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ON PRIME ENTIRE FUNCTIONS, II

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§1. **Introduction.** An entire function $F(z)=f\circ g(z)$ is said to be prime if every factorization of the above form implies that one of the functions $f(z)$ or $g(z)$ is linear.

In this paper we shall prove the following theorem on prime entire functions, which has been proved in our previous paper [3] in a less complete form.

THEOREM. *Let $F(z)$ be an entire function of order ρ $(>1/2)$, with only negative zeros. Assume that $n(r)\sim\lambda r^{\rho}$, $\lambda>0$, where $n(r)$ denotes the number of zeros of $F(z)$ in $|z|<r$. Further assume that there are two indices j and k such that a_j, a_k are zeros of $F(z)$ whose multiplicities p_j, p_k satisfy $(p_j, p_k)=1$. Then $F(z)$ is prime.*

Recently, Ozawa has conjectured in his paper [5] that the reciprocal of the Gamma Function is prime. We shall here give an affirmative answer to this conjecture, using the analogous technique to that used in the proof of the above theorem.

§2. In order to prove our theorem we shall refer to the following known results.

LEMMA 1. (Edrei [1]). *Let $f(z)$ be an entire function. Assume that there exists an unbounded sequence $\{h_\nu\}_{\nu=1}^{\infty}$ such that all the roots of the equations $f(z)=h_\nu$, $\nu=1, 2, \cdots$ be real. Then $f(z)$ is a polynomial of degree at most two.*

LEMMA 2. (Pólya [6]), *Suppose that $f(z)$, $g(z)$ are entire functions and that $\phi(z)=f\circ g(z)$ is of finite order. Then either $g(z)$ is a polynomial or $f(z)$ is of order zero.*

LEMMA 3. (Nevanlinna [4]). *If $\alpha>-1$ and ρ is a positive integer, then the canonical product $f(z;\rho,\alpha)$ with only negative zeros of genus q, whose number of zeros $n(r)$ in $|z|<r$ satisfies*

$$n(r)\sim\lambda r^{\rho}\cdot(\log r)^{\alpha}\qquad(\lambda>0),$$

has in $|\arg z|\leqq\pi-\delta$ $(0<\delta<\pi)$ the asymptotic behavior

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$$\log f(z;\ \rho, \alpha) \sim \lambda \frac{(-1)^\rho}{\alpha+1} z^\rho (\log z)^{\alpha+1},$$

and $q=\rho$.

LEMMA 4. (Edrei-Fuchs [2], Williamson [7]). *If an entire function* $f(z)$ *represented as*

$$f(z)=e^{\alpha_0 z^p+\cdots+\alpha_p} \prod E\left(\frac{z}{a_\nu},\ \rho\right),$$

where $E(u, p)$ *is the primary factor of genus* p, *satisfies*

$$N(r, 0;\ f)=o(T(r, f))$$

and if we put

$$c(r)=\alpha_0+\frac{1}{\rho}\sum_{|a_\nu|\leqq r}\frac{1}{a_\nu^{\rho}},$$

then, in a set $r=|z|$ *of density* 1, *we have*

$$\log |f(z)|=\mathrm{Re}\ \{c(2r)z^\rho\}+o(T(r, f)).$$

§3. **Proof of Theorem.** In the case that ρ $(>1/2)$ is not a positive integer, the theorem has been proved in our previous paper [3]. So it is sufficient to deal with the case that ρ is a positive integer.

Let $F(z)$ be $f\circ g(z)$. Assume that $f(w)$ is transcendental. If $f(w)=0$ has only a finite number of roots, then we can write

$$f(w)=p(w)e^{H(w)}$$

where $p(w)$ is a polynomial and $H(w)$ is also a polynomial, in view of $\rho<+\infty$. Since by Lemma 2 $g(z)$ is a polynomial, $F(z)=0$ has only a finite number of roots. This is a contradiction. Hence $f(w)=0$ has an infinite number of roots $\{w_n\}$, $w_n\to\infty$. Consider the equations $g(z)=w_n$, $n=1, 2, \cdots$. All their roots lie on the real negative axis. Then by Lemma 1 $g(z)$ is a polynomial of degree at most two. By a slight precise argument we have that $g(z)$ must be linear, and thus $F(z)$ is prime.

Suppose next that $F(z)=f\circ g(z)$ with a polynomial $f(w)$. In this case, we have

$$(1) \qquad F(z)=Ag_1(z)^{l_1}\cdots g_p(z)^{l_p}, \qquad g_j(z)=g(z)-w_j.$$

On the other hand by Lemma 3, since we have

$$F(z)=e^{p(z)} \prod E\left(\frac{z}{a_\nu},\ \rho\right)$$

where $p(z)$ is a polynomial with $\deg p(z)\leqq\rho$,

$$\log M(r, F)>kr^\rho \log r \qquad (r\geqq r_0)$$

where k is a positive constant and $M(r, F)=\max_{|z|=r}|F(z)|$. Hence we have in view of (1)

$$\log M(r, g_s)>k'r^\rho \log r \qquad (1\leqq s\leqq p,\ r\geqq r_0') \tag{2}$$

where k' is a positive constant. Now the assumption $n(r)\sim\lambda r^\rho$ and (2) imply that

$$N(r, g_s)=o(T(r, g_s)).$$

Therefore, we have by Lemma 4

$$\log |g_s(z)|=|c_s(2r)|r^\rho \cos (\alpha_r^{(s)}+\rho\theta)+o(c_s(2r)r^\rho),$$

in a set $E_s=\{|z|=r\}$ of density 1 where $\alpha_r^{(s)}=\arg c_s(2r)$, $z=re^{i\theta}$. By the definition of $c(r)$ it is clear that either $|\alpha_r^{(s)}|<\varepsilon$ or $|\alpha_r^{(s)}-\pi|<\varepsilon$ hold for $r\geqq r_0(\varepsilon)$ $(1\leqq s\leqq p)$. Hence, in view of $\rho\geqq 1$, we can choose a rectilinear ray issuing from the origin such that along the ray in a set $E_1\cap E_2$ of density 1,

$$g(z)\to w_1, \qquad g(z)\to w_2 \qquad (z\to\infty).$$

This is clearly a contradiction. Therefore we have

$$F(z)=A(g(z)-w_1)^{l_1}.$$

By the existence of two zeros whose multiplicities are coprime, l_1 must reduce to 1. Hence we have

$$F(z)=A(g(z)-w_1),$$

and the theorem has been established.

§4. Let $F(z)=1/\Gamma(z)$ be $f\circ g(z)$. If $f(w)$ is transcendental, then $F(z)$ is prime by the same arguments as in the proof of Theorem.

If $f(w)$ is a polynomial, then we have the factorization (1) in §3. Putting

$$\frac{1}{\Gamma(z)}=z\cdot\frac{1}{z\Gamma(z)},$$

it is easily seen by Lemma 3 that

$$\log M\left(r, \frac{1}{\Gamma(z)}\right)>kr\log r \qquad (r\geqq r_0).$$

On the other hand,

$$n\left(r, \frac{1}{\Gamma(r)}\right)\sim r,$$

and thus we have

$$N\left(r, \frac{1}{\Gamma(z)}\right)=o(T(r, g_s)).$$

Therefore, by proceeding as in the proof of Theorem, we can conclude that $1/\Gamma(z)$ is prime.

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