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ON THE GROWTH RATE OF COMPOSITIONS OF ENTIRE FUNCTIONS

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1. Let $f(z)$ be an entire function, $M(r, f)$ its maximum modulus on $|z|=r$ and $T(r, f)$ its Nevanlinna characteristic function. Recently Gross and Yang [4] proved the following:

Suppose that $f(z)$, $g(z)$ are entire functions such that

$$T(\alpha r, g)=o\{T(r, f)\} \quad as \quad r\to\infty \tag{1.1}$$

for some constant $\alpha>1$. Then for any non-constant entire function $h(z)$,

$$T(r, h\circ g)=o\{T(r, h\circ f)\} \quad as \quad r\to\infty$$

In this paper we shall consider the asymptotic behavior of the ratio $\log M(r, h\circ g)/\log M(r, h\circ f)$ replacing $T(r, \cdot)$ by $\log M(r, \cdot)$ in the above condition (1.1).

Our results are the following:

THEOREM 1. *Let $g(z)$ and $f(z)$ be entire functions such that*

$$\lim_{r\to\infty}\frac{\log M(\alpha r, g)}{\log M(r, f)}=0 \tag{1.2}$$

for some constant $\alpha>1$. Then for any non-constant entire function $h(z)$,

$$\lim_{r\to\infty}\frac{\log M(r, h\circ g)}{\log M(r, h\circ f)}=0.$$

THEOREM 2. *Let $g(z)$ and $f(z)$ be entire functions such that*

$$\lim_{r\to\infty}\frac{\log M(r, g)}{\log M(r, f)}=0. \tag{1.3}$$

Then for any non-constant entire function $h(z)$,

$$\varliminf_{r\to\infty}\frac{\log M(r, h\circ g)}{\log M(r, h\circ f)}=0.$$

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THEOREM 3. *Let $g(z)$ and $f(z)$ be entire functions satisfying* (1. 3). *Suppose that $f(z)$ is of finite order. Then for any non-constant entire function $h(z)$,*

$$\lim_{r\to\infty}\frac{\log M(r, h\circ g)}{\log M(r, h\circ f)}=0.$$

The next Theorem deals with the possibility still left open in Theorem 2.

THEOREM 4. *There exist entire function $g(z)$ and $f(z)$ such that*

$$\lim_{r\to\infty}\frac{\log M(r, g)}{\log M(r, f)}=0 \quad \text{and} \quad \overline{\lim_{r\to\infty}}\frac{\log M(r, \exp\circ g)}{\log M(r, \exp\circ f)}=\infty.$$

2. Lemmas. We start from the following lemmas which will be used in the proof of our Theorems.

LEMMA 1 ([3, 5, 6, 7]). *Let $h(z)$ and $f(z)$ be entire with $f(0)=0$. Let ρ satisfy $0<\rho<1$ and let $c(\rho)=(1-\rho)^2/4\rho$. Then for $r\geqq 0$,*

$$M(r, h\circ f)\geqq M(c(\rho)M(\rho r, f), h).$$

LEMMA 2 ([1, 3]). *Let $h(z)$ and $f(z)$ be entire. Then*

$$M(r, h\circ f)\geqq M((1+o(1))M(r, f), h) \qquad \text{as} \quad r\to\infty,$$

outside a set of r of finite logarithmic measure which depends, as does $o(1)$, on $f(z)$.

LEMMA 3. *For any transcendental entire function $f(z)$, there exists an entire function $g(z)$ such that*

$$\lim_{r\to\infty}\frac{\log M(r, g)}{\log M(r, f)}=0 \quad \text{and} \quad \lim_{r\to\infty}\frac{\log M(r, \exp\circ g)}{\log M(r, f)}=\infty. \tag{2. 1}$$

Proof. Let $f(z)$ be a transcendental entire function. Then Hadamard's three-circle theorem asserts that $\log M(r, f)$ is a convex, increasing function of $\log r$. Hence, by the well-known property of logarithmically convex function,

$$\log M(r, f)=\log M(r_0, f)+\int_{r_0}^{r}\frac{\psi(t)}{t}dt \qquad (r\geqq r_0), \tag{2. 2}$$

where $r_0>0$ and $\psi(t)$ is a non-negative, non-decreasing function of t. Since $f(z)$ is transcendental, we have

$$\lim_{r\to\infty}\frac{1}{\log r}\int_{r_0}^{r}\frac{\psi(t)}{t}dt=\infty \quad \text{and} \quad \lim_{t\to\infty}\psi(t)=\infty. \tag{2. 3}$$

