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SUBMANIFOLDS OF CODIMENSION 2 IN AN EVEN-DIMENSIONAL EUCLIDEAN SPACE

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§ 0. Introduction.

A structure induced on a submanifold of codimension 2 of an almost Hermitian manifold or on a hypersurface of an almost contact metric manifold, called an (f, g, u, v, λ)-structure, has been studied in [1, 5, 6, 7]. Okumura and one of the present authors [7] proved

THEOREM 0.1. *Let a complete differentiable submanifold M of codimension 2 of an even-dimensional Euclidean space E be such that the connection induced in the normal bundle of M is trivial. If the (f, g, u, v, λ)-structure induced on M is normal, then M is a sphere, a plane or a product of a sphere and a plane, provided that $\lambda(1-\lambda^2)$ is almost everywhere non-zero in M.*

For an orientable hypersurface M of an odd-dimensional sphere S^{2n+1}, we can choose the first unit normal C in the direction opposite to that of the radius vector of S^{2n+1} and hence the second unit normal D is automatically fixed. We denote by H the second fundamental tensor with respect to C and by K that with respect to D. Then H is the identity and the connection induced in the normal bundle of M is trivial. Moreover, the normality of the (f, g, u, v, λ)-structure induced on M naturally is equivalent to the condition $Kf-fK=0$. Thus we can deduce, from Theorem 0.1, the following

THEOREM 0.2. *If M is a complete orientable hypersurface of S^{2n+1} satisfying $fK-Kf=0$ and $\lambda\neq$constant, then M is a sphere of radius $1/\sqrt{1+\alpha^2}$, where α is some constant determined by the embedding, provided that $\lambda(1-\lambda^2)$ is almost everywhere non-zero in M.*

In [2], Blair, Ludden and one of the present authors proved:

THEOREM 0.3. *If M is a complete orientable hypersurface of S^{2n+1} of constant scalar curvature satisfying $Kf+fK=0$ and $\lambda\neq$constant, then M is a natural sphere S^{2n} or $S^n\times S^n$, provided that $n>2$ and $\lambda(1-\lambda^2)$ is almost everywhere non-zero in M* (See also [3]).

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The main purpose of the present paper is to study complete orientable submanifolds M of codimension 2 in an even-dimensional Euclidean space F which satisfies the conditions $Hf-fH=0$ and $Kf+fK=0$, the normal bundle of M being locally trivial. Our main result will be stated in Theorem 3.1.

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§1. Submanifolds of codimension 2 of an even-dimensional Euclidean space.

Let E be a $(2n+2)$-dimensional Euclidean space and X the position vector starting from the origin of E and ending at a point P of E. The E being even-dimensional, it can be regarded as a flat Kählerian manifold with the numerical structure tensor F: $F^2=-I$, where I denotes the unit tensor and $FY\cdot FZ=Y\cdot Z$ for arbitrary vector fields Y and Z, where the dot denotes the inner product of vectors of E.

Let M be a $2n$-dimensional orientable manifold covered by a system of coordinate neighborhoods $\{U; x^h\}$, where here and in the sequel the indices $h, i, j, \cdots$ run over the range $\{1, 2, \cdots, 2n\}$ and the summation convention will be used with respect to these indices.

We assume that M is immersed in E by X: $M\to E$ and put $X_i=\partial_i X$, $\partial_i=\partial/\partial x^i$. Then X_i are $2n$ linearly independent local vector fields tangent to $X(M)$ and $g_{ji}=X_j\cdot X_i$ are local components of the tensor representing the Riemannian metric induced on M from that of E.

We assume that we can take two globally defined mutually orthogonal unit normals C and D to $X(M)$ in such a way that $X_1, X_2, \cdots, X_{2n}, C, D$ give the positive orientation of E. In the sequel we identify $X(M)$ with M itself.

The transforms FX_i of X_i by F can be expressed as linear combinations of X_h, C and D, that is, we have equations of the form

$$FX_i=f_i{}^h X_h+u_i C+v_i D, \tag{1.1}$$

where $f_i{}^h$ are components of a tensor field of type (1, 1) and u_i, v_i are those of 1-forms of M, all globally defined on M. The transforms FC and FD of C and D by F can be expressed as

$$FC=-u^h X_h+\lambda D, \tag{1.2}$$

$$FD=-v^h X_h-\lambda C \tag{1.3}$$

respectively, where $u^h=u_i g^{ih}$, $v^h=v_i g^{ih}$ and λ is a function on M, because

$$X_i\cdot FC=FX_i\cdot F^2C=-FX_i\cdot C=-u_i,$$

$$X_i\cdot FD=FX_i\cdot F^2D=-FX_i\cdot D=-v_i,$$

$$FC\cdot D=F^2C\cdot FD=-C\cdot FD.$$

Applying F to (1. 1), (1. 2) and (1. 3) and using $F^2=-I$, (1. 1), (1. 2) and (1. 3), we find

$$
\begin{aligned}
&f_i{}^t f_t{}^h=-\delta_i^h+u_i u^h+v_i v^h,\\
&u_t f_i{}^t=+\lambda v_i, \qquad f_i{}^h u^i=-\lambda v^h,\\
&v_t f_i{}^t=-\lambda u_i, \qquad f_i{}^h v^i=+\lambda u^h,\\
&u_i u^i=v_i v^i=1-\lambda^2, \qquad u_i v^i=0
\end{aligned} \tag{1. 4}
$$

(cf. [7]). We also have, from (1. 1),

$$
g_{ts} f_j{}^t f_i{}^s=g_{ji}-u_j u_i-v_j v_i \tag{1. 5}
$$

by virtue of $FX_j\cdot FX_i=X_j\cdot X_i=g_{ji}$. We can easily see that $f_{ji}=f_j{}^t g_{ti}$ is skew-symmetric in lower indices j and i.

