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ON A PROBLEM OF R. NEVANLINNA CONCERNING SOME CLASS OF ENTIRE FUNCTIONS

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1. R. Nevanlinna proved

THEOREM A. *Let $f(z)$ be a meromorphic function and set*

$$K(f)=\overline{\lim_{r\to\infty}}\frac{N(r,0)+N(r,\infty)}{T(r,f)},$$

$$k(\lambda)=\inf K(f),$$

where f ranges over all meromorphic functions of order λ. Then

$$k(\lambda)=0 \qquad (\lambda=1,2,3,\cdots)$$

and $k(\lambda)>0$ for all other λ.

For non-integral λ, Nevanlinna found a positive lower bound for $k(\lambda)$. Realizing that this bound was not sharp, he posed the problem of determining the exact value of $k(\lambda)$.

Edrei-Fuchs [1, 2] proved that

$$k(\lambda)=\begin{cases} 1, & \text{if } 0\leqq\lambda\leqq\dfrac{1}{2}, \\ \sin\pi\lambda, & \text{if } \dfrac{1}{2}<\lambda<1, \end{cases}$$

and

$$k(\lambda)\geqq\frac{|\sin\pi\lambda|}{2.2\lambda+(1/2)|\sin\pi\lambda|}.$$

It is generally conjectured on the basis of the asymptotic behavior of the Lindelöf functions [5, p. 54] that the correct value of $k(\lambda)$ is given by

$$k(q,\lambda)=\begin{cases} \dfrac{|\sin\pi\lambda|}{q+|\sin\pi\lambda|}, & q\leqq\lambda\leqq q+\dfrac{1}{2}, \\ \dfrac{|\sin\pi\lambda|}{q+1}, & q+\dfrac{1}{2}<\lambda\leqq q+1. \end{cases}$$

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Recently Hellerstein-Williamson [4] have verified this conjecture for the class of entire functions having only negative zeros. They have proved

THEOREM B. *Let $f(z)$ be an entire function of genus q, order λ and lower order μ, having only negative zeros. Then we have*

$$K(f)=\overline{\lim_{r\to\infty}}\frac{N(r,0)}{T(r,f)}\geqq\max_{\lambda\geqq\rho\geqq\mu}k(q,\rho)\geqq\min_{\lambda\geqq\rho\geqq\mu}k(q,\rho)\geqq\underline{\lim_{r\to\infty}}\frac{N(r,0)}{T(r,f)}.$$

These inequalities are best possible.

In this paper we shall verify the above conjecture for a certain class of entire functions containing the class of entire functions having only negative zeros. We shall obtain

THEOREM 1. *Let $f(z)$ be an entire function of genus $q(\geqq 1)$, order λ and lower order μ. If every zero a_ν of $f(z)$ lies in a domain S_h for an arbitrarily fixed number $h>0$, where*

$$S_h=\{z;\ \mathrm{Re}\ z<0 \text{ and } |\mathrm{Im}\ z|<h\},$$

then we have $q\leqq\mu$ and

$$K(f)=\overline{\lim_{r\to\infty}}\frac{N(r,0)}{T(r,f)}\geqq\max_{\lambda\geqq\rho\geqq\mu}k(q,\rho)\geqq\min_{\lambda\geqq\rho\geqq\mu}k(q,\rho)\geqq\underline{\lim_{r\to\infty}}\frac{N(r,0)}{T(r,f)}.$$

These inequalities are best possible.

2. Let

$$f(z)=\prod\left(1-\frac{z}{a_n}\right)\Big/\prod\left(1-\frac{z}{b_n}\right)\qquad (\sum(|a_n|^{-1}+|b_n|^{-1})<\infty)$$

and set

$$\hat{f}(z)=\prod\left(1+\frac{z}{|a_n|}\right)\Big/\prod\left(1-\frac{z}{|b_n|}\right),\qquad F(z)=\prod\left(1+\frac{z}{|a_n|}\right)\prod\left(1+\frac{z}{|b_n|}\right).$$

Then it follows from Goldberg [3] and D. F. Shea (cf. [4]) that

$$T(r,f)\leqq T(r,\hat{f})\leqq T(r,F).$$

It shows that it is enough to consider the subclass of the entire functions having only negative zeros for the Nevanlinna problem for the class of meromorphic functions of genus zero. It is a question whether a similar situation is valid for all finite genera.

