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ON EXTREMAL PROBLEMS WHICH CORRESPOND TO ALGEBRAIC UNIVALENT FUNCTIONS

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1. Let S denote the class of functions $f(z)$ regular and univalent in $|z|<1$

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n.$$

Let V_n denote the n-th coefficient region for functions of this class [6, § 1.2]. Let $F=F(a_2, \bar{a}_2, \cdots, a_n, \bar{a}_n)$ be a real-valued function satisfying the conditions

a) F is defined in an open set O containing V_n,
b) F and F_ν are continuous in O,
c) $|\text{grad}\, F|=(\sum_{\nu=2}^{n}|F_\nu|^2)^{1/2}>0$ in O

where

$$F_\nu=\frac{1}{2}\left(\frac{\partial F}{\partial x_\nu}-i\frac{\partial F}{\partial y_\nu}\right),$$

$$x_\nu=\frac{1}{2}(a_\nu+\bar{a}_\nu),\ y_\nu=\frac{1}{2i}(a_\nu-\bar{a}_\nu).$$

Then the following result was given by Schaeffer and Spencer [6, Lemma VII]:

Every function $f(z)$ of class S belonging to a point $(a_2, \cdots, a_n)$ where F attains its maximum on V_n must satisfy the differential equation

$$\left(z\frac{f'(z)}{f(z)}\right)^2\sum_{\nu=1}^{n-1}\frac{A_\nu}{f(z)^\nu}=\sum_{\nu=-n+1}^{n-1}\frac{B_\nu}{z^\nu} \tag{1}$$

where

$$A_\nu=\sum_{k=\nu+1}^{n} a_k^{(\nu+1)}F_k,\ B_\nu=\sum_{k=1}^{n-\nu} k a_k F_{k+\nu},\ \nu=1, 2, \cdots, n-1,$$

(2)

$$B_0=\sum_{k=1}^{n}(k-1)a_k F_k,\ B_{-\nu}=\bar{B}_\nu$$

and

$$f(z)^\nu=\sum_{k=\nu}^{\infty} a_k^{(\nu)} z^k.$$

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The derivatives are taken at the point $(a_2, \cdots, a_n)$. Moreover this differential eqaution has the properties (i) $B_0>0$ and (ii) the right hand side of (1) is non-negative on $|z|=1$ with at least one zero there.

Further Schaeffer and Spencer showed that if a function $f(z)$ of class S satisfies more than one differential equation of the form (1) which has the properties (i) and (ii), then it is an algebraic function [6, Theorem V]. Moreover as in the proof of Lemma XXXI in [6] we have that if $f(z)$ is single-valued, then it is of the form

$$f(z)=\frac{z}{(1-e^{i\alpha}z)(1-e^{i\beta}z)}.$$

Ozawa proposed the following problem to the author orally: Determine the algebraic functions of class S, not being single-valued, which are extremal functions for certain two extremal problems

$$\max_S F(a_2, \bar{a}_2, \cdots, a_m, \bar{a}_m)$$

and

$$\max_S \tilde{F}(a_2, \bar{a}_2, \cdots, a_n, \bar{a}_n)$$

where $m<n$, and find corresponding functions F and $\tilde{F}$.

In this paper we shall consider two-valued algebraic functions and the cases $m=3, 5$. Here we remark that if an extremal function is two-valued, then m and n are odd.

2. In our study we use the following lemma which was proved by Ozawa. For the sake of completeness we shall prove it.

LEMMA. *If a two-valued algebraic function $w=f(z)$ of class S satisfies differential equations of the form*

$$\left(\frac{z}{w}\frac{dw}{dz}\right)^2 \sum_{\nu=1}^{m-1}\frac{A_\nu}{w^\nu}=\sum_{\nu=-m+1}^{m-1}\frac{B_\nu}{z^\nu}, \quad A_{m-1}=B_{m-1}\neq 0, \quad B_{-\nu}=\bar{B}_\nu \tag{3}$$

and

$$\left(\frac{z}{w}\frac{dw}{dz}\right)^2 \sum_{\nu=1}^{n-1}\frac{C_\nu}{w^\nu}=\sum_{\nu=-n+1}^{n-1}\frac{D_\nu}{z^\nu}, \quad C_{n-1}=D_{n-1}\neq 0, \quad D_{-\nu}=\bar{D}_\nu \tag{4}$$

where $n>m$, then it satisfies an algebraic equation of the form

$$P(z)w^2+\beta z^2w-z^2=0, \qquad P(z)=1+\alpha_1z+\alpha_2z^2+\alpha_3z^3+\alpha_4z^4,\ \alpha_4\neq 0.$$

Proof. $w=f(z)$ satisfies an irreducible algebraic equation

$$P(z)w^2+Q(z)w+R(z)=0$$

where $P(z)$, $Q(z)$ and $R(z)$ are polynomials of z. Dividing (3) by (4) we have

$$w^{n-m}\frac{A_1w^{m-2}+\cdots+A_{m-1}}{C_1w^{n-2}+\cdots+C_{n-1}}=z^{n-m}\frac{\bar{B}_{m-1}z^{2m-2}+\cdots+B_{m-1}}{\bar{D}_{n-1}z^{2n-2}+\cdots+D_{n-1}}.$$

Hence for the two branches w_1, w_2 at $z=0$ we have

$$w_1(1+\lambda_1w_1+\cdots)=z(1+\mu_1z+\cdots),$$

$$w_2(1+\lambda_1w_2+\cdots)=-z(1+\mu_1z+\cdots).$$

Then

$$w_1+w_2=-2\lambda_1z^2+O(z^3),$$

$$w_1w_2=-z^2+O(z^3).$$

Since to each value of w there correspond two values of z, P, Q, R have degree at most 4 and one has degree 4. We may assume that $P(0)=1$. Comparing the coefficients Q, R with w_1+w_2, w_1w_2 we have that

$$Q(z)=\beta z^2+\beta' z^3+\beta'' z^4, \qquad \beta=-2\lambda_1,$$

$$R(z)=\gamma z^2+\gamma' z^3+\gamma'' z^4, \qquad \gamma=-1.$$

Similar situation holds for $z=\infty$, $w=0$. Only differences appearing here are the conjugation for μ_j and the replacement of z by $t=z^{-1}$. Hence we have

$$\beta'=\beta''=\gamma'=\gamma''=0.$$

Thus we have the desired result.