The structure defined on M by such a set of a tensor field f of type (1, 1), a Riemannian metric g, two 1-forms u and v and a function λ satisfying (1. 4) and (1. 5) is called an (f, g, u, v, λ)-structure (cf. [6]).

We denote by $\{{}_j{}^h{}_i\}$ the Christoffel symbols formed with g_{ji} and by ∇_i the operator of covariant differentiation with respect to $\{{}_j{}^h{}_i\}$. Then equations of Gauss are

$$
\nabla_j X_i=\partial_j X_i-\left\{ {h \atop j\ i} \right\} X_h=h_{ji}C+k_{ji}D, \tag{1. 6}
$$

where h_{ji} and k_{ji} are components of the second fundamental tensors with respect to C and D respectively, and equations of Weingarten are

$$
\begin{aligned}
&\nabla_j C=\partial_j C=-h_j{}^h X_h+l_j D,\\
&\nabla_j D=\partial_j D=-k_j{}^h X_h-l_j C,
\end{aligned} \tag{1. 7}
$$

where $h_j{}^h$ and $k_j{}^h$ are given respectively by $h_j{}^h=h_{ji}g^{ih}$ and $k_j{}^h=k_{ji}g^{ih}$, and l_j are components of the third fundamental tensor, i.e., components of the connection induced on the normal bundle.

Now, differentiating (1. 1) covariantly and taking account of $\nabla_j F=0$ and of equations of Gauss and Weingarten, we obtain

$$
\nabla_j f_i{}^h=-h_{ji}u^h+h_j{}^h u_i-k_{ji}v^h+k_j{}^h v_i, \tag{1. 8}
$$

$$
\nabla_j u_i=-h_{jt}f_i{}^t-\lambda k_{ji}+l_j v_i, \tag{1. 9}
$$

$$
\nabla_j v_i=-k_{jt}f_i{}^t+\lambda h_{ji}-l_j u_i. \tag{1. 10}
$$

Similarly we have, from (1. 2),

$$\nabla_j\lambda = -h_{ji}v^i + k_{ji}u^i. \tag{1.11}$$

Now we put

$$S_{ji}{}^h = N_{ji}{}^h + (\nabla_j u_i - \nabla_i u_j)u^h + (\nabla_j v_i - \nabla_i v_j)v^h, \tag{1.12}$$

where $N_{ji}{}^h$ is the Nijenhuis tensor formed with $f_i{}^h$, i.e.,

$$N_{ji}{}^h = f_j{}^t\nabla_t f_i{}^h - f_i{}^t\nabla_t f_j{}^h - (\nabla_j f_i{}^t - \nabla_i f_j{}^t)f_t{}^h.$$

If the tensor $S_{ji}{}^h$ vanishes, the (f, g, u, v, λ)-structure is said to be *normal*. Substituting (1. 8), (1. 9) and (1. 10) into (1. 12), we find

$$\begin{aligned} S_{ji}{}^h =& (f_j{}^t h_t{}^h - h_j{}^t f_t{}^h)u_i - (f_i{}^t h_t{}^h - h_i{}^t f_t{}^h)u_j \\ &+ (f_j{}^t k_t{}^h - k_j{}^t f_t{}^h)v_i - (f_i{}^t k_t{}^h - k_i{}^t f_t{}^h)v_j \\ &+ (l_j v_i - l_i v_j)u^h - (l_j u_i - l_i u_j)v^h. \end{aligned} \tag{1.13}$$

In the sequel, we need the structure equations of the submanifold M, that is, the following equations of Gauss

$$K_{kjih} = h_{kh}h_{ji} - h_{jh}h_{ki} + k_{kh}k_{ji} - k_{jh}k_{ki}, \tag{1.14}$$

where K_{kjih} are covariant components of the curvature tensor of M, and equations of Codazzi and Ricci

$$\nabla_k h_{ji} - \nabla_j h_{ki} - l_k k_{ji} + l_j k_{ki} = 0, \tag{1.15}$$

$$\nabla_k k_{ji} - \nabla_j k_{ki} + l_k h_{ji} - l_j h_{ki} = 0, \tag{1.16}$$

$$\nabla_j l_i - \nabla_i l_j + h_{jt}k_i{}^t - h_{it}k_j{}^t = 0. \tag{1.17}$$

§2. The case in which f and H commute and f and K anticommute.

We suppose that f and H commute, i.e.,

$$f_j{}^t h_t{}^h - h_j{}^t f_t{}^h = 0, \tag{2.1}$$

which is equivalent to

$$h_{jt}f_i{}^t + h_{it}f_j{}^t = 0, \tag{2.2}$$

that is, $h_{jt}f_i{}^t$ is skew-symmetric. We suppose also that f and K anticommute, i.e.,

$$f_j{}^t k_t{}^h + k_j{}^t f_t{}^h = 0, \tag{2.3}$$

which is equivalent to

$$k_{jt}f_i{}^t - k_{it}f_j{}^t = 0, \tag{2.4}$$

that is, $k_{jt}f_i{}^t$ is symmetric. We first prove

PROPOSITION 2.1. *Let $X(M)$ be a submanifold of codimension 2 of E such that* (2.3) *is satisfied and the function λ is almost everywhere non-zero in M. Then*

$$k_t{}^t=0, \tag{2.5}$$

that is, the mean curvature vector is in the direction of C if it does not vanish.

