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THE AXIOM OF SPHERES IN KAEHLER GEOMETRY

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1. Introduction. Let M be an Hermitian manifold of complex dimension >1 with almost complex structure J and Riemannian metric g. A 2-dimensional subspace σ of M_m, the tangent space of M at m, is called a *holomorphic* (resp., *antiholomorphic*) *plane* if $J\sigma=\sigma$ (resp., $J\sigma$ is orthogonal to σ). M is said to satisfy the *axiom of holomorphic* (resp., *antiholomorphic*) *planes* if for every $m\in M$ and every holomorphic (resp., *antiholomorphic*) *plane* σ at m, there exists a totally geodesic submanifold N satisfying $m\in N$ and $N_m=\sigma$. Yano and Mogi [7] showed that a Kaehler manifold satisfying the axiom of holomorphic planes has constant holomorphic curvature. The same conclusion prevails for a Kaehler manifold satisfying the axiom of antiholomorphic planes, as was recently shown by Chen and Ogiue [2].

A Riemannian manifold M of (real) dimension $d\geqq 3$ is said to satisfy the *axiom of r-spheres* ($2\leqq r<d$) if for each $m\in M$ and any r-dimensional subspace S of M_m, there exists an r-dimensional umbilical submanifold N with parallel mean curvature vector field satisfying $m\in N$ and $N_m=S$. This notion was introduced by Leung and Nomizu [6] who proved that a manifold with this property for some fixed r, $2\leqq r<d$, has constant sectional curvature. This generalizes the well-known theorem of Cartan [1] concerning the axiom of r-planes.

For an Hermitian manifold M, one of the authors [3] recently introduced the *axiom of holomorphic* 2-*spheres* and generalized the theorem of Yano and Mogi. Similarly, Harada [5] has introduced the *axiom of antiholomorphic* 2-*spheres* and generalized the theorem of Chen and Ogiue.

A subspace S of M_m, where M is an Hermitian manifold, is said to be *holomorphic* (resp., *antiholomorphic*) if $JS=S$ (resp., JS is orthogonal to S). Let $d=\dim_C M$.

Axiom of holomorphic $2r$-planes (resp., **$2r$-spheres**). *For each $m\in M$ and $2r$-dimensional holomorphic subspace S of M_m, $1\leqq r<d$, there exists a totally geodesic submanifold* (resp., *umbilical submanifold with parallel mean curvature vector field*) *N satisfying $m\in N$ and $N_m=S$.*

Axiom of antiholomorphic r-planes (resp., **r-spheres**). *For each $m\in M$ and r-dimensional antiholomorphic subspace S of M_m, $2\leqq r<d$, there exists a totally geodesic submanifold* (resp., *umbilical submanifold with parallel mean curvature*

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vector field) N satisfying $m \in N$ and $N_m = S$.

We shall prove the following.

THEOREM 1. *Let M be a Kaehler manifold of complex dimension $d>1$ satisfying the axiom of holomorphic $2r$-spheres for some fixed r, $1 \leqq r < d$. Then, M has constant holomorphic curvature.*

COROLLARY. *Let M be a Kaehler manifold of complex dimension $d>1$ satisfying the axiom of holomorphic $2r$-planes for some fixed r, $1 \leqq r < d$. Then, M has constant holomorphic curvature.*

THEOREM 2. *Let M be a Kaehler manifold of complex dimension $d>1$ satisfying the axiom of antiholomorphic r-spheres for some fixed r, $2 \leqq r < d$. Then, M has constant holomorphic curvature.*

COROLLARY. *Let M be a Kaehler manifold of complex dimension $d>1$ satisfying the axiom of antiholomorphic r-planes for some fixed r, $2 \leqq r < d$. Then, M has constant holomorphic curvature.*