We put

$$\phi(r)=\sup_{r_0\leqq t\leqq r}\frac{\psi(t)}{\log\log t} \quad (r_0>e) \qquad \text{when} \quad \overline{\lim_{t\to\infty}}\frac{\psi(t)}{\log t}=\infty \tag{2. 4}$$

and

(2.5) $$\phi(r)=\sup_{r_0\leqq t\leqq r}\frac{\psi(t)}{\log\psi(t)}\quad(\psi(r_0)>1)\qquad\text{when}\quad\overline{\lim_{t\to\infty}}\frac{\psi(t)}{\log t}<\infty$$

Then it follows from (2.3), (2.4) and (2.5) that $\phi(r)$ is non-decreasing and

(2.6) $$\lim_{r\to\infty}\phi(r)=\infty\quad\text{and}\quad\lim_{r\to\infty}\frac{\phi(r)}{\psi(r)}=0.$$

Put

$$\Phi(r)=\int_{r_0}^{r}\frac{\phi(t)}{t}dt\qquad(r\geqq r_0).$$

Then we obtain

(2.7) $$\lim_{r\to\infty}\frac{\Phi(r)}{\log M(r,f)}=0$$

and

(2.8) $$\lim_{r\to\infty}\frac{\Phi(r)^2}{\log M(r,f)}=\infty.$$

In fact, it follows from (2.3) and (2.6) that

$$\overline{\lim_{r\to\infty}}\frac{\Phi(r)}{\log M(r,f)}=\overline{\lim_{r\to\infty}}\left(\int_{r_0}^{r}\frac{\phi(t)}{t}dt\Big/\int_{r_0}^{r}\frac{\psi(t)}{t}dt\right)\leqq\overline{\lim_{r\to\infty}}\frac{\phi(r)}{\psi(r)}=0,$$

which implies (2.7). Next taking (2.4) and (2.5) into account, we get

$$\Phi(r)^2\geqq\frac{1}{K(\log\log r)^2}\int_{r_0}^{r}\frac{\psi(t)}{t}dt\cdot\int_{r_0}^{r}\frac{\psi(t)}{t}dt$$

with a suitable constant K. Hence (2.8) follows from (2.2), (2.3) and the above inequality.

Now, the definition of $\Phi(r)$ and (2.6) yield that $\Phi(r)$ is increasing and convex in $\log r$ and $\Phi(r)\neq O(\log r)$ $(r\to\infty)$. Hence Clunie's theorem [2] asserts that there exists an entire function $g(z)$ such that

$$M(r,g)=\max_{|z|=r}\operatorname{Re}g(z)$$

and

(2.9) $$\log M(r,g)\sim\Phi(r)\qquad(r\to\infty).$$

and consequently

(2.10) $$\begin{aligned}\log M(r,\exp\circ g)&=\exp(\log M(r,g))\\&\geqq\frac{1}{2}(\log M(r,g))^2\sim\frac{1}{2}\Phi(r)^2\qquad(r\to\infty).\end{aligned}$$

Therefore (2.1) follows from (2.7), (2.8), (2.9) and (2.10).

3. Proof of Theorem 1. Choose $\rho>0$ such that $\alpha>1/\rho>1$ and assume, for convenience, that $f(0)=0$. The case $f(0)\neq 0$ can be dealt with as in the proof of Theorem 1 in [3]. Then

$$\overline{\lim_{r\to\infty}}\frac{\log M(\alpha r, g)}{\log M(r, f)}\geqq\overline{\lim_{r\to\infty}}\frac{\log M(r/\rho, g)}{\log M(r, f)}=\overline{\lim_{r\to\infty}}\frac{\log M(r, g)}{\log M(\rho r, f)}$$

$$=\overline{\lim_{r\to\infty}}\frac{\log M(r, g)}{\log M(\rho r, f)+\log c(\rho)}$$

and so from the condition (1.2)