Proof. Transvecting (2.4) with u^jv^i, we find $\lambda(k_{jt}u^ju^t+k_{it}v^iv^t)=0$, from which

$$k_{ji}u^ju^i+k_{ji}v^jv^i=0. \tag{2.6}$$

Transvecting next (2.4) with f^{ji}, we find $2k_{jt}(-g^{tj}+u^tu^j+v^tv^j)=0$, which implies $k_t{}^t=0$ by virtue of (2.6). But the mean curvature vector of $X(M)$ is given by

$$\frac{1}{2n}g^{ji}\nabla_jX_i=\frac{1}{2n}(h_t{}^tC+k_t{}^tD)=\frac{1}{2n}h_t{}^tC \tag{2.7}$$

and consequently, we see that the mean curvature vector is in the direction of C if it does not vanish.

We next prove

PROPOSITION 2.2. *Let $X(M)$ be a submanifold of codimension 2 of E such that* (2.1) *is satisfied and the function $\lambda(1-\lambda^2)$ is almost everywhere non-zero in M. Then we have*

$$\begin{aligned}(1-\lambda^2)h_{ji}u^i&=(h_{ts}u^tu^s)u_j,\\(1-\lambda^2)h_{ji}v^i&=(h_{ts}v^tv^s)v_j,\end{aligned} \tag{2.8}$$

where $h_{ts}u^tu^s=h_{ts}v^tv^s$ and consequently, at every point at which $1-\lambda^2\neq0$,

$$h_{ji}u^i=pu_j,\qquad h_{ji}v^i=pv_j, \tag{2.9}$$

p being given by

$$p=\frac{h_{ts}u^tu^s}{1-\lambda^2}=\frac{h_{ts}v^tv^s}{1-\lambda^2}.$$

Proof. Transvecting (2.2) with u^ju^i, we find $\lambda(-h_{jt}u^jv^t-h_{it}u^iv^t)=0$, from which

$$h_{ji}u^jv^i=0. \tag{2.10}$$

Transvecting (2.2) with u^jv^i, we obtain $\lambda(h_{jt}u^ju^t-h_{jt}v^jv^t)=0$, from which

$$h_{ji}u^ju^i=h_{ji}v^jv^i. \tag{2.11}$$

Transvecting (2.2) with $f_h{}^j$, we find

$$h_{ts}f_i{}^t f_h{}^s + h_{it}(-\delta_h^t + u_h u^t + v_h v^t) = 0,$$

or equivalently

$$h_{ts}f_i{}^t f_h{}^s - h_{ih} + (h_{it}u^t)u_h + (h_{it}v^t)v_h = 0, \tag{2.12}$$

from which, taking the skew-symmetric part,

$$(h_{it}u^t)u_h - (h_{ht}u^t)u_i + (h_{it}v^t)v_h - (h_{ht}v^t)v_i = 0.$$

Transvecting this with u^h, we find

$$(1-\lambda^2)h_{it}u^t = (h_{ts}u^t u^s)u_i + (h_{ts}u^t v^s)v_i,$$

from which, using (2.10),

$$(1-\lambda^2)h_{it}u^t = (h_{ts}u^t u^s)u_i.$$

Similarly, we can get

$$(1-\lambda^2)h_{it}v^t = (h_{ts}v^t v^s)v_i.$$

Thus we have (2.9). Consequently, Proposition 2.2 is proved.

From (2.9), (2.11) and (2.12), we have

$$h_{ih} = h_{ts}f_i{}^t f_h{}^s + p(u_i u_h + v_i v_h). \tag{2.13}$$

We also have

PROPOSITION 2.3. *Let $X(M)$ be a submanifold of codimension 2 of E such that* (2.3) *is satisfied and the function $\lambda(1-\lambda^2)$ is almost everywhere non-zero in M. Then*

$$\begin{aligned}(1-\lambda^2)k_{ji}u^i &= (k_{ts}u^t u^s)u_j + (k_{ts}u^t v^s)v_j,\\ (1-\lambda^2)k_{ji}v^i &= (k_{ts}u^t v^s)u_j + (k_{ts}v^t v^s)v_j,\end{aligned} \tag{2.14}$$

where $k_{ts}u^t u^s + k_{ts}v^t v^s = 0$, and consequently, at every point at which $1-\lambda^2 \neq 0$, we have

$$k_{ji}u^i = \alpha u_j + \beta v_j, \qquad k_{ji}v^i = \beta u_j - \alpha v_j, \tag{2.15}$$

α and β being given respectively by

$$\alpha = \frac{k_{ts}u^t u^s}{1-\lambda^2} = -\frac{k_{ts}v^t v^s}{1-\lambda^2}, \qquad \beta = \frac{k_{ts}u^t v^s}{1-\lambda^2}.$$

Proof. Transvecting (2.4) with $u^j v^i$, we find $\lambda(k_{jt}u^j u^t + k_{it}v^i v^t) = 0$, from which $k_{ji}u^j u^i + k_{ji}v^j v^i = 0$. Transvecting (2.4) with $f_h{}^j$, we find

$$k_{ts}f_i{}^t f_h{}^s - k_{it}(-\delta_h^t + u_h u^t + v_h v^t) = 0,$$

or equivalently

$$k_{ts}f_i{}^t f_h{}^s + k_{ih} - (k_{it}u^t)u_h - (k_{it}v^t)v_h = 0, \tag{2.16}$$

from which, taking the skew-symmetric part,

$$(k_{it}u^t)u_h - (k_{ht}u^t)u_i + (k_{it}v^t)v_h - (k_{ht}v^t)v_i = 0.$$

Transvecting this with u^h, we find

$$(1-\lambda^2)k_{it}u^t = (k_{ts}u^t u^s)u_i + (k_{ts}u^t v^s)v_i.$$

Similarly we can get

$$(1-\lambda^2)k_{it}v^t = (k_{ts}u^t v^s)u_i + (k_{ts}v^t v^s)v_i.$$

Thus, taking account of $k_{ji}u^j u^i + k_{ji}v^j v^i = 0$, we get (2.15). Consequently, Proposition 2.3 is proved.

