3.18}$$

if $a \geqq a_6$.

3.7. Put $v'(a, \eta)=\inf\{\operatorname{Im} f(z) \,|\, z \in \xi(a, \eta)\}$, and $v''(a, \eta)=\sup\{\operatorname{Im} f(z) \,|\, z \in \xi(a, \eta)\}$.

LEMMA 8.

$$\lim_{a \to \infty}(v''(a, \eta)-v'(a, \eta))=0\,,$$

uniform convergence for $|\eta|<\pi/2$.

Proof. Suppose the assertion is false. There exist ε $(0<\varepsilon<\pi/2)$, a_n $(>a_6$ and $\to\infty)$, and η_n $(|\eta_n|<\pi/2)$ such that $v''(a_n, \eta_n)-v'(a_n, \eta_n) \geqq \varepsilon$. Take δ $(0<\delta<\pi/2)$ and fix it for a moment. For sufficiently large n, we apply (3.15), Lemma

1) Professor Warschawski pointed out that the proof of the existence of $\lim \operatorname{Im}(z-f(z))$ is immediate if we apply a well-known property of the Poisson integral. In fact, the bounded harmonic function $\operatorname{Im}(z-f(z))$ converges to 0 as $z=x+i(\pi/2) \to \infty$ and $z=x-i(\pi/2) \to \infty$. That this implies $\operatorname{Im}(z-f(z))=0$ as $S \ni z \to \infty$ is a conseqnence of Schwarz's theorem for the Poisson integral, applied on transforming S onto the unit disk. Neverthpless, we shall present our alternative proof based on the method of module, because it may be utilized in future where more general domains will be considered.

7, and (3.13) to obtain

$$D(a_n)\subset\left\{w\,|\,|\mathrm{Im}\,w|<\frac{\pi}{2}+\delta\right\} \tag{3.19}$$

$$x'(a_n+1)-x'(a_n)<1+\delta \tag{3.20}$$

$$x''(a_n)-x'(a_n)<\delta\,. \tag{3.21}$$

Put $\Xi_n=\{f(\xi(a_n, \eta))\,|\,|\eta|<\pi/2\}$. On considering $f^{-1}(\Xi_n)$ we have immediately

$$m(\Xi_n)\geqq\frac{\pi}{x''(a_n+1)-x'(a_n)}\geqq\frac{\pi}{1+2\delta}\,, \tag{3.22}$$

where $m(\Xi_n)$ is the module of Ξ_n. On the other hand, on the w-plane, put $v_n=(1/2)(v'(a_n, \eta_n)+v''(a_n, \eta_n))$ and $Q_n=\{w\,|\,a_n<\mathrm{Re}\,w<a_n+1,\ |\mathrm{Im}\,w-v_n|<\varepsilon/6\}$. The density $\rho_n(w)$ being equal to $(1+\varepsilon^2/9)^{-1/2}$ for $w\in Q_n$, 1 for $w\in D(a_n, a_n+1)-Q_n$, and 0 elsewhere satisfies $\int_\xi \rho_n|dw|>1$ for every $\xi\in\Xi_n$. Thus

$$m(\Xi_n)\leqq\iint\rho_n^2 dudu\leqq\pi+2\delta-\frac{\varepsilon^3}{3(9+\varepsilon^2)}\,.$$

On combining this with (3.22), and on letting $\delta\to0$, we obtain

$$\pi\leqq\pi-\frac{\varepsilon^3}{3(9+\varepsilon^2)}\,,$$

a contradiction.

3.8. LEMMA 9.

$$\lim_{a\to\infty} v'(a, \eta)=\lim_{a\to\infty} v''(a, \eta)=\eta\,,$$

uniform convergence for $|\eta|<\pi/2$.

Proof. Suppose the assertion is not true. There exist ε $(0<\varepsilon<\pi/2)$, a_n $(>a_6$ and $\to\infty)$, and η_n $(|\eta_n|<\pi/2)$ such that either

$$\eta_n-\frac{\varepsilon}{2}\geqq v'(a_n, \eta_n)\,, \tag{3.23}$$

or

$$\eta_n+\frac{\varepsilon}{2}\leqq v''(a_n, \eta_n)\,. \tag{3.23'}$$

Without loss of generality, we may assume that the former occurs for every n. Take a δ $(0<\delta<\pi/2)$ and fix it for a monent. Let n be sufficiently large so that (3.19), (3.20), and (3.21) hold, and furthermore, $v''(a_n, \eta)-v'(a_n, \eta)<\varepsilon/4$ for all η $(|\eta|<\pi/2)$. Then

$$v''(a_n, \eta_n)<\eta_n-\frac{\varepsilon}{4}\,.$$

Let $\Xi_n'=\{f(\xi(a_n, \eta))\,|\,-\pi/2<\eta<\eta_n\}$. We have immediately $m(\Xi_n')\leqq\eta_n+\pi/2+\delta-\varepsilon/4$ and $m(f^{-1}(\Xi_n'))\geqq(\eta_n+\pi/2)/(1+2\delta)$. Thus

$$\frac{\eta_n+\frac{\pi}{2}}{1+2\delta}\leqq\eta_n+\frac{\pi}{2}+\delta-\frac{\varepsilon}{4}.$$

Now, let $\delta\to 0$. On taking a cluster value η^* of η_n's, we obtain

$$\eta^*+\frac{\pi}{2}\leqq\eta^*+\frac{\pi}{2}-\frac{\varepsilon}{2},$$

a contradiction.

3.9. Proof of Proposition 4 is now complete. In fact, by (3.18) and Lemma 9, we see that, for any $\varepsilon>0$, there exists an a_7 ($\geqq a_6$) such that, for any η with $|\eta|<\pi/2$, the f-image of $\{z|a_7<\operatorname{Re} z,\ \operatorname{Im} z=\eta\}$ is contained in $\{w||\operatorname{Im} w-\eta|<\varepsilon\}$. This means

$$\lim_{S\ni z\to+\infty}\operatorname{Im}(z-f(z))=0\,.$$

Together with 3.5°, we obtain

$$\lim_{S\in z\to+\infty}(z-f(z))=\beta\neq\pm\infty\,.$$

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