3. In this section we prove the following

THEOREM 1. *Let* $F(a_2, \bar{a}_2, a_3, \bar{a}_3)$ *and* $\tilde{F}(a_2, \bar{a}_2, \cdots, a_n, \bar{a}_n)$ $(n>3)$ *be real-valued functions satisfying the conditions* a), b), c) *and* d) $F_3\neq 0$, $\tilde{F}_n\neq 0$. *If* $f(z)$ *is an extremal function for the extremal problems*

$$\max_S F(a_2, \bar{a}_2, a_3, \bar{a}_3)$$

and

$$\max_S \tilde{F}(a_2, \bar{a}_2, \cdots, a_n, \bar{a}_n),$$

then $f(z)$ *is of the form*

$$f(z)=\frac{z}{(1-e^{i\alpha}z)(1-e^{i\beta}z)}.$$

Proof. By the result of Schaeffer and Spencer, $w=f(z)$ satisfies differential equations of the form

$$(5) \qquad \left(\frac{z}{w}\frac{dw}{dz}\right)^2\sum_{\nu=1}^{2}\frac{A_\nu}{w^\nu}=\sum_{\nu=-2}^{2}\frac{B_\nu}{z^\nu}, \qquad A_2=B_2\neq 0, B_{-\nu}=\bar{B}_\nu$$

and

$$\left(\frac{z}{w}\frac{dw}{dz}\right)^2\sum_{\nu=1}^{2n-1}\frac{C_\nu}{w^\nu}=\sum_{\nu=-n+1}^{n-1}\frac{D_\nu}{z^\nu}, \qquad C_{n-1}=D_{n-1}\neq 0,\ D_{-\nu}=\bar{D}_\nu$$

which have the properties (i) and (ii). Then as in the proof of Lemma XXXI in [6] we have that $f(z)$ is either a two-valued algebraic function or of the form

$$f(z)=\frac{z}{(1-e^{i\alpha}z)(1-e^{i\beta}z)}.$$

We assume that $f(z)$ is a two-valued algebraic function. By Lemma, $w=f(z)$ satisfies an algebraic equation of the form

(6) $$P(z)w^2+\beta z^2w-z^2=0, \qquad P(z)=1+\alpha_1z+\alpha_2z^2+\alpha_3z^3+\alpha_4z^4,\ \alpha_4\neq 0.$$

Putting $\zeta=w^{-1}$ we can write this as

(7) $$\zeta^2-\beta\zeta-z^{-2}P=0,$$

and differentiating we have

$$\frac{d\zeta}{dz}=\frac{P'z-2P}{z^3(2\zeta-\beta)}.$$

Inserting this in (5) we have

$$\frac{A_1+A_2\zeta}{\zeta(2\zeta-\beta)^2}=\frac{z^2S}{(P'z-2P)^2}$$

where $S=B_2+B_1z+B_0z^2+\bar{B}_1z^3+\bar{B}_2z^4$. Using (7) this reduces to the form

$$\frac{A_1+A_2\zeta}{(\beta^2z^2+4P)\zeta}=\frac{S}{(P'z-2P)^2}.$$

Since $f(z)$ is not single-valued, we have

$$\frac{A_2}{\beta^2z^2+4P}=\frac{S}{(P'z-2P)^2}.$$

Putting $T=\beta^2z^2+4P$ we have

$$A_2(T'z-2T)^2=16ST.$$

This implies that all zeros of β^2z^2+4P are multiple, and hence that

$$\beta^2z^2+4P=4\alpha_4(z-a)^2(z-b)^2.$$

Hence we can write (6) as

$$4\alpha_4(z-a)^2(z-b)^2w^2=z^2(\beta w-2)^2.$$

This contradicts that $f(z)$ is two-valued. Thus we have the desired result.

4. In the sequel we are concerned with the case $m=5$. Firstly we determine the two-valued algebraic functions of class S which are extremal functions for certain two extremal problems

$$\max_S F(a_2, \bar{a}_2, \cdots, a_5, \bar{a}_5)$$

and

$$\max_S \tilde{F}(a_2, \bar{a}_2, \cdots, a_n, \bar{a}_n) \qquad (n>5).$$

THEOREM 2. *Let* $F(a_2, \bar{a}_2, \cdots, a_5, \bar{a}_5)$ *and* $\tilde{F}(a_2, \bar{a}_2, \cdots, a_n, \bar{a}_n)$ $(n>5)$, *be real-valued functions satisfying the conditions* a), b), c) *and* d) $F_5 \neq 0$, $\tilde{F}_n \neq 0$. *If* $f(z)$ *is a two-valued algebraic function which is an extremal function for the extremal problems*

$$\max_S F(a_2, \bar{a}_2, \cdots, a_5, \bar{a}_5)$$

and

$$\max_S \tilde{F}(a_2, \bar{a}_2, \cdots, a_n, \bar{a}_n),$$

then it satisfies an algebraic equation of the form

$$\{1+e^{-i\theta}\alpha z+(a_2^2-2a_3-e^{-i\theta}a_2\alpha)z^2-\bar{\alpha}z^3-e^{-i\theta}z^4\}w^2-(e^{-i\theta}\alpha+2a_2)z^2w-z^2=0, \tag{8}$$
$$w=f(z)$$

where a_ν *is the* ν*-th coefficient of* $f(z)$, θ *is a real number and* α *is a complex number.*

Proof. By the result of Schaeffer and Spencer, $w=f(z)$ satisfies differential equations of the form

$$\left(\frac{z}{w}\frac{dw}{dz}\right)^2 \sum_{\nu=1}^{4} \frac{A_\nu}{w^\nu} = \sum_{\nu=-4}^{4} \frac{B_\nu}{z^\nu}, \qquad A_4=B_4 \neq 0,\ B_{-\nu}=\bar{B}_\nu \tag{9}$$

and

$$\left(\frac{z}{w}\frac{dw}{dz}\right)^2 \sum_{\nu=1}^{n-1} \frac{C_\nu}{w^\nu} = \sum_{\nu=-n+1}^{n-1} \frac{D_\nu}{z^\nu}, \qquad C_{n-1}=D_{n-1} \neq 0,\ D_{-\nu}=\bar{D}_\nu$$

which have the properties (i) and (ii). Hence by Lemma $w=f(z)$ satisfies an algebraic equation of the form

$$\zeta^2-\beta\zeta-z^{-2}P=0, \qquad P=1+\alpha_1 z+\alpha_2 z^2+\alpha_3 z^3+\alpha_4 z^4,\ \alpha_4 \neq 0,\ \zeta=w^{-1}. \tag{10}$$