From (2.14) and (2.16), we have

$$k_{ih} = -k_{ts}f_i{}^t f_h{}^s + \alpha(u_i u_h - v_i v_h) + \beta(u_i v_h + u_h v_i). \tag{2.17}$$

We next prove

PROPOSITION 2.4. *Let $X(M)$ be a submanifold of codimension 2 of E such that the global unit normals C and D are parallel in the normal bundle. Assume that* (2.1) *and* (2.3) *are satisfied and the function $\lambda(1-\lambda^2)$ is almost everywhere non-zero in M. Then*

$$h_t{}^t = constant, \tag{2.18}$$

that is, the mean curvature of $X(M)$ is constant.

Proof. Since C and D are parallel in the normal bundle, the third fundamental tensor l_j vanishes identically. Then, differentiating the first equation of (2.9) covariantly, we find, by using (1.9) with $l_j=0$,

$$(\nabla_j h_i{}^t)u_t + h_i{}^t(-h_{js}f_t{}^s - \lambda k_{jt}) = (\nabla_j p)u_i + p(-h_{jt}f_i{}^t - \lambda k_{ji}), \tag{2.19}$$

from which, using equation (1.15) of Codazzi with $l_j=0$, we have

$$2h_j{}^t h_i{}^s f_{ts} = (\nabla_j p)u_i - (\nabla_i p)u_j - 2p h_{jt} f_i{}^t. \tag{2.20}$$

Transvecting this with u^i, we find $(1-\lambda^2)\nabla_j p = (u^i \nabla_i p)u_j$. In the same way, we can prove, from the second equation of (2.9), $(1-\lambda^2)\nabla_j p = (v^i \nabla_i p)v_j$. The last two equations imply that $\nabla_j p = 0$, from which we have

$$p = \text{const.} \tag{2.21}$$

Thus (2.20) becomes $h_j{}^t h_i{}^s f_{st} = p h_{jt} f_i{}^t$, or $h_j{}^t h_t{}^s f_{is} = p h_{jt} f_i{}^t$, from which, transvecting with $f_h{}^i$, we find

$$h_j{}^t h_t{}^s(-g_{sh}+u_s u_h+v_s v_h)=p h_{jt}(-\delta_h^t+u_h u^t+v_h v^t).$$

Therefore, using (2. 9), we have

$$h_{jt}h_h{}^t=p h_{jh}, \qquad \text{or} \qquad h_t{}^h h_i{}^t=p h_i{}^h. \tag{2. 22}$$

Denote by ρ an eigenvalue of $h_i{}^h$ and by w^h the corresponding eigenvector. Then we have $h_i{}^t w^i=\rho w^t$, from which, applying $h_t{}^h$ and using (2. 22), $p h_i{}^h w^i=\rho h_t{}^h w^t$, $p\rho=\rho^2$, that is,

$$\rho=0 \qquad \text{or} \qquad \rho=p. \tag{2. 23}$$

Thus the second fundamental tensor $h_i{}^h$ has only two constant eigenvalues. Let m be the multiplicity of the eigenvalue p, p being assumed to be non-zero, then m is constant, and we have

$$h_t{}^t=mp, \tag{2. 24}$$

which is a constant. But the mean curvature is given by

$$\frac{1}{2n}h_t{}^t=\frac{1}{2n}mp$$

and consequently is a constant too. Therefore Proposition 2. 4 is proved.

We can prove

PROPOSITION 2. 5. *Under the same assumptions as stated in Proposition* 2. 4, *we have*

$$\nabla_k h_{ji}=0. \tag{2. 25}$$

Proof. Differentiating (2. 22) covariantly, we find

$$(\nabla_k h_{jt})h_h{}^t+(\nabla_k h_h{}^t)h_{jt}=p\nabla_k h_{jh}. \tag{2. 26}$$

Thus, using $\nabla_k h_{jh}=\nabla_j h_{kh}$, which is a direct consequence of (1. 15) with $l_j=0$, we have from (2. 26)

$$(\nabla_k h_h{}^t)h_{jt}-(\nabla_j h_h{}^t)h_{kt}=0,$$

and, interchanging the indices h and k,

$$(\nabla_h h_k{}^t)h_{jt}-(\nabla_j h_k{}^t)h_{ht}=0. \tag{2. 27}$$

Adding (2. 26) and (2. 27), we find

$$2(\nabla_k h_h{}^t)h_{jt}=p\nabla_k h_{jh}. \tag{2. 28}$$

Transvecting (2. 28) with $h_l{}^j$ and using (2. 22), we have

$$h_l{}^t(\nabla_k h_{ht})=0.$$

Thus, (2. 28) implies that

(2. 29) $$p\nabla_k h_{jh}=0.$$

Since p is constant, (2. 29) implies that $\nabla_k h_{ji}=0$, if $p\neq 0$. On the other hand, from (2. 22), we have $h_{ji}h^{ji}=ph_t{}^t$. Thus, if $p=0$, we have $h_{ji}=0$ and hence $\nabla_k h_{ji}=0$. Therefore, in any case, we have $\nabla_k h_{ji}=0$. This completes the proof of the proposition.