In this paper we shall show that a similar situation is true for the problem in the class of entire functions satisfying the conditions of Theorem 1, that is, THEOREM 2. *Let*

$$g(z)=\prod_{\nu=1}^{\infty} E\left(\frac{z}{a_\nu},\ q\right)$$

be a canonical product of genus $q(\geqq 1)$ *and set*

$$\hat{g}(z)=\prod_{\nu=1}^{\infty} E\left(-\frac{z}{|a_\nu|},\ q\right).$$

If all a_ν *belong to* S_h, *then we have*

$$\lim_{r\to\infty}\frac{T(r,\,g)}{T(r,\,\hat{g})}=1.$$

Theorem 1 is a consequence of Theorem 2 and Theorem B. Hence we shall only prove Theorem 2 in the following lines.

3. *Proof of Theorem* 2. From the definition of the Weierstrass primary factor, we have

$$\log|E(u,\,q)|=-\mathrm{Re}\sum_{l=q+1}^{\infty}\frac{u^l}{l},\qquad \text{if}\quad |u|\leqq\frac{q}{q+1}, \tag{3.1}$$

and

$$\log|E(u,\,q)|=\log|1-u|+\mathrm{Re}\sum_{l=1}^{q}\frac{u^l}{l},\qquad \text{if}\quad |u|>\frac{q}{q+1}. \tag{3.2}$$

Put $z=re^{i\theta}$ and $a_\nu=r_\nu e^{i\theta_\nu}$. Since $a_\nu\in S_h$, we have

$$|\pi-\theta_\nu|\leqq\frac{\pi}{2}\sin|\pi-\theta_\nu|\leqq\frac{\pi}{2}\,\frac{h}{r_\nu}=\frac{A}{r_\nu},$$

where $A=(\pi/2)h$, and

$$|(-1)^l\cos l\theta-\cos l(\theta-\theta_\nu)|\leqq 2\left|\sin\frac{l(\pi-\theta_\nu)}{2}\right|\leqq l|\pi-\theta_\nu|\leqq\frac{lA}{r_\nu}. \tag{3.3}$$

If $|z/a_\nu|\leqq q/(q+1)$, then it follows from (3.1) and (3.3) that

$$\left|\log\left|E\left(\frac{z}{a_\nu},\ q\right)\right|-\log\left|E\left(-\frac{z}{|a_\nu|},\ q\right)\right|\right|$$

$$=\left|\sum_{l=q+1}^{\infty}\frac{1}{l}\left(\frac{r}{r_\nu}\right)^l((-1)^l\cos l\theta-\cos l(\theta-\theta_\nu))\right|$$

$$\leqq\sum_{l=q+1}^{\infty}\frac{A}{r_\nu}\left(\frac{r}{r_\nu}\right)^l=\frac{A}{r_\nu}\left(\frac{r}{r_\nu}\right)^{q+1}\frac{1}{1-r/r_\nu}\leqq\frac{qA}{r_\nu^{q+1}}r^q.$$

Hence we obtain

$$\left|\log\left|E\left(\frac{z}{a_\nu},\ q\right)\right|-\log\left|E\left(-\frac{z}{|a_\nu|},\ q\right)\right|\right|\leqq\frac{qA}{r_\nu^{q+1}}r^q,\quad\text{for}\quad\left|\frac{z}{a_\nu}\right|\leqq\frac{q}{q+1}.\tag{3. 4}$$

If $|z/a_\nu|>q/(q+1)$ and $|\theta|<\pi$, then it follows from (3. 2) and (3. 3) that

$$\log\left|E\left(\frac{z}{a_\nu},\ q\right)\right|-\log\left|E\left(-\frac{z}{|a_\nu|},\ q\right)\right|$$

$$=\frac{1}{2}\log\left|\frac{1-2(r/r_\nu)\cos(\theta-\theta_\nu)+(r/r_\nu)^2}{1+2(r/r_\nu)\cos\theta+(r/r_\nu)^2}\right|+\sum_{l=1}^{q}\frac{1}{l}\left(\frac{r}{r_\nu}\right)^l(\cos l(\theta-\theta_\nu)-(-1)^l\cos l\theta)$$

$$\leqq\frac{1}{2}\log\left|1-\frac{2(r/r_\nu)(\cos(\theta-\theta_\nu)+\cos\theta)}{1+2(r/r_\nu)\cos\theta+(r/r_\nu)^2}\right|+\frac{A}{r_\nu}\sum_{l=1}^{q}\left(\frac{r}{r_\nu}\right)^l$$

$$\leqq\frac{1}{2}\log\left|1+\frac{2A(r/r_\nu^2)}{1+2(r/r_\nu)\cos\theta+(r/r_\nu)^2}\right|+\frac{A}{r_\nu}\sum_{l=1}^{q}\left(1+\frac{1}{q}\right)^{q-l}\left(\frac{r}{r_\nu}\right)^q.$$