Differentiating we have

$$\frac{d\zeta}{dz}=\frac{P'z-2P}{z^3(2\zeta-\beta)}. \tag{11}$$

Inserting (11) in (9) we have

$$\frac{A_4\zeta^3+A_3\zeta^2+A_2\zeta+A_1}{\zeta(2\zeta-\beta)^2}=\frac{S}{(P'z-2P)^2}$$

where $S=B_4+B_3z+B_2z^2+B_1z^3+B_0z^4+\bar{B}_1z^5+\bar{B}_2z^6+\bar{B}_3z^7+\bar{B}_4z^8$. Using (10) this reduces to the form

$$\frac{L_1\zeta+L_0}{M_1\zeta}=\frac{S}{(P'z-2P)^2}$$

where

$$L_1=A_2+\beta A_3+\beta^2A_4+XA_4,$$

$$L_0=A_1+X(A_3+\beta A_4),$$

$$M_1=\beta^2+4X,$$

$$X=z^{-2}P(z).$$

Since $f(z)$ is not single-valued, we have

$$L_0=0 \tag{12}$$

and

$$\frac{L_1}{M_1}=\frac{S}{(P'z-2P)^2}. \tag{13}$$

Since $P(z)$ is a polynomial of degree 4, (12) implies that $A_1=0$ and $\beta=-A_3A_4^{-1}$. Hence we can write (13) as

$$\frac{A_4^3P+A_2A_4^2z^2}{4A_4^2P+A_3^2z^2}=\frac{S}{(P'z-2P)^2}. \tag{14}$$

Suppose that there is no common zero of $A_4^3P+A_2A_4^2z^2$ and $4A_4^2P+A_3^2z^2$. Then (14) reduces to the form

$$(P'z-2P)^2=S^*(4A_4^2P+A_3^2z^2)$$

where S^* is a polynomial of degree 4. Putting $T=4A_4^2P+A_3^2z^2$ we have

$$(T'z-2T)^2=16A_4^4S^*T.$$

This implies that all zeros of $4A_4^2P+A_3^2z^2$ are multiple, whence we have

$$4A_4^2P+A_3^2z^2=4A_4^2\alpha_4(z-a)^2(z-b)^2.$$

Hence we can write (10) as

$$4\alpha_4(z-a)^2(z-b)^2w^2=z^2(\beta w-2)^2.$$

This contradicts that $f(z)$ is two-valued.

Let z_0 be a common zero of $A_4^3P+A_2A_4^2z^2$ and $4A_4^2P+A_3^2z^2$. Then we have

$$(4A_2A_4-A_3^2)z_0^2=0.$$

Since $P(0)\neq 0$, we have the relation

$$4A_2A_4=A_3^2. \tag{15}$$

Hence (14) reduces to the form

$$4S=A_4(P'z-2P)^2.$$

We may assume that $A_4=B_4=e^{i\theta}$. By this equation we have the relation

$$\begin{aligned} &B_3=\bar{B}_1e^{i\theta},\\ &4B_2=B_3^2e^{-i\theta},\\ &2B_0=|B_3|^2+4. \end{aligned} \tag{16}$$

Using the relations (15) and (16) we can write (9) as

$$\left(\frac{z}{w}\frac{dw}{dz}\right)^2\left(\frac{1}{w^2}+\frac{e^{-i\theta}A_3}{2w}\right)^2=\left(\frac{1}{z^2}+\frac{e^{-i\theta}B_3}{2z}+\frac{\bar{B}_3}{2}z+e^{-i\theta}z^2\right)^2.$$

We integrate and find

$$\{1+e^{-i\theta}B_3z+(3a_2^2-2a_3-e^{-i\theta}a_2A_3)z^2-\bar{B}_3z^3-e^{-i\theta}z^4\}w^2-e^{-i\theta}A_3z^2w-z^2=0$$

where a_ν is the ν-th coefficient of $f(z)$. Since $A_3=B_3+2e^{i\theta}a_2$, we have the desired result by putting $B_3=\alpha$.

REMARK. Suppose that the polynomial

$$P(w,z)=\{1+e^{-i\theta}\alpha z+(a_2^2-2a_3-e^{-i\theta}a_2\alpha)z^2-\bar{\alpha}z^3-e^{-i\theta}z^4\}w^2-(e^{-i\theta}\alpha+2a_2)z^2w-z^2$$

is reducible. We may assume that $P(w,z)$ has the factorization

$$P(w,z)=\{p(z)w+z\}\{(p(z)-(e^{-i\theta}\alpha+2a_2)z)w-z\},\ p(z)=\lambda z^2+\mu z+\nu.$$

Then we have the relations

$$\begin{aligned} &\lambda^2=-e^{-i\theta}, \qquad \nu^2=1,\\ &2\lambda\mu-\lambda(e^{-i\theta}\alpha+2a_2)=-\bar{\alpha},\\ &2\mu\nu-\nu(e^{-i\theta}\alpha+2a_2)=e^{-i\theta}\alpha,\\ &\mu^2+2\lambda\nu-\mu(e^{-i\theta}\alpha+2a_2)=a_2^2-2a_3-e^{-i\theta}a_2\alpha. \end{aligned}$$

Hence there are two cases

i) $\alpha=ie^{i3\theta/2}\bar{\alpha}$,

$$P(w,z)=\{(1-a_2z+ie^{-i\theta/2}z^2)w-z\}\{(1+(e^{-i\theta}\alpha+a_2)z+ie^{-i\theta/2}z^2)w+z\},$$

ii) $\alpha=-ie^{i3\theta/2}\bar{\alpha}$,

$$P(w, z)=\{(1-a_2z-ie^{-i\theta/2}z^2)w-z\}\{(1+(e^{-i\theta}\alpha+a_2)z-ie^{-i\theta/2}z^2)w+z\}.$$

However in these cases there are two-valued algebraic functions of S satisfying (8). For instance in the case $\alpha=0$, $e^{-i\theta}=-1$ the two-valued algebraic function $w=z\{1-(\varepsilon+\bar{\varepsilon})z^2+z^4\}^{-1/2}$, $|\varepsilon|=1$, satisfies (8).

5. Next we construct an extremal problem concerning the first four coefficients $a_2, \cdots, a_5$ for which the algebraic functions of class S satisfying (8) are extremal.