We are now going to prove formulas (2. 39) and (2. 40) which will be useful in the sequel. Substituting (2. 9) and (2. 15) into (1. 11), we have

(2. 30) $$\nabla_i\lambda=\alpha u_i+(\beta-p)v_i.$$

Differentiating (2. 30) covariantly, we find

$$\nabla_j\nabla_i\lambda=(\nabla_j\alpha)u_i+\alpha\nabla_j u_i+(\nabla_j\beta)v_i+(\beta-p)\nabla_j v_i,$$

and, hence, using (1. 9) and (1. 10),

(2. 31) $$\nabla_j\nabla_i\lambda=(\nabla_j\alpha)u_i+\alpha(-h_{jt}f_i{}^t-\lambda k_{ji})+(\nabla_j\beta)v_i+(\beta-p)(-k_{jt}f_i{}^t+\lambda h_{ji}),$$

from which, taking the skew-symmetric part,

(2. 32) $$0=(\nabla_j\alpha)u_i-(\nabla_i\alpha)u_j-2\alpha h_{jt}f_i{}^t+(\nabla_j\beta)v_i-(\nabla_i\beta)v_j.$$

Transvecting (2. 32) with u^i, we find

(2. 33) $$(1-\lambda^2)(\nabla_j\alpha)=(u^t\nabla_t\alpha)u_j+(u^t\nabla_t\beta-2\alpha\lambda p)v_j.$$

Transvecting (2. 32) with v^i, we find

(2. 34) $$(1-\lambda^2)(\nabla_j\beta)=(v^t\nabla_t\alpha+2\alpha\lambda p)u_j+(v^t\nabla_t\beta)v_j.$$

Multiplying (2. 32) by $1-\lambda^2$ and subsituting (2. 33) and (2. 34) in the equation obtained, we find

(2. 35) $$2\alpha(1-\lambda^2)h_{jt}f_i{}^t=-(u^t\nabla_t\beta-v^t\nabla_t\alpha-4\alpha\lambda p)(u_jv_i-u_iv_j),$$

from which, transvecting with u^i, $-2\alpha\lambda p=u^t\nabla_t\beta-v^t\nabla_t\alpha-4\alpha\lambda p$, or equivalently

(2. 36) $$u^t\nabla_t\beta-v^t\nabla_t\alpha=2\alpha\lambda p.$$

Thus (2. 35) becomes

(2. 37) $$\alpha(1-\lambda^2)h_{jt}f_i{}^t=\alpha\lambda p(u_jv_i-u_iv_j),$$

from which, transvecting with $f_h{}^i$, $\alpha(1-\lambda^2)h_{jt}(-\delta_h^t+u_hu^t+v_hv^t)=-\alpha\lambda^2 p(u_ju_h+v_jv_h)$, or equivalently

$$\alpha(1-\lambda^2)h_{jh}=\alpha p(u_j u_h+v_j v_h). \tag{2.38}$$

Transvecting (2.38) with g^{jh}, we find $\alpha(1-\lambda^2)h_t{}^t=2\alpha p(1-\lambda^2)$, from which, using (2.24),

$$\alpha(m-2)p=0. \tag{2.39}$$

Thus, since m and p are constant, we have

$$(m-2)p=0 \quad \text{or} \quad \alpha=0. \tag{2.40}$$

We shall consider three cases, that is, Case I where $m\neq 2, p\neq 0$, Case II where $m=2$ and Case III where $p=0$. These cases with some additional assumptions will be discussed in §3.

In the next step, we prove

PROPOSITION 2.6. *Under the same assumptions as stated in Proposition* 2.4, *we have*

$$(1-\lambda^2)(k_j{}^t k_{ti}+\beta h_{ji})=[\alpha^2+\beta(\beta+p)](u_j u_i+v_j v_i). \tag{2.41}$$

Proof. Differentiating the second equation of (2.15) covariantly, we find

$$\begin{aligned}&(\nabla_j k_i{}^t)v_t+k_i{}^t(-k_{js}f_t{}^s+\lambda h_{jt})\\&=(\nabla_j\beta)u_i+\beta(-h_{jt}f_i{}^t-\lambda k_{ji})-(\nabla_j\alpha)v_i-\alpha(-k_{jt}f_i{}^t+\lambda h_{ji}).\end{aligned}$$

Taking the skew-symmetric part and using $\nabla_k k_{ji}=\nabla_j k_{ki}$, which is a direct consequence of (1.16) with $l_j=0$, we find

$$2k_j{}^t k_i{}^s f_{ts}=(\nabla_j\beta)u_i-(\nabla_i\beta)u_j-(\nabla_j\alpha)v_i+(\nabla_i\alpha)v_j-2\beta h_{jt}f_i{}^t.$$

Multiplying this equation by $1-\lambda^2$ and substituting (2.33) and (2.34) into the equation obtained, we find

$$2(1-\lambda^2)k_j{}^t k_i{}^s f_{ts}=-(u^t\nabla_t\alpha+v^t\nabla_t\beta)(u_j v_i-u_i v_j)-2(1-\lambda^2)\beta h_{jt}f_i{}^t,$$

or equivalently

$$2(1-\lambda^2)k_j{}^t k_t{}^s f_{is}=-(u^t\nabla_t\alpha+v^t\nabla_t\beta)(u_j v_i-u_i v_j)-2(1-\lambda^2)\beta h_{jt}f_i{}^t, \tag{2.42}$$

from which, transvecting with $u^j v^i$ and using (2.9) and (2.15),

$$u^t\nabla_t\alpha+v^t\nabla_t\beta=-2\lambda[\alpha^2+\beta(\beta+p)]. \tag{2.43}$$

