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PICARD CONSTANT OF A FINITELY SHEETED COVERING SURFACE

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§ 1. Introduction.

Let R be an open Riemann surface and $M(R)$ the set of non-constant meromorphic functions on R. Let f be a member of $M(R)$ and $P(f)$ the number of lacunary values of f. Let $P(R)$ be

$$\sup_{f\in M(R)} P(f).$$

This is called the Picard constant of R. It is known that $P(R)\geqq 2$ and $P(R)$ is conformally invariant. If R is an n-sheeted covering surface of $|z|<\infty$, then $2\leqq P(R)\leqq 2n$ [4].

In this paper we shall consider the following problem:

Problem. Determine the Picard constant of a finitely sheeted covering surface of $|z|<\infty$.

This problem is very difficult to solve, in general. We shall restrict ourvelves to an n-sheeted covering surface R which is called regularly branched, that is, a surface which has no branch point other than those of order $n-1$.

Ozawa [5] has proved the following result:

If R is a two-sheeted covering surface of $|z|<\infty$ and if $P(R)=4$, then R is essentially equivalent to the surface defined by an algebroid function y such that $y^2=(e^H-\alpha)(e^H-\beta)$, where H is an entire function and α, β are constants satisfying $\alpha\beta(\alpha-\beta)\neq 0$.

Niino and Hiromi [1] have proved the following result:

If R is a three-sheeted regularly branched covering surface and if $P(R)\geqq 5$, then $P(R)=6$ and R is essentially equivalent to the surface defined by $y^3=(e^H-\alpha)\times(e^H-\beta)^2$, where H is an entire function and α, β and non-zero constants satisfying $\alpha\neq\beta$.

In § 2 we shall consider a preliminary result on $P(f)$.

In § 3 we shall prove a generalization of the above results.

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In §4 we shall prove a theorem concerning the Picard constant of a surface defined by $y^n = g(z)$.

2. Let f be an n-valued algebroid function. Assume that $P(f) \geqq n+2$ and f is entire. Then the defining equation of f is of the form

$$F(f, z) \equiv f^n - S_1(z) f^{n-1} + S_2(z) f^{n-2} + \cdots + (-1)^n S_n(z) = 0, \tag{1}$$

where $\{S_j(z)\}$ are entire functions. Let $\{\alpha_j\}$ be finite lacunary values of f. Then

$$F(\alpha_j, z) = e^{H_j}, \quad 1 \leqq j \leqq l, \ H_j \not\equiv \text{constant}, \quad l+1 \leqq j \leqq k, \ H_j \equiv \text{constant}, \tag{2}$$

where α_j for $1 \leqq j \leqq l$ are exceptional values of the second kind and remaining α_j are those of the first kind. Here $k \geqq n+1$ and $l \leqq n$. (Remark: the inequality $l \leqq n$ is due to Rémoundos [6])

Pick up $n+1$ members $\{\beta_1, \beta_2, \cdots, \beta_{n+1}\}$ from $\{\alpha_j\}$, and let L_j be the function or constant H_j which corresponds to β_j.

Then, from (2),

$$\begin{aligned} &\beta_1^n - S_1 \beta_1^{n-1} + S_2 \beta_1^{n-2} + \cdots + (-1)^n S_n = e^{L_1}, \\ &\beta_2^n - S_1 \beta_2^{n-1} + S_2 \beta_2^{n-2} + \cdots + (-1)^n S_n = e^{L_2}, \\ &\cdots\cdots\cdots\cdots\cdots, \\ &\beta_{n+1}^n - S_1 \beta_{n+1}^{n-1} + S_2 \beta_{n+1}^{n-2} + \cdots + (-1)^n S_n = e^{L_{n+1}}. \end{aligned} \tag{3}$$

Therefore,

$$\begin{aligned} &(\beta_1^n - e^{L_1}) - S_1 \beta_1^{n-1} + S_2 \beta_1^{n-2} + \cdots + (-1)^n S_n = 0, \\ &(\beta_2^n - e^{L_2}) - S_1 \beta_2^{n-1} + S_2 \beta_2^{n-2} + \cdots + (-1)^n S_n = 0, \\ &\cdots\cdots\cdots\cdots\cdots, \\ &(\beta_{n+1}^n - e^{L_{n+1}}) - S_1 \beta_{n+1}^{n-1} + S_2 \beta_{n+1}^{n-2} + \cdots + (-1)^n S_n = 0. \end{aligned} \tag{4}$$

This linear system has a non-trivial solution $(1, -S_1, S_2, \cdots, (-1)^n S_n)$. Hence

$$\operatorname{Det} \begin{pmatrix} \beta_1^n - e^{L_1}, & \beta_1^{n-1}, \beta_1^{n-2}, \cdots, 1 \\ \beta_2^n - e^{L_2}, & \beta_2^{n-1}, \beta_2^{n-2}, \cdots, 1 \\ \cdots\cdots\cdots\cdots\cdots & \\ \beta_{n+1}^n - e^{L_{n+1}}, & \beta_{n+1}^{n-1}, \beta_{n+1}^{n-2}, \cdots, 1 \end{pmatrix} \equiv 0. \tag{5}$$

In this equation (5), the coefficient of e^{L_j} is the determinant of Vandermonde, and so it is not zero.