Let $w=f(z)$ be an algebraic function of class S satisfying (8). Then it satisfies the differential equation

$$\left(\frac{z}{w}\frac{dw}{dz}\right)^2\left\{\frac{e^{i\theta}}{w^4}+\frac{\alpha+2e^{i\theta}a_2}{w^3}+\frac{e^{-i\theta}(\alpha+2e^{i\theta}a_2)^2}{4w^2}\right\}$$

$$=\frac{e^{i\theta}}{z^4}+\frac{\alpha}{z^3}+\frac{e^{-i\theta}\alpha^2}{4z^2}+\frac{e^{i\theta}\bar{\alpha}}{z}+\frac{|\alpha|^2}{2}+2+e^{-i\theta}\alpha z+\frac{e^{i\theta}\bar{\alpha}^2}{4}z^2$$

$$+\bar{\alpha}z^3+e^{-i\theta}z^4.$$

Now we put in the relation (2)

$$A_1=0,\ A_2=4^{-1}e^{-i\theta}\alpha^2+a_2\alpha+e^{i\theta}a_2^2,\ A_3=\alpha+2e^{i\theta}a_2,\ A_4=e^{i\theta}.$$

Then we have by eliminating F_ν $(\nu=2, 3, 4, 5)$

$$B_0=e^{i\theta}(4a_5-8a_2a_4+16a_2^2a_3-6a_2^4-6a_3^2)+(3a_4-6a_2a_3+3a_2^3)\alpha$$

$$+e^{-i\theta}\left(\frac{1}{2}a_3-\frac{1}{2}a_2^2\right)\alpha^2.$$

We shall show that

$$\max_S F,$$

$$F=\Re\left\{e^{i\theta}(4a_5-8a_2a_4+16a_2^2a_3-6a_2^4-6a_3^2)+\frac{4}{3}(3a_4-6a_2a_3+3a_2^3)\alpha\right.$$

$$\left.+2e^{-i\theta}\left(\frac{1}{2}a_3-\frac{1}{2}a_2^2\right)\alpha^2\right.$$

is a desired extremal problem.

THEOREM 3. *In* S

$$\Re\left[e^{i\theta}\left(a_5-2a_2a_4+4a_2^2a_3-\frac{3}{2}a_2^4-\frac{3}{2}a_3^2\right)+(a_4-2a_2a_3+a_2^3)\alpha+e^{-i\theta}\left(\frac{1}{4}a_3-\frac{1}{4}a_2^2\right)\alpha^2\right]$$

$$\leqq\frac{1}{2}+\frac{1}{4}|\alpha|^2.$$

Equality occurs only for the algebraic functions of class S satisfying

$$\{1+e^{-i\theta}\alpha z+(a_2^2-2a_3-e^{-i\theta}a_2\alpha)z^2-\bar{\alpha}z^3-e^{-i\theta}z^4\}w^2-(e^{-i\theta}\alpha+2a_2)z^2w-z^2=0.$$

Proof. By the result of Schaeffer and Spencer, every extremal function $w=f(z)$ satisfies the differential equation

$$\left(\frac{z}{w}\frac{dw}{dz}\right)^2\sum_{\nu=1}^{4}\frac{A_\nu}{w^\nu}=\sum_{\nu=-4}^{4}\frac{B_\nu}{z^\nu}, \tag{17}$$

where

$$A_1=0,\ A_2=4^{-1}e^{-i\theta}\alpha^2+a_2\alpha+e^{i\theta}a_2^2,\ A_3=\alpha+2e^{i\theta}a_2,\ A_4=e^{i\theta},$$

$$B_1=e^{i\theta}(2a_4-4a_2a_3+2a_2^3)+(a_3-a_2^2)\alpha,\ B_2=4^{-1}e^{-i\theta}\alpha^2,\ B_3=\alpha,\ B_4=e^{i\theta},$$

$$B_0=e^{i\theta}(4a_5-8a_2a_4+16a_2^2a_3-6a_2^4-6a_3^2)+(3a_4-6a_2a_3+3a_2^3)\,\alpha+\frac{1}{2}\,e^{-i\theta}(a_3-a_2^2)\alpha^2.$$

Since $4A_2A_4=A_3^2$, we can write (17) as

$$\left(\frac{z}{w}\frac{dw}{dz}\right)^2\left\{\frac{1}{w^2}+\frac{e^{-i\theta}(\alpha+2e^{i\theta}a_2)}{2w}\right\}^2=\left(\frac{1}{z^2}+\frac{e^{-i\theta}\alpha}{2z}+\frac{\bar{\alpha}}{2}z+e^{-i\theta}z^2\right)^2.$$

We integrate and find

$$\{1+e^{-i\theta}\alpha z+(a_2^2-2a_3-e^{-i\theta}a_2\alpha)z^2-\bar{\alpha}z^3-e^{-i\theta}z^4\}w^2-(e^{-i\theta}\alpha+2a_2)z^2w-z^2=0.$$

Hence the coefficients of $f(z)$ satisfy the relations

$$2a_4-4a_2a_3+2a_2^3-e^{-i\theta}(a_2^2-a_3)\alpha-\bar{\alpha}=0$$

and

$$2a_5-4a_2a_4+8a_2^2a_3-3a_2^4-3a_3^2-e^{-i\theta}+e^{-i\theta}(a_2^3-2a_2a_3+a_4)\alpha=0.$$

Therefore we have

$$\Re\left\{e^{i\theta}\left(a_5-2a_2a_4+4a_2^2a_3-\frac{3}{2}a_2^4-\frac{3}{2}a_3^2\right)+(a_4-2a_2a_3+a_2^3)\alpha+e^{-i\theta}\left(\frac{1}{4}a_3-\frac{1}{4}a_2^2\right)\alpha^2\right\}$$

$$=\frac{1}{2}+\frac{1}{4}|\alpha|^2.$$

Thus we have the desired result.

6. Now we show that for some n $(n>5)$ there is an extremal problem concerning the first $n-1$ coefficients $a_2,\cdots,a_n$ for which the algebraic functions of class S satisfying (8) are extremal.