Thus (2.42) becomes

$$(1-\lambda^2)k_j{}^t k_t{}^s f_{is}=\lambda[\alpha^2+\beta(\beta+p)](u_j v_i-u_i v_j)-(1-\lambda^2)\beta h_{jt}f_i{}^t.$$

Transvecting this with $f_h{}^i$, we find

$$(1-\lambda^2)k_j{}^t k_t{}^s(-g_{sh}+u_s u_h+v_s v_h)=-\lambda^2[\alpha^2+\beta(\beta+p)](u_j u_h+v_j v_h)-(1-\lambda^2)\beta h_{jt}(-\delta_h^t+u_h u^t+v_h v^t),$$

from which, using (2. 9) and (2. 15),

$$(1-\lambda^2)(k_j{}^t k_{th}+\beta h_{jh})=-[\alpha^2+\beta(\beta+p)](u_j u_h+v_j v_h),$$

which proves proposition 2. 6.

We have, from equation (1. 14) of Gauss,

$$K_{ji}=h_t{}^t h_{ji}-h_{jt}h_i{}^t-k_{jt}k_i{}^t, \tag{2. 44}$$

from which, using (2. 22),

$$K_{ji}=(h_t{}^t-p)h_{ji}-k_{jt}k_i{}^t. \tag{2. 45}$$

From (2. 41) and (2. 45), we find

$$(1-\lambda^2)K_{ji}=(1-\lambda^2)(h_t{}^t-p+\beta)h_{ji}-[\alpha^2+\beta(\beta+p)](u_j u_i+v_j v_i), \tag{2. 46}$$

from which, transvecting with g^{ji},

$$g^{ji}K_{ji}=(h_t{}^t-p+\beta)h_s{}^s-2[\alpha^2+\beta(\beta+p)], \tag{2. 47}$$

which gives the scalar curvature of M.

§ 3. Complete submanifolds with constant scalar curvature.

We assume, here and in the sequel, that the submanifold M is complete and the scalar curvature $g^{ji}K_{ji}$ of M is constant. We have mentioned in § 2 three Cases I, II and III. First we consider Case I where $m\neq 2$ and $p\neq 0$. We find $\alpha=0$ from (2. 39) with $(m-2)p\neq 0$. The scalar curvature $g^{ji}K_{ji}$ being constant, we see from (2. 47) that β is constant. Thus (2. 43) implies $\lambda\beta(\beta+p)=0$, that is,

$$\beta=0 \qquad \text{or} \qquad \beta=-p. \tag{3. 1}$$

Then (2. 41) becomes

$$k_{jt}k_i{}^t=-\beta h_{ji}, \tag{3. 2}$$

because $\alpha=0$ and $\beta(\beta+p)=0$. Differentiating (3. 2) covariantly and taking account of (2. 25), we find

$$(\nabla_k k_{jt})k_i{}^t+k_{jt}(\nabla_k k_i{}^t)=0, \tag{3. 3}$$

from which, using equation (1. 16) of Codazzi with $l_j=0$, $k_{jt}(\nabla_k k_i{}^t)-k_{kt}(\nabla_j k_i{}^t)=0$, or equivalently

$$k_{jt}(\nabla_i k_k{}^t)-k_{it}(\nabla_j k_k{}^t)=0. \tag{3. 4}$$

Adding (3. 3) and (3. 4), we have $k_{jt}(\nabla_k k_i{}^t)=0$. Transvecting this with $k_h{}^j$ and using (3. 2), we obtain

$$\beta h_h{}^t(\nabla_k k_{it})=0. \tag{3. 5}$$

Since β is constant, (3. 5) implies $h_h{}^t(\nabla_k k_{it})=0$ if $\beta\neq 0$. On the other hand, taking account of (3. 2), we have trivially $h_h{}^t(\nabla_k k_{it})=0$ if $\beta=0$. Therefore, in any case, we have

$$h_h{}^t(\nabla_k k_{it})=0. \tag{3. 6}$$

We see, from (1. 17) with $l_j=0$, that $h_i{}^h$ and $k_i{}^h$ are commutative, that is, $h_t{}^i k_j{}^t - k_t{}^i h_j{}^t=0$. Thus, using (2. 5), (2. 22) and (3. 2), we see that the second fundamental tensors $h_i{}^h$ and $k_i{}^h$ have at each point of M respectively the forms

$$(h_j{}^h)=\left(\begin{array}{cccc|c} p & & & & \\ & p & & & 0 \\ & & \ddots & & \\ & & & p & \\ \hline & 0 & & & 0 \end{array}\right), \quad (k_i{}^h)=\left(\begin{array}{cccccc|c} q & & & & & & \\ & q & & & & & \\ & & \ddots & & & & \\ & & & q & & & 0 \\ & & & & -q & & \\ & & & & & \ddots & \\ & & & & & & -q \\ \hline & & 0 & & & & 0 \end{array}\right) \tag{3. 7}$$

with respect to a suitable orthonormal frame, because of (3. 1), where $q=p$ if $\beta=-p$ and $q=0$ if $\beta=0$. Thus, we can choose in any coordinate neighborhood of M, since $\dim M=2n$, a field of frames $\{e_{(1)}, e_{(2)}, \cdots, e_{(2n)}\}$ such that

$$\begin{aligned} &h_i{}^h e^i{}_{(\gamma)}=pe^h{}_{(\gamma)}, && (\gamma=1, 2, \cdots, m), \\ &k_i{}^h e^i{}_{(\mu)}=qe^h{}_{(\mu)}, && (\mu=1, 2, \cdots, m/2), \\ &k_i{}^h e^i{}_{(\nu)}=-qe^h{}_{(\nu)}, && (\nu=m/2+1, \cdots, m). \end{aligned} \tag{3. 8}$$

