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MINIMALITY IN FAMILIES OF SOLUTIONS OF $\Delta u = Pu$ ON RIEMANN SURFACES

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I. Introduction.

Consider a Riemann surface R and the space $HD(R)$ of all harmonic functions with finite Dirichlet integral. The monotone closure of $HD(R)$ is denoted by $\widetilde{HD}(R)$. C. Constantinescu and A. Cornea in 1958 started the study of minimal functions in $\widetilde{HD}(R)$. In 1960, Nakai ([2], also cf. [7]) introduced a representing measure on the Royden boundary Γ associated to R and a kernel on $R \times \Gamma$ which serve to represent $\widetilde{HD}(R)$. One significant result is that $\widetilde{HD}$-minimal functions correspond to atoms on Γ.

It was Ozawa [5] who first considered the solutions of $\Delta u = Pu$ on R where P is a nonnegative density. Glasner and Katz [1] have recently shown that solutions of $\Delta u = Pu$ on R can also be studied in terms of their behavior on Γ. Using their machinery, one can obtain analogues of Nakai's results for the space $PE(R)$ of solutions with finite energy integral and its monotone closure $\widetilde{PE}(R)$. In particular, a representing measure on Γ and a kernel on $R \times \Gamma$ can be constructed for solutions so that $\widetilde{PE}$-minimal functions can be characterized analogously.

In view of this, a natural question is: what is the relation between the $\widetilde{HD}$- and $\widetilde{PE}$-minimality? Or equivalently, if a point on Γ is atomic with respect to one measure, will it be atomic with respect to another? In this paper it is shown that the answer is virtually yes. This answer is encouraging because it suggests that there is a topological property of Γ which can be associated with $\widetilde{HD}$- or $\widetilde{PE}$-minimality. Finding such a property would have important implications in the study of quasi-conformal or quasi-isometric invariants.

As a remark, all the results in this paper can be carried over to Riemannian manifolds.

II. Preliminaries.

II 1. We consider an open Riemann surface R and the equation $\Delta u = Pu$ on R, where P is a nonnegative density. For simplicity, solutions of $\Delta u = Pu$ will be called solutions. Let $M(R)$ be the Royden algebra associated with R which is the

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set of all bounded Tonelli functions on R with finite Dirichlet integral. The Royden compactification R^* is the unique compact Hausdorff space such that R is open and dense in R^*, functions in $M(R)$ have continuous extensions to R^* and $M(R)$ separates points of R^*. $\Gamma=R^*\backslash R$ is the Royden boundary. If we let $M_\Delta(R)$ be the BD-closure (i.e. the closure with respect to the topology generated by the compact bounded convergence and the Dirichlet semi-norm) of the set $M_0(R)$ of all functions in $M(R)$ with compact support, then $\Delta=\{q\in\Gamma: f(q)=0 \text{ for all } f\in M_\Delta(R)\}$ is the harmonic boundary of R. Because it is uniformly dense in the set of all continuous functions on R^*, $M(R)$ has the important *Urysohn property*: *for any two disjoint compact sets* K_1, K_2 *in* R^* *and any two distinct real numbers* r_1, r_2 *with* $r_1<r_2$, *there is a function* $f\in M(R)$ *such that* $r_1\leqq f\leqq r_2$ *and* $f|K_i=r_i$, $i=1, 2$. For the details of this section see [1], [2] or [7].

II 2. A modification of Nakai's result ([7], p. 168) is the *minimum principle*: *if u is any supersolution on R bounded from below such that for all* $q\in\Delta$ $\liminf_{z\in R, z\to q}u(z)\geqq c$ *for some nonpositive number c, then* $u\geqq c$.

II 3. A subset of Δ introduced in [1] is crucial for solutions, i.e. $\Delta^P=\{q\in\Delta$: q has a neighborhood U in R^* with $\int_{U\cap R}Pdxdy<\infty\}$. Δ^P is open in Δ. Functions in $E(R)$ vanish on $\Delta\backslash\Delta^P$, where $E(R)$ is the subalgebra of $M(R)$ of all bounded Tonelli functions f on R with finite energy integral

$$E(f)=E_R(f)=\int_R df\wedge{}^*df+\int_R f^2P\,dxdy.$$

$E(R)|\Delta^P$ is uniformly dense in the set of all continuous functions on Δ^P vanishing at infinity in view of the Stone-Weierstrass theorem. $E(R)$ is BE-complete, i.e. complete with respect to the topology generated by the compact bounded convergence and the energy semi-norm.