Without loss of generality, we may assume that the first m members $\beta_1, \beta_2, \cdots, \beta_m$ are lacunary values of the second kind and remaining β_j are those of the first kind. Then, we have

(6) $$a_0=a_1e^{L_1}+a_2e^{L_2}+\cdots+a_me^{L_m},\qquad a_1a_2\cdots a_m\neq 0.$$

Hence, by the impossibility of Borel's identity (cf. [3]), we can divide the set $\{L_j\}$ into some classes A_ν, any one of which contains more than two members, such that for any L_j, $L_k\in A_\nu$, $L_j-L_k\equiv$constant, and for any $L_j\in A_\nu$, $L_k\in A_\mu$ ($\nu\neq\mu$), $L_j-L_k\not\equiv$constant.

Now, divide the set $\{H_j\}$ into classes which have the same property of the above partition of $\{L_j\}$.

By the assumption $P(f)\geqq n+2$, we have $K=k-(n+1)\geqq 0$. If some class A_ν contains fewer than $K+2$ members, then we can obtain the equation (3) which contains only one member of this class A_ν. Then the above argument shows that another member belongs to A_ν. This is a contradiction.

Hence, any one of these classes contains at least $K+2$ members.

This fact implies that, if $2(K+2)>l$, the difference of any two of $\{H_j\}_{j=1,\dots,l}$ is constant.

Therefore, if $2(K+2)>n\geqq l$ (i.e. $k>(3/2)n-1$), the difference of any two of $\{H_j\}$, which correspond to the lacunary values of the second kind, is constant.

Let f be an n-valued entire algebroid function satisfying $P(f)>(3/2)n$. From the above fact, the equation (3) may be written in the following form:

(7) $$\begin{aligned}&\beta_1^n-S_1\beta_1^{n-1}+S_2\beta_1^{n-2}+\cdots+(-1)^nS_n=\gamma_1e^H,\\&\cdots\cdots\cdots\cdots\cdots,\\&\beta_m^n-S_1\beta_m^{n-1}+S_2\beta_m^{n-2}+\cdots+(-1)^nS_n=\gamma_me^H,\\&\beta_{m+1}^n-S_1\beta_{m+1}^{n-1}+S_2\beta_{m+1}^{n-2}+\cdots+(-1)^nS_n=\gamma_{m+1},\\&\cdots\cdots\cdots\cdots\cdots,\\&\beta_{n+1}^n-S_1\beta_{n+1}^{n-1}+S_2\beta_{n+1}^{n-2}+\cdots+(-1)^nS_n=\gamma_{n+1},\end{aligned}$$

where H is a non-constant entire function and $\gamma_1,\gamma_2,\cdots,\gamma_{n+1}$ are non-zero constants.

Then, we have

(8) $$(-1)^jS_j=a_je^H+b_j,\qquad a_j,\ b_j\ \text{being constants},\qquad j=1,2,\cdots,n.$$

Substituting (8) into (1),

(9) $$F(f,z)\equiv G_1(f)+G_2(f)e^H=0,$$

where

$$G_1(f)=f^n+b_1f^{n-1}+b_2f^{n-2}+\cdots+b_n,$$

$$G_2(f)=a_1f^{n-1}+a_2f^{n-2}+\cdots+a_n.$$

The algebraic equations $G_1(z)=0$, $G_2(z)=0$ have no common root, because of the irreducibility of $F(f,z)$. And, the roots of $G_1(z)=0$ are lacunary values of the second kind of f, and the roots of $G_2(z)=0$ are lacunary values of the first kind of f. Moreover, f has no other finite lacunary value. In fact, a function $b+ae^H$ $(ab\neq 0)$ has at least one zero (Picard's small theorem).