If we denote by $\mathcal{D}$ the distribution spanned by $e_{(1)}, e_{(2)}, \cdots$, and $e_{(m)}$, then $\mathcal{D}$ is a global dirtribution because $p\neq 0$. We denote by $\bar{\mathcal{D}}$ the orthogonal complement of $\mathcal{D}$. The distribution $\bar{\mathcal{D}}$ is locally spanned by $e_{(m+1)}, \cdots, e_{(2n)}$. Since $\nabla_k h_j{}^h=0$ and $p\neq 0$, the distribution $\mathcal{D}$ is integrable and the integral manifolds of $\mathcal{D}$ are totally geodesic in M. Thus $\bar{\mathcal{D}}$ is also integrable and the integral manifolds of $\bar{\mathcal{D}}$ are totally geodesic in M.

If we denote by $\bar{V}$ an arbitrary maximal integral submanifold of $\bar{\mathcal{D}}$, then we see, taking account of (3. 7), that $\bar{V}$ is totally geodesic in E^{2n+2}. On the other hand, $\bar{V}$ is complete because $\bar{V}$ is totally geodesic in M which is complete. Thus, $\bar{V}$ is a plane E^{2n-m} in E^{2n+2}.

Let V be the maximal integral submanifold of $\mathcal{D}$ passing through a point P of M. Then we see that V is complete and lies on an $(m+2)$-dimensional plane E^{m+2} which is orthogonal to $\bar{V}$ passing through P, where $\bar{V}$ is a $(2n-m)$-dimensional plane E^{2n-m}. Hence, taking account of (3.7), we see that the submanifold V, which is immersed in E^{m+2}, has the second fundamental tensors $\bar{h}_b{}^a$ and $\bar{k}_b{}^a$ of the forms

$$(3.9)\qquad (\bar{h}_b{}^a)=\begin{pmatrix} p & & & & 0\\ & p & & & \\ & & p & & \\ & & & \ddots & \\ 0 & & & & p\end{pmatrix},\qquad (\bar{k}_b{}^a)=\begin{pmatrix} q & & & & & & 0\\ & q & & & & & \\ & & \ddots & & & & \\ & & & q & & & \\ & & & & -q & & \\ & & & & & \ddots & \\ 0 & & & & & & -q\end{pmatrix}$$

with respect to the local frame $\{e_{(1)}, e_{(2)}, \cdots, e_{(m)}\}$ and the unit normals C and D, where C and D are contained in E^{m+2} along V, the indices $a, b, c, \cdots$ running over the range $\{1, 2, \cdots, m\}$.

According to (3.1), we first consider the case where $\beta=-p$, which implies $q=p$. If we take account of (3.6), we see that the distributions Δ^+ spanned by $\{e_{(1)}, e_{(2)}, \cdots, e_{(m/2)}\}$ and Δ^- spanned by $\{e_{(m/2+1)}, \cdots, e_{(m)}\}$ are both parallel along V. Consequently, since V is complete, we can easily verify the following fact: the submanifold V is congruent in E^{m+2} to the submanifold $S^{m/2}(1/\sqrt{2}|p|)\times S^{m/2}(1/\sqrt{2}|p|)$, which is natually imbedded in E^{m+2} (cf. Yano and Ishihara [4]). Next, we consider the case where $\beta=0$, which implies $q=0$, then we see, using (3.9) with $q=0$, that V is totally umbilical in E^{m+2} and complete. Thus, in this case, V is congruent to $S^m(1/|p|)$ in E^{m+2}, which is natually imbedded in E^{m+2}.

Summing up the arguments developed above, we can conclude that in Case I, the submanifold M is congruent in E^{2n+2} to $S^m(r)\times E^{2n-m}$ or $S^{m/2}(r)\times S^{m/2}(r)\times E^{2n-m}$, r being a positive number, which is natually imbedded in E^{2n+2}, where $S^k(r)$ denotes a k-dimensional sphere of radius r.

In the next step, we consider Case II where $m=2$. In this case we see that $p\neq 0$. From (2.24), we find that

$$(3.10)\qquad h_t{}^t=2p,$$

which implies that

$$(3.11)\qquad (1-\lambda^2)h_{ji}=p(u_ju_i+v_jv_i),$$

by virtue of (2.9). Taking account of (3.10), we have, from (2.46),

$$(3.12)\qquad g^{ji}K_{ji}=2(p^2-\alpha^2-\beta^2).$$

Substituting (3.11) into (2.41) and using $p\neq 0$, we find

$$(3.13)\qquad k_j{}^tk_{ti}=\frac{1}{p}(\alpha^2+\beta^2)h_{ji}.$$

In Case I, we found that (3.2) with $\beta=$const. implies (3.6). The scalar curvature $g^{ji}K_{ji}$ being constant, we see from (3.12) that $\alpha^2+\beta^2$ is constant. Thus, in the same way as developed in Case I, we can prove that (3. 13) implies $h_h{}^t(\nabla_k k_{it})=0$.