Hence we obtain

$$\begin{aligned}&\log\left|E\left(\frac{z}{a_\nu},\ q\right)\right|-\log\left|E\left(-\frac{z}{|a_\nu|},\ q\right)\right|\\ &\leqq\frac{1}{2}\log\left|1+\frac{2A(r/r_\nu^2)}{1+2(r/r_\nu)\cos\theta+(r/r_\nu)^2}\right|+\frac{qeA}{r_\nu^{q+1}}r^q,\quad\text{for}\quad\left|\frac{z}{a_\nu}\right|>\frac{q}{q+1},\ |\theta|<\pi.\end{aligned}\tag{3. 5}$$

Further if $|z/a_\nu|\geqq(q+1)/q$, then we have

$$1+2\frac{r}{r_\nu}\cos\theta+\left(\frac{r}{r_\nu}\right)^2=\left(\frac{r}{r_\nu}-1\right)^2+2\frac{r}{r_\nu}(1+\cos\theta)\geqq\frac{1}{q^2},$$

and consequently, from (3. 5),

$$\log\left|E\left(\frac{z}{a_\nu},\ q\right)\right|-\log\left|E\left(-\frac{z}{|a_\nu|},\ q\right)\right|\leqq(eAq^2+qeA)\frac{1}{r_\nu^{q+1}}r^q,\ \text{for}\ \left|\frac{z}{a_\nu}\right|\geqq\frac{q+1}{q}.\tag{3. 6}$$

Since the genus of $g(z)$ is q, we have $\sum r_\nu^{-q-1}=A_1<\infty$. Using (3. 4), (3. 5) and (3. 6) we find that with $\sigma=(q+1)/q$,

$$\log|g(z)|\leqq\log|\hat{g}(z)|+\frac{1}{2}\sum_{\sigma^{-1}r<r_\nu<\sigma r}\log\left(1+\frac{2A(r/r_\nu^2)}{1+2(r/r_\nu)\cos\theta+(r/r_\nu)^2}\right)+2q^2eAA_1r^q\ \text{for}\ |\theta|<\pi,$$

and consequently

$$T(r,g)\leqq T(r,\hat{g})+\sum_{\sigma^{-1}r<r_\nu<\sigma r}\frac{1}{2\pi}\int_0^\pi\log\left(1+\frac{2A(r/r_\nu^2)}{1+2(r/r_\nu)\cos\theta+(r/r_\nu)^2}\right)d\theta+A_2r^q,\tag{3. 7}$$

where $A_2=2q^2eAA_1$.

We also know that for real a,

$$\int_0^\pi \log(1+2a\cos\theta+a^2)d\theta=\begin{cases} 0, & \text{for } |a|\leqq 1, \\ \pi\log a^2, & \text{for } |a|>1, \end{cases} \tag{3.8}$$

and

$$\int_0^\pi \log(a+\cos\theta)d\theta=\pi\log(a+\sqrt{a^2-1})-\pi\log 2, \quad \text{for} \quad a>1.$$

Hence we have

$$\begin{aligned}&\int_0^\pi \log\left(1+2\frac{r}{r_\nu}\cos\theta+\frac{r^2}{r_\nu^2}+2A\frac{r}{r_\nu^2}\right)d\theta \\ &=-\pi\log 2+\pi\log\left(1+\frac{r^2}{r_\nu^2}+2A\frac{r}{r_\nu^2}+\sqrt{\left(1-\frac{r^2}{r_\nu^2}\right)^2+4A\frac{r}{r_\nu^2}\left(1+\frac{r^2}{r_\nu^2}\right)+4A^2\frac{r^2}{r_\nu^4}}\right).\end{aligned} \tag{3.9}$$

If $1\geqq r/r_\nu>q/(q+1)$, then (3. 8) and (3. 9) imply

$$\begin{aligned}&\int_0^\pi \log\left(1+\frac{2A(r/r_\nu^2)}{1+2(r/r_\nu)\cos\theta+(r/r_\nu)^2}\right)d\theta \\ &\leqq -\pi\log 2+\pi\log\left(1+\frac{r^2}{r_\nu^2}+2A\frac{r}{r_\nu^2}+\left(1-\frac{r^2}{r_\nu^2}\right)+2\sqrt{2A}\frac{r^{1/2}}{r_\nu}+2A\frac{r}{r_\nu^2}\right) \\ &\leqq -\pi\log 2+\pi\log\left(2+2A\frac{1}{r_\nu}+2\sqrt{2A}\frac{1}{r_\nu^{1/2}}+2A\frac{1}{r_\nu}\right) \\ &\leqq 2A\pi\frac{1}{r_\nu}+\sqrt{2A}\pi\frac{1}{r_\nu^{1/2}}\leqq 2eA\pi\frac{1}{r_\nu^{q+1}}r^q+e\sqrt{2A}\pi\frac{1}{r_\nu^{q+1/2}}r^q.\end{aligned}$$