Let Σ denote the class of functions $g(z)$ univalent in $|z|>1$, regular apart from a simple pole at the point at infinity and having expansion at that point

$$g(z)=z+b_0+\sum_{n=1}^{\infty}\frac{b_n}{z^n}.$$

Let $G_\mu(w)$ be the μ-th Faber polynomial which is defined by

$$G_\mu(g(z))=z^\mu+\sum_{\nu=1}^{\infty}\frac{\beta_{\mu\nu}}{z^\nu}.$$

Then Grunsky's inequality [1] has the form

$$\left|\sum_{\mu,\nu=1}^{N}\nu\beta_{\mu\nu}x_\mu x_\nu\right|\leqq\sum_{\nu=1}^{N}\nu|x_\nu|^2.$$

Let $f(z)$ be a function of class S and put

$$f(z^{-1})^{-1}=z+b_0+\sum_{n=1}^{\infty}\frac{b_n}{z^n}\qquad(|z|>1).$$

Applying Grunsky's inequality with $N=8$, $x_1=x_3=x_5=x_7=0$ to the function $g(z)=f(z^{-2})^{-1/2}$, we have

$$|G(x_2, x_4, x_6, x_8; b_1, b_2, \cdots, b_7)|\leqq|x_2|^2+2|x_4|^2+3|x_6|^2+4|x_8|^2,$$

where

$$\begin{aligned}
&G(x_2, x_4, x_6, x_8; b_1, b_2, \cdots, b_7)\\
&=x_2^2b_1+4x_2x_4b_2+6x_2x_6b_3+8x_2x_8b_4\\
&\quad+2x_4^2(2b_3+b_1^2)+12x_4x_6(b_4+b_1b_2)+8x_4x_8(2b_5+2b_1b_3+b_2^2)\\
&\quad+3x_6^2(3b_5+3b_1b_3+3b_2^2+b_1^3)+24x_6x_8(b_6+b_1b_4+2b_2b_3+b_1^2b_2)\\
&\quad+4x_8^2(4b_7+4b_1b_5+8b_2b_4+6b_3^2+4b_1^2b_3+8b_1b_2^2+b_1^4).
\end{aligned}$$

We seek for the values x_2, x_4, x_6, x_8 such that $G(x_2, x_4, x_6, x_8; b_1, b_2, \cdots, b_7)$ attains the value $|x_2|^2+2|x_4|^2+3|x_6|^2+4|x_8|^2$ at the algebraic functions satisfying (8). The coefficients of the algebraic functions satisfying (8) satisfy the relations

$$\begin{aligned}
&2b_2+e^{-i\theta}b_1\alpha+\bar{\alpha}=0,\\
&2b_3+b_1^2+e^{-i\theta}b_2\alpha+e^{-i\theta}=0,\\
&2b_4+2b_1b_2+e^{-i\theta}b_3\alpha=0,\qquad(18)\\
&2b_5+2b_1b_3+b_2^2+e^{-i\theta}b_4\alpha=0,\\
&2b_6+2b_1b_4+2b_2b_3+e^{-i\theta}b_5\alpha=0
\end{aligned}$$

and

$$2b_7+2b_1b_5+2b_2b_4+b_3^2+e^{-i\theta}b_6\alpha=0.$$

Using these relations, we can find that $x_2=-2e^{i\theta}\bar{\alpha}$, $x_4=e^{-i\theta}\alpha^2$, $x_6=2\alpha$ and $x_8=e^{i\theta}$ are desired numbers, namely

$$G(-2e^{i\theta}\bar{\alpha}, e^{-i\theta}\alpha^2, 2\alpha, e^{i\theta}; b_1, b_2, \cdots, b_7)$$
$$=|-2e^{i\theta}\bar{\alpha}|^2+2|e^{-i\theta}\alpha^2|^2+3|2\alpha|^2+4|e^{i\theta}|^2$$

at the algebraic functions satisfying (8). Thus we have the inequality

$$\begin{aligned}&\Re\{e^{i2\theta}(16b_7+16b_1b_5+32b_2b_4+24b_3^2+16b_1^2b_3+32b_1b_2^2+4b_1^4)\\&\quad+e^{i\theta}(48b_6+48b_1b_4+96b_2b_3+48b_1^2b_2)\alpha\\&\quad+(52b_5+52b_1b_3+44b_2^2+12b_1^3)\alpha^2+e^{-i\theta}(24b_4+24b_1b_2)\alpha^3\\&\quad+e^{-i2\theta}(4b_3+2b_1^2)\alpha^4-16e^{i2\theta}b_4\bar{\alpha}+4e^{i2\theta}b_1\bar{\alpha}^2-24e^{i\theta}b_3|\alpha|^2-8b_2\alpha|\alpha|^2\}\\&\leqq 4+16|\alpha|^2+2|\alpha|^4\end{aligned}$$

in S. Equality occurs for the algebraic functions satisfying (8). Rewriting with the coefficients of $f(z)$ we have the following

THEOREM 4. *In S*

$$\begin{aligned}\Re\Big\{&e^{i2\theta}\Big(-a_9+2a_2a_8-4a_2^2a_7+8a_2^3a_6-16a_2^4a_5+30a_2^5a_4-50a_2^6a_3+\frac{35}{4}a_2^8\\&\quad-12a_2a_3a_6-16a_2a_4a_5+48a_2a_3^2a_4+36a_2^2a_3a_5+21a_2^2a_4^2\\&\quad-52a_2^2a_3^3-88a_2^3a_3a_4+87a_2^4a_3^2-10a_3a_4^2+3a_3a_7-9a_3^2a_5\\&\quad+\frac{19}{4}a_3^4+4a_4a_6+\frac{5}{2}a_5^2\Big)\\&+e^{i\theta}(-3a_8+6a_2a_7-12a_2^2a_6+24a_2^3a_5-45a_2^4a_4+75a_2^5a_3-15a_2^7-36a_2a_3a_5\\&\quad-21a_2a_4^2+39a_2a_3^3+99a_2^2a_3a_4-108a_2^3a_3^2+9a_3a_6-24a_3^2a_4+12a_4a_5)\alpha\\&+\Big(-\frac{13}{4}a_7+\frac{13}{2}a_2a_6-13a_2^2a_5+25a_2^3a_4-\frac{85}{2}a_2^4a_3+10a_2^6-37a_2a_3a_4\\&\quad+\frac{183}{4}a_2^2a_3^2+\frac{39}{4}a_3a_5-\frac{29}{4}a_3^3+6a_4^2\Big)\alpha^2\\&+e^{-i\theta}\Big(-\frac{3}{2}a_6+3a_2a_5-6a_2^2a_4+\frac{21}{2}a_2^3a_3-3a_2^5-\frac{15}{2}a_2a_3^2+\frac{9}{2}a_3a_4\Big)\alpha^3\\&+e^{-i2\theta}\Big(-\frac{1}{4}a_5+\frac{1}{2}a_2a_4-a_2^2a_3+\frac{3}{8}a_2^4+\frac{3}{8}a_3^2\Big)\alpha^4\\&+e^{i2\theta}(a_6-2a_2a_5+3a_2^2a_4-4a_2^3a_3+a_2^5-2a_3a_4+3a_2a_3^2)\bar{\alpha}\\&+e^{i2\theta}\Big(-\frac{1}{4}a_3+\frac{1}{4}a_2^2\Big)\bar{\alpha}^2\end{aligned}$$