II 4. The PE-projection is denoted by π^P. For any $f\in E(R)$, it gives the unique PE-function (i.e. solution with finite energy integral) π^Pf with $\pi^Pf|\Delta=f|\Delta$. For the existence of such solution see Theorem 3 of [1]. Note that if $P\equiv0$, π^0 is the HD-projection which gives the unique HD-functions (i.e. harmonic functions with finite Dirichlet integral).

III. Representing measure and its features.

III 1. Following the pattern that Nakai has established in [2] (or [7], p. 171) for harmonic functions and being aware of the results of Glasner-Katz ([1], Theorem 3), we can construct a positive bounded regular Borel representing measure m^P on Γ centered at $z_0\in R$ having support equal to the closure of Δ^P characterized by $u(z_0)=\int_\Gamma u\,dm^P$ for every PE-function u. Moreover, using Harnack's inequality we can also construct a nonnegative kernel $K^P(z, q)$ on $R\times\Gamma$ with the property $u(z)=\int_\Gamma K^P(z, q)u(q)dm^P(q)$ for all $z\in R$ and all PE-function u. It can be shown that $K^P(z_0, q)=1$ for all $q\in\Delta^P$ and $K^P(z, q)$ is a solution on R

almost everywhere with respect to m^P.

III 2. Nakai's characterizations of $\widetilde{HD}$-functions and $\widetilde{HD}$-minimal functions are also valid, mutatis mutandis, for $\widetilde{PE}$-functions and $\widetilde{PE}$-minimal functions.

DEFINITION. A nonnegative solution is called a $\widetilde{PE}$-function if it is the infimum of a downward directed family of PE-functions. The collection of all $\widetilde{PE}$-functions on R is denoted by $\widetilde{PE}(R)$.

LEMMA. *If χ_C is the characteristic function of a compact subset C of Δ^P, then $u(z)=\int_\Gamma K^P(z, q)\chi_C(q)dm^P(q)$ is a $\widetilde{PE}$-function.*

The proof of the following theorem can be carried out as in Nakai's [2] (or [7]) except that bounded upper semicontinuous functions on Δ^P do not necessarily correspond to $\widetilde{PE}$-functions. $PE(R)$ has no order unit in general, so the work of Nakai does not carry over formally. However, it is possible to get by with the help of the lemma.

THEOREM. *If $u\in\widetilde{PE}(R)$, then*

$$u(z)=\int_\Gamma K^P(z, q)(\limsup_{x\in R, x\to q}u(x))dm^P(q).$$

DEFINITION. A nonzero $\widetilde{PE}$-function u is called a $\widetilde{PE}$-minimal function if for any $\widetilde{PE}$-function v such that $u\geqq v$, we have $cu=v$ for some constant c.

THEOREM. *There exists a $\widetilde{PE}$-minimal function on R if and only if there exists a point in Δ^P with positive m^P-measure.*

More precisely, if u is $\widetilde{PE}$-minimal, then there is a point $q_0\in\Delta^P$ such that $m^P(q_0)>0$ and $u(z)=aK^P(z, q_0)$ for some positive constant a. Conversely, if $m^P(q_0)>0$ for some $q_0\in\Delta^P$, then $K^P(z, q_0)$ is a $\widetilde{PE}$-minimal function.

IV. An intrinsic property of minimal functions.

IV 1.

THEOREM. *For any connected set $S\subset\Delta^P$, $m^P(S)>0$ if and only if $m^0(S)>0$.*

Proof. Since the representing measures are regular, we may assume without loss of generality that S is compact.

Suppose that $m^P(S)>0$. By the regularity of the representing measures, there exists a sequence $\{U_n\}$ of open sets in R^* such that $S\subset\bar{U}_{n+1}\subset U_n$, $\lim m^P(U_n\cap\Gamma)=m^P(S)$ and $\lim m^0(U_n\cap\Gamma)=m^0(S)$. We may assume that $\int_{U_n\cap R}Pdxdy<\infty$ for all n because S is compact. By the Urysohn property, for every n there is an $f_n\in M(R)$ such that $0\leqq f_n\leqq 1$, $f_n|\bar{U}_{n+1}=1$ and $f_n|R^*\setminus U_n=0$. Clearly $f_n\in E(R)$ and $f_n\geqq f_{n+1}$ for all n. By the choice of $\{U_n\}$, $\lim f_n|\Gamma=\chi_S$ almost everywhere with respect to both m^P and m^0, where χ_S is the characteristic function of S. Note that $\pi^0 f_n \in HBD(R)$, $\pi^P f_n \in PBE(R)$ and $\pi^0 f_n|\Delta=\pi^P f_n|\Delta=f_n|\Delta$. $\pi^0 f_n$, being a nonnegative harmonic function, is a supersolution. And so is $\pi^0 f_n-\pi^P f_n$. Observe that for all $q\in\Delta$, $\lim\inf_{z\in R, z\to q}(\pi^0 f_n-\pi^P f_n)(z)=f_n(q)-f_n(q)=0$ and thus $\pi^0 f_n \geqq \pi^P f_n$ by the minimum principle. Note that $\{\pi^0 f_n\}$ and $\{\pi^P f_n\}$ are both decreasing sequences and