2. Preliminaries. We consider a Kaehler manifold (M, g) as a Riemannian manifold with metric g admitting a skew-symmetric linear transformation field J satisfying $J^2 = -I$ (identity) and $DJ = 0$, where D is the covariant differentiation operator of the Levi-Civita connection of g. For any tangent vectors $X, Y \in M_m$, the curvature transformation is defined by

$$R(X, Y) = D_{[X,Y]} - D_X D_Y + D_Y D_X ,$$

and the curvature at a 2-dimensional subspace σ of M_m is given by

$$K(\sigma) = K(X, Y) = g(R(X, Y)X, Y)$$

for an arbitrary orthonormal basis $\{X, Y\}$ of σ. If σ is a holomorphic plane, $\{X, JX\}$ is an orthonormal basis of σ for X an arbitrary unit vector in σ. The curvature transformation satisfies the relations

$$R(X, JY) = -R(JX, Y) \tag{2.1}$$

$$K(X, JY) = K(JX, Y) . \tag{2.2}$$

For a submanifold N of a Riemannian manifold M, let D' denote the induced connection on N and let $D^{\perp}$ denote the connection in the normal bundle of N in M. The second fundamental form h is defined by

$$D_X Y = D'_X Y + h(X, Y) ,$$

where X and Y are vector fields tangent to N. Thus, h is a normal bundle-valued symmetric tensor field of type $(0, 2)$ on N. If ξ is a vector field normal to N, a linear transformation field A_ξ on N is defined by

$$D_X\xi = D^{\perp}_X\xi - A_\xi X\,,$$

where X is tangent to N. We have the well-known relation

$$g(h(X, Y), \xi) = g(A_\xi X, Y)\,.$$

The normal form of Codazzi's equation is

$$(R(X, Y)Z)_n = (\nabla_Y h)(X, Z) - (\nabla_X h)(Y, Z)\,, \tag{2.3}$$

where X, Y and Z are tangent to N, the subscript n denotes the normal component, and ∇h is the van der Waerden-Bortolotti covariant derivative of h with respect to the covariant derivative operators D' and $D^{\perp}$ given by

$$(\nabla_X h)(Y, Z) = D^{\perp}_X(h(Y, Z)) - h(D'_X Y, Z) - h(Y, D'_X Z)\,.$$

The *mean curvature normal* H of N in M is defined by

$$\text{trace}\, A_\xi = \dim N \cdot g(\xi, H)$$

for arbitrary ξ normal to N. H is *parallel* (in the normal bundle) if $D^{\perp}_X H = 0$. The submanifold N is *umbilical* in M if

$$h(X, Y) = g(X, Y)H\,,$$

and it is *totally geodesic* if it is umbilical and H vanishes. The following lemma will be required.

LEMMA 2.1. *Let N be an umbilical submanifold of a Riemannian manifold M. Then, $D^{\perp}_X H = 0$ if and only if $\nabla_X h = 0$, where X is an arbitrary vector field tangent to N.*

Proof. Let X, Y, Z be arbitrary vector fields tangent to N. Then,

$$\begin{aligned}(\nabla_X h)(Y, Z) &= D^{\perp}_X(h(Y, Z)) - h(D'_X Y, Z) - h(Y, D'_X Z)\\ &= (Xg(Y, Z))H + g(Y, Z)D^{\perp}_X H - g(D'_X Y, Z)H - g(Y, D'_X Z)H\\ &= (D'_X g)(Y, Z)H + g(Y, Z)D^{\perp}_X H\\ &= g(Y, Z)D^{\perp}_X H.\end{aligned}$$

3. Proofs of theorems. We shall require the following well-known fact whose proof we give for the sake of completeness.

LEMMA 3.1. *Let M be a Kaehler manifold of real dimension $\geqq 4$. If $g(R(X, Y)JX, X) = 0$ for every orthonormal triple $X, Y, JX \in M_m$ and for every $m \in M$, then M has constant holomorphic curvature.*

Proof. If $X, Y, JX \in M_m$ are orthonormal, so are $(X+Y)/\sqrt{2}$, $J(X+Y)/\sqrt{2}$, $J(X-Y)/\sqrt{2}$. Applying the hypothesis to this triple and using relations (2.1) and (2.2), we get $K(X, JX)=K(Y, JY)$. The Kaehlerian analogue of Schur's theorem then gives the lemma.