$$+e^{i\theta}\left(\frac{3}{2}a_5-3a_2a_4+\frac{9}{2}a_2^2a_3-\frac{3}{2}a_2^4-\frac{3}{2}a_3^2\right)|\alpha|^2$$

$$+\left(\frac{1}{2}a_4-a_2a_3+\frac{1}{2}a_2^3\right)\alpha|\alpha|^2\Big\}$$

$$\leqq\frac{1}{4}+|\alpha|^2+\frac{1}{8}|\alpha|^4.$$

Equality occurs for the algebraic functions satisfying

$$\{1+e^{-i\theta}\alpha z+(a_2^2-2a_3-e^{-i\theta}a_2\alpha)z^2-\bar{\alpha}z^3-e^{-i\theta}z^4\}w^2-(e^{-i\theta}\alpha+2a_2)z^2w-z^2=0.$$

7. Finally we consider the case $m=5$, $n=7$. Let $w=f(z)$ be a two-valued algebraic function of class S satisfying an algebraic equation of the form

$$\begin{aligned} &P(z)w^2+\beta z^2w-z^2=0,\\ &P(z)=1+e^{-i\theta}\alpha z+(a_2^2-2a_3-e^{-i\theta}a_2\alpha)z^2-\bar{\alpha}z^3-e^{-i\theta}z^4,\\ &\beta=-e^{-i\theta}\alpha-2a_2. \end{aligned}\tag{19}$$

Further let $w=f(z)$ satisfy a differential equation of the form

$$\left(\frac{z}{w}\frac{dw}{dz}\right)^2\sum_{\nu=1}^{6}\frac{C_\nu}{z^\nu}=\sum_{\nu=-6}^{6}\frac{D_\nu}{z^\nu},\qquad C_6=D_6\neq0,\ D_{-\nu}=\bar{D}_\nu\tag{20}$$

which has the properties (i) and (ii). Then as in the proof of Theorem 2 we have

$$\frac{L_1\zeta+L_0}{M_1\zeta}=\frac{S}{z^2(P'z-2P)^2},\qquad \zeta^{-1}=f(z)$$

where

$$\begin{aligned} &L_1=C_2+\beta C_3+\beta^2C_4+\beta^3C_5+\beta^4C_6+(C_4+2\beta C_5+3\beta^2C_6)X+C_6X^2,\\ &L_0=C_1+(C_3+\beta C_4+\beta^2C_5+\beta^3C_6)X+(C_5+2\beta C_6)X^2,\\ &M_1=\beta^2+4X,\\ &X=z^{-2}P(z),\\ &S=D_6+D_5z+\cdots+D_0z^6+\cdots+\bar{D}_5z^{11}+\bar{D}_6z^{12}. \end{aligned}$$

Since $f(z)$ is two-valued, we have

$$L_0=0\tag{21}$$

and

$$\frac{L_1}{M_1}=\frac{S}{z^2(P'z-2P)^2}.\tag{22}$$

By (21) we have

$$
\begin{aligned}
&C_5+2\beta C_6=0,\\
(23)\qquad &C_3+\beta C_4+\beta^2C_5+\beta^3C_6=0,\\
&C_1=0,
\end{aligned}
$$

whence we can write (22) as

$$
(24)\qquad \frac{4C_6^3P^2+(4C_4C_6^2-C_5^2C_6)z^2P+4C_2C_6^2z^4}{16C_6^2P+C_5^2z^2}=\frac{S}{(P'z-2P)^2}.
$$

Suppose that the numerator and the denominator of the left hand side of (24) have no common zero. Then (24) reduces to the form

$$
(P'z-2P)^2=S^*(16C_6^2P+C_5^2z^2)
$$

where S^* is a polynomial of degree 4. Putting $T=16C_6^2P+C_5^2z^2$ we have

$$
(T'z-2T)^2=256C_6^4S^*T.
$$

This implies that all zeros of $16C_6^2P+C_5^2z^2$ are multiple. Hence we can write (19) as

$$
-4e^{-i\theta}(z-a)^2(z-b)^2w^2=z^2(\beta w-2)^2.
$$

This is a contradiction. Let z_0 be a common zero of the numerator and the denominator of the left hand side of (24). Then we have

$$
(256C_2C_6^3-16C_4C_5^2C_6+5C_5^4)z_0^2=0.
$$

Since $P(0)\neq 0$, we have $z_0\neq 0$, whence

$$
(25)\qquad 256C_2C_6^3-16C_4C_5^2C_6+5C_5^4=0.
$$

Hence (24) reduces to the form

$$
64C_6S=\{16C_6^2P+(16C_4C_6-5C_5^2)z^2\}(P'z-2P)^2.
$$

By this equation and the relations (23), (25) we obtain the following relations by putting $C_6=e^{i\varphi}$ and $D_4=\gamma$

$$
\begin{aligned}
&C_6=e^{i\varphi},\ C_5=4e^{i\varphi}a_2+2e^{i(\varphi-\theta)}\alpha,\ C_4=e^{i\varphi}(4a_2^2+2a_3)+6e^{i(\varphi-\theta)}a_2\alpha+\gamma,\\
&C_3=4e^{i\varphi}a_2a_3+e^{i(\varphi-\theta)}(4a_2^2+2a_3)\alpha-e^{i(\varphi-3\theta)}\alpha^3+2a_2\gamma+e^{-i\theta}\alpha\gamma,\\
&C_2=e^{i\varphi}(2a_2^2a_3-a_2^4)+2e^{i(\varphi-\theta)}a_2a_3\alpha+e^{i(\varphi-2\theta)}\left(\frac{1}{2}a_3-\frac{1}{2}a_2^2\right)\alpha^2\\
&\qquad -e^{i(\varphi-3\theta)}a_2\alpha^3-\frac{5}{16}e^{i(\varphi-4\theta)}\alpha^4+a_2^2\gamma+e^{-i\theta}a_2\alpha\gamma+\frac{1}{4}e^{-i2\theta}\alpha^2\gamma,\\
&C_1=0
\end{aligned}
$$