Summing up these facts, we have the following theorem:

THEOREM 1. *Let f be an n-valued entire algebroid function satisfying $P(f)>(3/2)n$. Then there exist an entire function H and constants $a_1, a_2, \cdots, a_n$; $b_1, b_2, \cdots, b_n$, such that the defining equation of f is $F(f, z)\equiv G_1(f)+G_2(f)e^H=0$, where $G_1(f)=f^n+b_1f^{n-1}+b_2f^{n-2}+\cdots+b_n$ and $G_2(f)=a_1f^{n-1}+a_2f^{n-2}+\cdots+a_n$. Furthermore, the roots of the algebraic equation $G_2(z)=0$ are lacunary values of the first kind, and the roots of $G_1(z)=0$ are those of the second kind, and f has no other lacunary value. Moreover, these two algebraic equations have no common root.*

§ **3.** We shall prove the following theorem:

THEOREM 2. *Let R be an n-sheeted regularly branched covering surface of $|z|<\infty$, and if $P(R)>(3/2)n$, then $P(R)=2n$ and R can be represented by an algebroid function y such that $y^n=(e^H-\alpha)(e^H-\beta)^{n-1}$, where H is a non-constant entire function and α, β are constants satisfying $\alpha\beta(\alpha-\beta)\neq 0$.*

Proof. By the assumption, there exists an algebroid function f on R such that $P(f)>(3/2)n$. We may assume that f is entire. Then, f may be regarded as a function defined by the equation of type (9). By the way, (9) is irreducible, and therefore the existence domain of f is equivalent to R.

We shall define an algebraic function f_0, which is associated to f, by the equation:

(10) $$F(f_0, z)\equiv G_1(f_0)+zG_2(f_0)=0.$$

In this case, we can see easily

(11) $$f=f_0\circ e^H.$$

A simple application of Nevanlinna's ramification relation shows that

(12) for any $a\in\{0<|z|<\infty\}$, the equation $a=e^{H(z)}$ has at least one simple root z_0.

REMARK. More precisely, Hiromi and Ozawa [2] have proved that $N_1(r, a-e^H)\sim m(r, e^H)$ as $r\to\infty$, where $N_1(r, a-e^H)$ is the counting function of simple zeros of the function $a-e^H$.

From the assumption of regularly branched property of R, f has no algebraic singularity other than those of order $n-1$. Considering this fact together with (11) and (12), we can conclude that f_0 has no singularity other than algebraic singularities of order $n-1$ over $0<|z|<\infty$.

By the way, (10) may be written in the following form:

(13) $$z=-\frac{G_1(f_0)}{G_2(f_0)}.$$

Therefore, f_0 is an algebraic function of genus zero. From these properties of f_0 and Hurwitz's formula for a covering surface, essentially, f_0 must be an algebraic function y such that $y^n=(z-\alpha)(z-\beta)^{n-1}$, where $\alpha\beta(\alpha-\beta)\neq 0$. Hence, f is essentially equal to y such that $y^n=(e^H-\alpha)(e^H-\beta)^{n-1}$.

Thus we have proved that, if R is regularly branched and if $P(R)>(3/2)n$, R is equivalent to the surface defined by an algebroid function y such that $y^n=(e^H-\alpha)(e^H-\beta)^{n-1}$, where H is an entire function and α, β are constants satisfying $\alpha\beta(\alpha-\beta)\neq 0$.

On the other hand, on the surface defined by $y^n=(e^H-\alpha)(e^H-\beta)^{n-1}$, there exists an algebroid function $\sqrt[n]{(e^H-\alpha)(e^H-\beta)^{n-1}}/(e^H-\beta)$, which omits $2n$ values (i.e. the n-th roots of 1 and those of $\alpha/\beta\neq 1$). Then $P(R)=2n$. Q. E. D.

§4. By an analogous argument, we shall prove the following theorem:

THEOREM 3. *Let R be an n-sheeted covering surface of $|z|<\infty$ defined by an algebroid function y such that $y^n=g(z)$, where $g(z)$ is a meromorphic function. If $P(R)=2n$, and if n is odd, then R can be represented by an algebroid function f such that $f^n=(e^H-\alpha)(e^H-\beta)^{n-1}$, where H is a non-constant entire function and α, β are constants satisfying $\alpha\beta(\alpha-\beta)\neq 0$.*

Proof. There exists a function f on R such that $P(f)=2n$. We may assume that f is defined by the equation of type (9). Let f_0 be an algebraic function defined by (11) from this function f. The function f represents R.

Investigating branch points of the surface $y^n=g(z)$, we can see that the total order of algebraic singularlities of f, which exist over one point, is equal to $P(n/P-1)$, where P is a divisor of n.

Therefore, f_0 has also the same property (by (11) and (12)) and f_0 has no singularlity over 0 and ∞ (by theorem 1).

Hence

$$P\left(\frac{n}{P}-1\right)=n-P\geqq\frac{n}{2} \tag{14}$$

and by Hurwitz's formula

$$\sum(\text{order of ramification of ramified points})=2n-2. \tag{15}$$

Therefore, f_0 is ramified over at most three points. But, if there are three such points, n must be even. In fact, in such a case, there must exist three divisors p, q and r of n such that

$$p+q+r=n+2 \qquad \text{(by (14) and (15))}. \tag{16}$$

If n is odd, then $p, q, r\leqq n/3$. But, under this condition, (16) cannot be satisfied. Thus, f_0 has two algebraic singularlities of order $n-1$. This fact completes the proof (cf. the proof of theorem 2). Q. E. D.

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