$$\int K^0(z, q) f_n(q) dm^0(q)=\pi^0 f_n(z)\geqq \pi^P f_n(z)=\int K^P(z, q) f_n(q) dm^P(q).$$

As n tends to ∞, we have by the monotone convergence theorem that

$$\int K^0(z, q)\chi_S(q) dm^0(q)\geqq \int K^P(z, q)\chi_S(q) dm^P(q).$$

Setting $z=z_0$ gives $m^0(S)\geqq m^P(S)>0$ where z_0 is the center of m^P and m^0. Thus the necessity is proved.

To prove the sufficiency, we need the subsequent lemmas.

IV 2.

LEMMA. *Let $S\subset\Delta$ be connected and compact. If $m^0(S)>0$, then for all open set U in R^* containing S, there is an open set V in R^* such that $S\subset V\subset U$ and $V\cap R$ is a region in R with piecewise smooth boundary.*

This is a modification of Proposition 9 in [4].

IV 3. Consider any region G in R. Let ∂G be its boundary. The closures of G and ∂G in R^* are denoted by $\bar{G}$ and $\overline{\partial G}$ respectively. Let $bG=(\bar{G}\backslash\overline{\partial G})\cap\Gamma$. Clearly $\bar{G}\backslash G=\overline{\partial G}\cup bG$. G, being itself a Riemann surface, has its own Royden compactification G^* and Royden boundary $\Gamma_G=G^*\backslash G$.

LEMMA. *There is a unique continuous mapping j from G^* onto $\bar{G}$ fixing G elementwise. Moreover, $\Gamma_G=j^{-1}(\overline{\partial G})\cup j^{-1}(bG)$ and j is a homeomorphism between $G\cup j^{-1}(bG)$ and $G\cup bG$.*

This lemma is due to Nakai ([4]; Propositions 7, 8).

IV 4. Now choose the center z_0 of the representing measure m^P to be in G. If we denote the representing measure on Γ_G centered at z_0 by m_G^P, then the following is true.

LEMMA. *Let G be a region in R with piecewise smooth boundary such that $\int_G P\,dxdy<\infty$. Let E be any Borel subset of bG, then $m^P(E)>0$ if and only if $m_G^P(j^{-1}(E))>0$.*

Nakai ([4], Proposition 8) has established this for $P\equiv 0$, but his proof can be generalized.

IV 5. The following lemma is due to Royden ([6], Proposition 11; also see Nakai [3]).

LEMMA. *If G is a region in R such that $\int_G P dxdy<\infty$, then there is an isomorphism T of $HB(G)$ onto $PB(G)$ with the following properties*:

1) $\{u_n\}\subset HB(G)$ *is a decreasing sequence with limit* $u\in HB(G)$ *if and only if* $\{Tu_n\}\subset PB(G)$ *is a decreasing sequence with limit* $Tu\in PB(G)$.

2) *Let* $\{G_i\}$ *be an exhaustion of G consisting of relatively compact regions with piecewise smooth boundary. For any* $u\in HB(G)$, *let* T_iu *be the continuous function on G such that* $T_iu|G_i\in PB(G_i)$ *and* $T_iu|G\backslash G_i=u|G\backslash G_i$. *Then* $\{T_iu\}$ *converges to Tu uniformly on compact subsets of G.*

3) $\sup_G|Tu|=\sup_G|u|$ *for all* $u\in HB(G)$.