and

$$
(26)\quad
\begin{aligned}
&D_6=e^{i\varphi},\ D_5=2e^{i(\varphi-\theta)}\alpha,\ D_4=\gamma,\ D_3=e^{-i\varphi}\alpha^3+e^{-i\theta}\alpha\gamma,\\
&D_2=e^{i(\varphi-\theta)}+\frac{1}{2}e^{i(\varphi-\theta)}|\alpha|^2+\frac{5}{16}e^{-i(\varphi+\theta)}\alpha^4+\frac{1}{4}e^{-i2\theta}\alpha^2\gamma,\\
&D_1=-2e^{-i(\varphi-\theta)}\alpha+e^{-i(\varphi-\theta)}\alpha|\alpha|^2+\bar{\alpha}\gamma,\\
&D_0=\left(\frac{1}{2}|\alpha|^2+2\right)\left(e^{-i\theta}\gamma+\frac{5}{4}e^{-i\varphi}\alpha^2\right)-\frac{3}{4}(e^{-i\varphi}\alpha^2+e^{i\varphi}\bar{\alpha}^2),\\
&e^{i(2\varphi-3\theta)}=-1.
\end{aligned}
$$

Since the differential equation (20) has the properties (i) and (ii), γ must satisfy the conditions

(I) $D_0>0$,

(II) $D_0+2\sum_{\nu=1}^{6}|D_\nu|\cos(\nu t-\Phi_\nu)\geqq 0 \qquad (0\leqq t\leqq 2\pi), \qquad D_\nu=|D_\nu|e^{i\Phi_\nu}$.

On the other hand as in § 5 we can construct an extremal problem whose every extremal function satisfies a differential equation of the form

$$\left(\frac{z}{w}\frac{dw}{dz}\right)^2\sum_{\nu=1}^{6}\frac{\tilde{A}_\nu}{w^\nu}=\sum_{\nu=-6}^{6}\frac{\tilde{B}_\nu}{z^\nu},$$

where

$$\tilde{A}_\nu=C_\nu,\ \tilde{B}_{-\nu}=\bar{\tilde{B}}_\nu, \qquad \nu=1,2,\cdots,6.$$

In fact we put $A_\nu=C_\nu$ $(\nu=1,2,\cdots,6)$ in the relation (2). Then we have by eliminating F_ν $(\nu=2,\cdots,7)$

$$
\begin{aligned}
B_0=\ &e^{i\varphi}X_0+e^{i(\varphi-\theta)}X_1\alpha+e^{i(\varphi-2\theta)}X_2\alpha^2+e^{i(\varphi-3\theta)}X_3\alpha^3+e^{i(\varphi-4\theta)}X_4\alpha^4\\
&+X_5\gamma+e^{-i\theta}X_6\alpha\gamma+e^{-i2\theta}X_7\alpha^2\gamma,\\
X_0=\ &6a_7-12a_2a_6+24a_2^2a_5-48a_2^3a_4+78a_2^4a_3-18a_2^6-84a_2^2a_3^2+72a_2a_3a_4\\
&-18a_3a_5-12a_4^2+12a_3^3,\\
X_1=\ &10a_6-20a_2a_5+40a_2^2a_4-70a_2^3a_3+20a_2^5+50a_2a_3^2-30a_3a_4,\\
X_2=\ &a_3^2-2a_2^2a_3+a_2^4, \qquad X_3=-3a_4+6a_2a_3-3a_2^3, \qquad X_4=\frac{5}{8}a_2^2-\frac{5}{8}a_3,\\
X_5=\ &4a_5-8a_2a_4+16a_2^2a_3-6a_2^4-6a_3^2, \qquad X_6=3a_4-6a_2a_3+3a_2^3,\\
X_7=\ &\frac{1}{2}a_3-\frac{1}{2}a_2^2.
\end{aligned}
$$

We put

$$\mathcal{F}=\Re\frac{1}{3}\Big\{e^{i\varphi}X_0+\frac{6}{5}e^{i(\varphi-\theta)}X_1\alpha+\frac{3}{2}e^{i(\varphi-2\theta)}X_2\alpha^2+2e^{i(\varphi-3\theta)}X_3\alpha^3$$
$$+3e^{i(\varphi-4\theta)}X_4\alpha^4+\frac{3}{2}X_5\gamma+2e^{-i\theta}X_6\alpha\gamma+3e^{-i2\theta}X_7\alpha^2\gamma\Big\}.$$

Then we can verify that max $\mathcal{F}$ is a desired extremal problem. We can not decide whether the algebraic functions of class S satisfying (8) are extremal for this problem in S or not. However by using the general coefficient theorem [4] we can prove that the algebraic functions of class S satisfying (8) are extremal for this problem in a certain subclass of S.

In the sequel we denote by

$$f^*(z)=z+\sum_{n=2}^{\infty}a_n^*z^n$$

the functions of class S satisfying (8) and denote by

$$g^*(z)=z+b_0^*+\sum_{n=1}^{\infty}\frac{b_n^*}{z^n}$$

the functions of class Σ satisfying

$$z^2w^2+(e^{-i\theta}\alpha-2b_0^*)z^2w+\{e^{-i\theta}+\bar{\alpha}z-(2b_1^*-b_0^{*2}+e^{-i\theta}b_0^*\alpha)z^2-e^{-i\theta}\alpha z^3-z^4\}=0. \tag{27}$$

Let $S(\alpha,\theta)$ denote the class of functions $f(z)\in S$ with expansion at the origin

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n$$

where $a_3-a_2^2=a_3^*-a_2^{*2}$ for a certain $f^*(z)$. Let $\Sigma(\alpha,\theta)$ denote the class of functions $g(z)\in\Sigma$ with expansion at the point at infinity

$$g(z)=z+b_0+\sum_{n=1}^{\infty}\frac{b_n}{z^n}$$

where $b_1=b_1^*$ for a certain $g^*(z)$.