If $((q+1)/q)>r/r_\nu>1$, then (3. 8) and (3. 9) imply

$$\begin{aligned}&\int_0^\pi \log\left(1+\frac{2A(r/r_\nu^2)}{1+2(r/r_\nu)\cos\theta+(r/r_\nu)^2}\right)d\theta \\ &=-\pi\log 2+\pi\log\left(\frac{r_\nu^2}{r^2}+1+2A\frac{1}{r}+\sqrt{\left(\frac{r_\nu^2}{r^2}-1\right)^2+4A\frac{1}{r}\left(\frac{r_\nu^2}{r^2}+1\right)+4A^2\frac{1}{r^2}}\right) \\ &\leqq 2A\pi\frac{1}{r_\nu}+\sqrt{2A}\pi\frac{1}{r_\nu^{1/2}}\leqq 2A\pi\frac{1}{r_\nu^{q+1}}r^q+\sqrt{2A}\pi\frac{1}{r_\nu^{q+1/2}}r^q.\end{aligned}$$

Using these inequalities in (3. 7), we deduce that

$$T(r,g)\leqq T(r,\hat{g})+A_3\left(\sum_{\sigma^{-1}r<r_\nu<\sigma r} r_\nu^{-q-1/2}\right)r^q+A_4r^q \tag{3.10}$$

with suitable positive constants A_3 and A_4.

Now it follows that

$$\sum_{\sigma^{-1}r<r_\nu<\sigma r} r_\nu^{-q-1/2}=\int_{\sigma^{-1}r}^{\sigma r} t^{-q-1/2}dn(t,0)\leqq\frac{n(\sigma r,0)}{(\sigma r)^{q+1/2}}+\left(q+\frac{1}{2}\right)\int_{\sigma^{-1}r}^{\sigma r}\frac{n(t,0)}{t^{q+3/2}}dt. \tag{3.11}$$

On the other hand, since $\hat{g}(z)$ has only negative zeros, it follows from Edrei-Fuchs [1, p. 308] that

$$\lim_{r\to\infty}\frac{r^q}{T(r,\hat{g})}=0 \tag{3.12}$$

and with a positive constant C,

$$T(r,\hat{g})\geqq Cr^{q+1}\int_0^\infty\frac{n(t,0)}{t^{q+1}}\frac{dt}{t+r},$$

and consequently

$$T(r,\hat{g})\geqq Cr^{q+1}\int_{\sigma^{-1}r}^{\sigma r}\frac{n(t,0)}{t^{q+3/2}}\frac{t^{1/2}}{t+r}dt\geqq\frac{C}{(\sigma+1)\sigma^{1/2}}r^{q+1/2}\int_{\sigma^{-1}r}^{\sigma r}\frac{n(t,0)}{t^{q+3/2}}dt \tag{3.13}$$

and

$$T(r,\hat{g})\geqq Cr^{q+1}\int_{\sigma r}^\infty\frac{n(t,0)}{t^{q+1}}\frac{dt}{t+r}\geqq\frac{C}{(q+1)(1+\sigma^{-1})\sigma^{q+1}}n(\sigma r,0). \tag{3.14}$$

Therefore from (3. 10)–(3. 14) we obtain

$$\overline{\lim_{r\to\infty}}\frac{T(r,g)}{T(r,\hat{g})}\leqq 1. \tag{3.15}$$

Next, by virtue of the same argument as above, we deduce that

$$T(r,\hat{g})\leqq T(r,g)+\sum_{\sigma^{-1}r<r_\nu<\sigma r}\frac{1}{4\pi}\int_0^{2\pi}\log\left(1+\frac{2A(r/r_\nu^2)}{1-2(r/r_\nu)\cos(\theta-\theta_\nu)+(r/r_\nu)^2}\right)d\theta+A_2r^q,$$

and hence

$$T(r,\hat{g})\leqq T(r,g)+A_3\Bigl(\sum_{\sigma^{-1}r<r_\nu<\sigma r}r_\nu^{-q-1/2}\Bigr)r^q+A_4r^q.$$

Therefore from (3. 11)–(3. 14) we obtain

$$\underline{\lim_{r\to\infty}}\frac{T(r,g)}{T(r,\hat{g})}\geqq 1. \tag{3.16}$$

Consequently (3. 15) and (3. 16) imply

$$\lim_{r\to\infty}\frac{T(r,g)}{T(r,\hat{g})}=1,$$

which completes the proof of Theorem 2.

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