COROLLARY. $u\in HBD(G)$ *if and only if* $Tu\in PBD(G)$ *for all* $u\in HB(G)$. *In this case* $u|\Delta_G=Tu|\Delta_G$, *where Δ_G is the harmonic boundary of G.*

Proof. Note that the sequence $\{T_iu\}$ converges to Tu in the compact bounded convergence topology. If $u\in HBD(G)$, then by the hypothesis and the Dirichlet principle for $\Delta u=Pu$ ([6], Lemma 8) $E_G(T_{i+j}u)\leqq E_G(T_iu)\leqq E_G(u)<\infty$. Moreover, Green's formula implies that $0=E_{G_{i+j}}(T_{i+j}u, T_{i+j}u-T_iu)=E_G(T_{i+j}u, T_{i+j}u-T_iu)$ $=E_G(T_{i+j}u)-E_G(T_{i+j}u, T_iu)$. Therefore $0\leqq E_G(T_iu-T_{i+j}u)=E_G(T_iu)-2E_G(T_iu, T_{i+j}u)$ $+E_G(T_{i+j}u)=E_G(T_iu)-E_G(T_{i+j}u)$. Thus $\{T_iu\}$ is BE-Cauchy. Since $E(G)$ is BE-complete, we have $Tu\in PBE(G)$. Furthermore, it is clear that $T_iu-u\in M_0(G)$. Consequently $Tu-u\in M_\Delta(G)$ for $M_\Delta(G)$ is the BD-closure of $M_0(G)$. Hence $(Tu-u)|\Delta_G$ $=0$, i.e., $u|\Delta_G=Tu|\Delta_G$.

The proof of the sufficiency is similar. Q.E.D.

IV 6. Now we are ready to complete the proof of Theorem IV 1. Suppose that $m^0(S)>0$ where S is assumed to be connected and compact. By IV 2 and because $S\subset\Delta^P$, there is an open set V in R^* containing S such that $G=V\cap R$ is a region in R with piecewise smooth boundary and $\int_G P dxdy<\infty$. Note that $S\subset bG\cap\Delta$ and recall the continuous mapping j from G^* into $\bar{G}$ given in IV 3 which is a homeomorphism from $G\cup j^{-1}(bG)$ to $G\cup bG$. It follows from IV 4 that $S'=j^{-1}(S)$ is a compact subset of Δ_G with $m_G^0(S')>0$.

Let $K_G^P(z, q)$ be the associated kernel of m_G^P and $\chi_{S'}$ the characteristic function of S'. By Lemma III 2

$$u(z)=\int_{\Gamma_G} K_G^0(z, q)\chi_{S'}(q)dm_G^0(q)$$

is a bounded harmonic function in $\widetilde{HD}(G)$ with $\sup u\geqq u(z_0)=m_G^0(S')>0$. Let $\{U_n\}$ be a sequence of open sets in G^* containing S' such that $\bar{U}_{n+1}\subset U_n$, $m_G^0(S')$ $=\lim m_G^0(U_n\cap\Gamma_G)$ and $m_G^P(S')=\lim m_G^P(U_n\cap\Gamma_G)$. By the Urysohn property, for each n there is an $f_n\in M(G)$ such that $0\leqq f_n\leqq 1$, $f_n|\bar{U}_{n+1}=1$ and $f_n|G^*\backslash U_n=0$. Clearly, $\{f_n\}$ is a decreasing sequence converging to $\chi_{S'}$ on Γ_G almost everywhere with respect to both m_G^0 and m_G^P. Let

$$u_n=\pi_G^0 f_n$$

where π_G^0 is the HD-projection on G (cf. II 4). Note that $u_n \in HBD(G)$ and $u_n|\Delta_G = f_n|\Delta_G$. Thus $\{u_n\}$ is also a decreasing sequence by the minimum principle. The monotone convergence theorem implies that

$$\lim u_n(z)=\lim \int_{\Gamma_G} K_G^0(z, q)u_n(q)dm_G^0(q)=\int_{\Gamma_G} K_G^0(z, q)\chi_{S'}(q)dm_G^0(q)=u(z).$$

Consider the isomorphism T given in IV 5. Note that $Tu_n \in PBE(G)$, $Tu_n|\Delta_G = u_n|\Delta_G$ and $Tu=\lim Tu_n$. Thus

$$Tu(z)=\lim \int_{\Gamma_G} K_G^P(z, q)u_n(q)dm_G^P(q)=\int_{\Gamma_G} K_G^P(z, q)\chi_{S'}(q)dm_G^P(q)$$

by the monotone convergence theorem once more. Since $\sup Tu=\sup u>0$ by IV 5, we have that $0<Tu(z_0)=m_G^P(S')=m_G^P(j^{-1}(S))$.

Now it follows from IV 4 again that $m^P(S)>0$. Q.E.D.

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