THEOREM 5. *If γ satisfies the conditions* (I), (II), *then in* $\Sigma(\alpha,\theta)$

$$\begin{aligned}&\Re\Big\{-e^{i\varphi}(b_5+b_1b_3+b_2^2)-e^{i(\varphi-\theta)}(2b_4+2b_1b_2)\alpha+\frac{1}{4}e^{i(\varphi-2\theta)}b_1^2\alpha^2\\&\quad+e^{i(\varphi-3\theta)}b_2\alpha^3+\frac{5}{16}e^{i(\varphi-4\theta)}b_1\alpha^4-\Big(b_3+\frac{1}{2}b_1^2\Big)\gamma-e^{-i\theta}b_2\alpha\gamma\\&\quad-\frac{1}{4}e^{-i2\theta}b_1\alpha^2\gamma\Big\}\\&\leqq\Big(\frac{1}{4}|\alpha|^2+\frac{1}{2}\Big)\Big(e^{-i\theta}\gamma+\frac{5}{4}e^{-i\varphi}\alpha^2\Big)-\frac{1}{8}\Re\{2e^{-i\varphi}\alpha^2+e^{i\varphi}\bar{\alpha}^2\}\end{aligned}\tag{28}$$

where $e^{i(2\varphi-3\theta)}=-1$. *Equality occurs for the functions of class* Σ *satisfying* (27). *Further in* $S(\alpha,\theta)$

$$\mathcal{F}\leqq\left(\frac{1}{2}|\alpha|^{2}+1\right)\left(e^{-i\theta}\gamma+\frac{5}{4}e^{-i\varphi}\alpha^{2}\right)-\frac{1}{4}\Re\{2e^{-i\varphi}\alpha^{2}+e^{i\varphi}\bar{\alpha}^{2}\}\tag{29}$$

where $e^{i(2\varphi-3\theta)}=-1$. *Equality occurs for the functions of class* S *satisfying* (8).

Proof. Let

$$g(z)=z+b_0+\sum_{n=1}^{\infty}\frac{b_n}{z^n}$$

be a function of class $\Sigma(\alpha,\theta)$ and let

$$g^*(z)=z+b_0^*+\sum_{n=1}^{\infty}\frac{b_n^*}{z^n}$$

be a function of class Σ satisfying (27) such that $b_1=b_1^*$. We may assume that $b_0=b_0^*=0$. $w=g^*(z)$ satisfies the differential equation

$$\left(\frac{z}{w}\frac{dw}{dz}\right)^2\sum_{\nu=2}^{6}C_\nu w^\nu=\sum_{\nu=-6}^{6}D_\nu z^\nu,\qquad D_{-\nu}=\bar{D}_\nu\tag{30}$$

where

$$e^{i(2\varphi-3\theta)}=-1,$$

$$C_6=e^{i\varphi},\qquad C_5=2e^{i(\varphi-\theta)}\alpha,\qquad C_4=\gamma-2e^{i\varphi}b_1^*,$$

$$C_3=-2e^{i(\varphi-\theta)}b_1^*\alpha-e^{i(\varphi-3\theta)}\alpha^3+e^{-i\theta}\alpha\gamma,$$

$$C_2=-\frac{1}{2}e^{i(\varphi-2\theta)}b_1^*\alpha^2-\frac{5}{16}e^{i(\varphi-4\theta)}\alpha^4+\frac{1}{4}e^{-i2\theta}\alpha^2\gamma$$

and $D_0, D_1, \cdots, D_6$ are the same as in (26). The right hand side of (30) is non-negative on $|z|=1$. Hence the image of $|z|>1$ under $w=g^*(z)$ is an admissible domain with respect to the quadratic differential

$$-\left(\sum_{n=2}^{6}C_n w^{n-2}\right)dw^2.$$

Then by the general coeffieient theorem [4] we have

$$\Re\{-(C_6c_5+C_5c_4+C_4c_3+C_3c_2+C_6c_2^2)\}\leqq 0$$

where

$$g\circ g^{*-1}(w)=w+\sum_{n=2}^{\infty}\frac{c_n}{w^n}.$$

Rewriting with the coefficients of $g^*(z)$ and $g(z)$ we have

$$\mathfrak{R}\{-e^{i\varphi}(b_5-b_5^*+b_1^*b_3-b_1^*b_3^*+b_2^2-b_2^{*2})-e^{i(\varphi-\theta)}(2b_4-2b_4^*+2b_1^*b_2-2b_1^*b_2^*)\alpha$$
$$+e^{i(\varphi-3\theta)}(b_2-b_2^*)\alpha^3-(b_3-b_3^*)\gamma-e^{-i\theta}(b_2-b_2^*)\alpha\gamma\}\leqq 0.$$

Since $b_1=b_1^*$, we obtain the inequality (28) by using the relation (18).

Next let $f(z)$ be a function of class $S(\alpha,\theta)$. Then $f(z^{-1})^{-1}$ belongs to $\Sigma(\alpha,\theta)$. Hence we obtain the inequality (29) by rewriting (28) with the coefficients of $f(z)$.

COROLLARY 1. *Let $\lambda\geqq 2$ and let*

$$g(z)=z+b_0+\sum_{n=1}^{\infty}\frac{b_n}{z^n}$$

be a function of class Σ whose coefficient b_1 is real. Then

$$\mathfrak{R}\left\{\pm(b_5+b_1b_3+b_2^2)+\lambda\left(b_3+\frac{1}{2}b_1^2\right)\right\}\leqq\frac{1}{2}\lambda. \tag{31}$$

Equality occurs for the functions of class Σ satisfying

$$z^2w^2-2b_0^*z^2w-\{z^4+(2b_1^*-b_0^{*2})z^2+1\}=0. \tag{32}$$

Proof. Put $\alpha=0$ and $\theta=\pi$. Then $e^{i\varphi}=\pm 1$ and the conditions (I), (II) reduce to the condition that $\gamma\leqq -2$. Hence we have the inequality (31) in $\Sigma(0,\pi)$ by putting $\lambda=-\gamma$. On the other hand the function

$$g_\varepsilon(z)=z\left(1+\frac{2\varepsilon}{z^2}+\frac{1}{z^4}\right)^{1/2}$$
$$=z+\frac{\varepsilon}{z}+\cdots\qquad(-1\leqq\varepsilon\leqq 1)$$

satisfies (32). Hence $g(z)$ belongs to $\Sigma(0,\pi)$. Thus we obtain the desired result.

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