Since $h_i{}^h$ and $k_i{}^h$ are commutative, using (2. 5), (2. 22) and (3. 13), we see that the second fundamental tensors $h_i{}^h$ and $k_i{}^h$ have at each point of M the form

$$(3.14)\qquad (h_i{}^h)=\left(\begin{array}{cc:c} p & & \\ & p & 0 \\ \hdashline & 0 & 0 \end{array}\right),\qquad (k_i{}^h)=\left(\begin{array}{cc:c} q & & \\ & -q & 0 \\ \hdashline & 0 & 0 \end{array}\right)$$

with respect to a suitable orthonormal frame, where $q=\sqrt{\alpha^2+\beta^2}$ is constant. Thus we can choose in any coordinate neighborhood of M, where $\dim M=2n$, a field of frames $\{e_{(1)}, e_{(2)}, \cdots, e_{(2n)}\}$ such that

$$(3.15)\qquad \begin{aligned} h_i{}^h e^i{}_{(1)} &= p e^h{}_{(1)}, & h_i{}^h e^i{}_{(2)} &= p e^h{}_{(2)}, \\ k_i{}^h e^i{}_{(1)} &= q e^h{}_{(1)}, & k_i{}^h e^i{}_{(2)} &= -q e^h{}_{(2)}. \end{aligned}$$

As in Case I, by using $h_h{}^t(\nabla_k k_{it})=0$, (3. 13), (3. 14) and (3. 15), we can prove that, when $q=0$, the submanifold M is congruent in E^{2n+2} to $S^2(1/|p|)\times E^{2n-2}$, and when $q\neq 0$, to $S^1(1/\sqrt{2}q)\times S^1(1/\sqrt{2}q)\times E^{2n-2}$. Therefore, we can conclude that in Case II, the submanifold M is congruent to $S^2(r)\times E^{2n-2}$ or $S^1(r)\times S^1(r)\times E^{2n-2}$, r being a positive number, which is naturally imbedded in E^{2n+2}.

Finally, we consider Case III where $p=0$. In this case (2. 22) implies $h_{ji}=0$. Thus, the submanifold M lies on hypersurface E^{2n+1} of E^{2n+2}. Taking account of $h_{ji}=0$ and $l_j=0$, we can write (2. 41) and (2. 46) as

$$(3.16)\qquad (1-\lambda^2)k_j{}^t k_{ti}=(\alpha^2+\beta^2)(u_j u_i+v_j v_i),$$

$$(3.17)\qquad g^{ji}K_{ji}=-2(\alpha^2+\beta^2),$$

respectively, where $\alpha^2+\beta^2$ is constant because of $g^{ji}K_{ji}=\text{const}$. The tensor $k_i{}^h$ is the second fundamental tensor of M immersed in the hypersurface E^{2n+1} with respect to the normal D. We now suppose that $\alpha^2+\beta^2\neq 0$ and restrict ourselves to the open set M_0 $(\subset M)$ where $1-\lambda^2\neq 0$. Then, taking account of (2. 5) and (3. 16), we see that $k_i{}^h$ has at each point of M_0 the form

$$(3.18)\qquad (k_i{}^h)=\left(\begin{array}{cc:c} q & 0 & \\ 0 & -q & 0 \\ \hdashline & 0 & 0 \end{array}\right),\qquad q=\sqrt{\alpha^2+\beta^2}$$

with respect to a suitable orthonormal frame. Therefore we can choose in any coordinate neighborhood of M_0 a field of frames $\{e_{(1)}, e_{(2)}, \cdots, e_{(2n)}\}$, with respect to which (3. 8) holds, where $e_{(1)}$ and $e_{(2)}$ are linear combinations of u^h and v^h. On the other hand, we can easily see, by using (1. 9) and (1. 10) with $h_{ji}=0$ and $l_j=0$, that the distribution spanned in M_0 by u^h and v^h is integrable and totally geodesic in M_0. Thus, the distribution spanned in M_0 by $e_{(1)}$ and $e_{(2)}$ is also integrable and its integral manifolds are totally geodesic in M_0. Therefore, according to the same arguments developed in discussing the Cases I and II, we can conclude the fact: In Case III, the open submanifold M_0 is locally isometric to $S^1(r)\times S^1(r)\times E^{2n-2}$, which is locally flat. Thus the scalar curvature $g^{ji}K_{ji}$ of M vanishes identically in M_0 and hence in M because of the continuity of $g^{ji}K_{ji}$. Since $g^{ji}K_{ji}=0$ in M, (3. 17) implies $\alpha^2+\beta^2=0$, which contradicts the assumption that $\alpha^2+\beta^2\neq 0$. Consequently, we see that $\alpha^2+\beta^2=0$ in Case III. Therefore we find, from (3. 16), that $k_{ji}=0$ holds identically in M. Thus, M is totally geodesic in the hyperplane E^{2n+1} and consequently is congruent to a plane E^{2n} $(\subset E^{2n+1}\subset E^{2n+2})$.

Summing up the conclusions obtained in Cases I, II and III, we have

THEOREM 3. 1. *Let M be a complete submanifold of codimension 2 in an even-dimensional Euclidean space E^{2n+2} such that the scalar curvature of M is constant and there are global unit normals C and D to M which are parallel in the normal bundle. If $fH=Hf$ and $fK=-Kf$ hold, where H and K are the second fundamental tensors of M respectively with respect to C and D, f being the tensor field of type* $(1, 1)$ *appearing in the induced structure* (f, g, u, v, λ) *of M, then M is in E^{2n+2}, provided that $\lambda(1-\lambda^2)$ is non-zero almost everywhere in M, congruent to one of the following submanifolds*:

$$E^{2n},\ S^{2n}(r),\ S^n(r)\times S^n(r),\ S^l(r)\times E^{2n-l} \qquad (l=1, 2, \cdots, 2n-1),$$

$$S^k(r)\times S^k(r)\times E^{2n-2k} \qquad (k=1, 2, \cdots, n-1),$$

where $S^k(r)$ denotes a k-dimensional sphere of radius r (>0) imbedded natrually in E^{2n+2}.

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