PROPOSITION 3.2. *Let M be a Kaehler manifold of complex dimension $d>1$ having the property that for each $m\in M$ and every holomorphic $2r$-dimensional subspace S of M_m, for some fixed r, $1\leqq r<d$, there exists a submanifold N satisfying $m\in N$, $N_m=S$ and $\nabla h=0$. Then, M has constant holomorphic curvature.*

Proof. At an arbitrary point $m\in M$, let X, Y, JX be orthonormal vectors. Let S be a $2r$-dimensional holomorphic subspace of M_m with $X, JX\in S$ and Y orthogonal to S. Let N be a submanifold satisfying $m\in N$, $N_m=S$ and $\nabla h=0$. In particular, we have

$$(\nabla_X h)(JX, X)=0\,, \qquad (\nabla_{JX} h)(X, X)=0\,.$$

Substituting in (2.3), we get $(R(X, JX)X)_n=0$; hence, $g(R(X, JX)X, Y)=0$. The proposition now follows from Lemma 3.1.

PROPOSITION 3.3. *Let M be a Kaehler manifold of complex dimension $d>1$ having the property that for each $m\in M$ and every antiholomorphic r-dimensional subspace S of M_m for some fixed r, $2\leqq r<d$, there exists a submanifold N satisfying $m\in N$, $N_m=S$ and $\nabla h=0$. Then, M has constant holomorphic curvature.*

Proof. At an arbitrary point $m\in M$, let X, Y, JX be orthonormal at m. Let S be an r-dimensional antiholomorphic subspace of M_m with $X, Y\in S$ and JX orthogonal to S. Proceeding as in the proof of Proposition 3.2, we get $(\nabla_X h)(Y, X)=(\nabla_Y h)(X, X)=0$, and hence $(R(X, Y)X)_n=0$, so that $g(R(X, Y)X, JX)=0$. Again, Lemma 3.1 completes the proof.

Theorems 1 and 2 now follow from Lemma 2.1 and Propositions 3.2 and 3.3.

Remarks. (a) The original proof of the theorem of Leung and Nomizu [6] uses the tangential form of Codazzi's equation. It is easy to establish the Riemannian analogue of Propositions 3.2 and 3.3, thereby providing a simplification by using the normal form of Codazzi's equation and Lemma 2.1.

(b) If an umbilical submanifold N of a Kaehler manifold (M, g) is complex, then it is totally geodesic. Indeed, an arbitrary complex submanifold of (M, g) is known to be minimal, that is $H=0$. On the other hand, the mean curvature vector field of a $2r$-dimensional $(1\leqq r<\dim_C M)$ umbilical submanifold N of a space of constant holomorphic curvature is a parallel field. In fact, if X and ξ are any vector fields tangent and normal to N, respectively, then $g(R(X, JX)\xi, JX)=0$. Hence, from the tangential form of Codazzi's equation $X\cdot g(\xi, H)=g(D_X^{\perp}\xi, H)$, from which $D_X^{\perp}H=0$. The umbilical submanifolds of a Kaehler manifold of constant holomorphic curvature K are known to be of three types:

(i) Kaehler submanifolds of constant holomorphic curvature K,
(ii) totally real submanifolds of constant sectional curvature $K/4$,
(iii) umbilical submanifolds of submanifolds of type (ii). (This classification

is given by Chen and Ogiue in a forthcoming paper. Here, a submanifold N is *totally real* if for any X tangent to N, JX is orthogonal to N.)

(c) The case $r=2$ of the corollary to Theorem 1 is of interest because a holomorphic 4-dimensional subspace S of M_m is spanned by the vectors in a pair (σ, σ') of holomorphic planes. It is on just such a pair that one of the authors and Kobayashi [4] defined the concept of biholomorphic curvature.

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