$$\lim_{r\to\infty}\frac{\log M(r, g)}{\log c(\rho)M(\rho r, f)}=0. \tag{3.1}$$

Hence there is $r_0>0$ such that for all $r>r_0$

$$M(r, g)<c(\rho)M(\rho r, f). \tag{3.2}$$

$\log M(r, h)$ is an increasing convex function of $\log r$, so that $\log M(r, h)/\log r$ is finally increasing and hence Lemma 1 and (3.2) yield

$$\frac{\log M(r, h\circ g)}{\log M(r, h\circ f)}\leqq\frac{\log M(M(r, g), h)}{\log M(c(\rho)M(\rho r, f), h)}\leqq\frac{\log M(r, g)}{\log c(\rho)M(\rho r, f)}$$

for all large r. Therefore Theorem 1 follows from this inequality and (3.1).

4. Proof of Theorem 2. The condition (1.3) implies

$$\lim_{r\to\infty}\frac{\log M(r, g)}{\log (M(r, f)/2)}=0 \tag{4.1}$$

and so there exists $r_0>0$ such that for all $r>r_0$

$$M(r, g)<\frac{M(r, f)}{2}. \tag{4.2}$$

It follows from Lemma 2 that there is a set E of finite logarithmic measure such that

$$\log M(r, h\circ f)\geqq\log M((1+o(1))M(r, f), h) \qquad r\to\infty;\ r\notin E.$$

Hence using (4.1) and (4.2) and noting that $\log M(r, h)/\log r$ is increasing we obtain

$$\frac{\log M(r, h\circ g)}{\log M(r, h\circ f)}\leqq\frac{\log M(M(r, g), h)}{\log M((1+o(1))M(r, f), h)}\leqq\frac{\log M(r, g)}{\log (1+o(1))M(r, f)}$$

$$\to 0 \qquad \text{as} \quad r\to\infty;\ r\notin E$$

and consequently

$$\lim_{r\to\infty}\frac{\log M(r, h\circ g)}{\log M(r, h\circ f)}=0,$$

which is the desired result.

5. Proof of Theorem 3. We may suppose, without loss of generality, that $f(0)=0$ (cf. [3]). Let λ be the order of $f(z)$. Take β such that $\beta>\lambda-1$. Since $\log M(r, f)$ is convex in $\log r$, we get

$$\log M(r, f)\sim\log M(r-r^{-\beta}, f) \qquad (r\to\infty) \tag{5. 1}$$

(cf. [3]). Hence (1. 3) and (5. 1) imply

$$\lim_{r\to\infty}\frac{\log M(r, g)}{\log(r^{-2(1+\beta)}M(r-r^{-\beta}, f)/4)}=0. \tag{5. 2}$$

We put $\rho=(r-r^{-\beta})/r$. Then we have $c(\rho)>r^{-2(1+\beta)}/4$. Hence it follows from Lemma 1 and (5. 2) that

$$\begin{aligned}\frac{\log M(r, h\circ g)}{\log M(r, h\circ f)}&\leqq\frac{\log M(M(r, g), h)}{\log M(r^{-2(1+\beta)}M(r-r^{-\beta}, f)/4, h)}\\&\leqq\frac{\log M(r, g)}{\log(r^{-2(1+\beta)}M(r-r^{\beta}, f)/4)}\to 0 \qquad (r\to\infty),\end{aligned}$$

from which Theorem 3 follows.

6. Proof of Pheorem 4. Let $f(z)$ be a transcendental entire function such that

$$\overline{\lim_{r\to\infty}}\frac{\log M(r, f)}{\log M(r, \exp\circ f)}=\infty. \tag{6. 1}$$

The existence of such a function $f(z)$ was shown by Clunie [3]. For the entire function $f(z)$, from Lemma 3, there exists an entire function $g(z)$ satisfying (2. 1). The entire functions $f(z)$ and $g(z)$ are our desired functions. In fact, (2. 1) and (6. 1) imply

$$\overline{\lim_{r\to\infty}}\frac{\log M(r, \exp\circ g)}{\log M(r, \exp\circ f)}\geqq\underline{\lim_{r\to\infty}}\frac{\log M(r, \exp\circ g)}{\log M(r, f)}\ \overline{\lim_{r\to\infty}}\frac{\log M(r, f)}{\log M(r, \exp\circ f)}=\infty.$$

Thus the proof of Theorem 4 is complete.

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