KÔDAI MATH. SEM. REP
24 (1972), 34—49

DEFICIENCIES OF AN ENTIRE ALGEBROID FUNCTION OF FINITE ORDER

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§1. Recently Niino-Ozawa [1], [2] has established some curious results for a two- or three- or four-valued entire algebroid function. A typical theorem is the following:

Theorem. *Let $f(z)$ be a two-valued entire transcendental algebroid function and a_1, a_2 and a_3 be different finite numbers satisfying*

$$\delta(a_1, f)+\delta(a_2, f)+\delta(a_3, f)>2.$$

Then at least one of $\{a_j\}$ is a Picard exceptional value of f.

This result discloses the remarkable fact that the condition only on the deficiencies implies the existence of a Picard exceptional value in the two-valued case and there is a big gap between the distribution of deficiencies of entire algebroid functions and that of one-valued entire functions.

In this paper we shall relax somewhat the condition on the deficiencies as follows:

Theorem 1. *Let $f(z)$ be a two-valued entire transcendental algebroid function of finite order by an irreducible equation*

$$F(z, f)\equiv f^2+A_1f+A_0=0,$$

where A_1 and A_0 are entire functions in $|z|<\infty$. Let a_1, a_2 and a_3 be three different finite numbers satisfying

$$\Delta(a_1, f)+\delta(a_2, f)+\delta(a_3, f)>2,$$

where $\delta(a, f)$ and $\Delta(a, f)$ indicate the Nevanlinna-Selberg deficiency and Valiron deficiency of f at a respectively. Then at least one of $\{a_j\}$ is a Picard exceptional value of f or more precisely it occurs either

(a) $$\delta(a_1, f)=1, \quad \delta(a_2, f)=\delta(a_3, f)>\frac{1}{2} \quad or$$

(b) $$\delta(a_2, f)=1, \quad \Delta(a_1, f)=\Delta(a_3, f)>\frac{1}{2}.$$

Received January 14, 1971.

Further if there is another deficiency of f at a_4 then

(a)′ $\delta(a_4, f) \leqq 1 - \delta(a_2, f)$ *or*

(b)′ $\delta(a_4, f) \leqq 1 - \Delta(a_1, f)$ *corresponding to the cases* (a) *or* (b).

THEOREM 2. *Let $f(z)$ be a three-valued transcendental entire algebroid function of finite order defined by an irreducible equation*

$$F(z, f) \equiv f^3 + A_2 f^2 + A_1 f + A_0 = 0,$$

where A_2, A_1 and A_0 are entire functions. Let a_1, a_2, a_3 and a_4 be four different finite numbers satisfying

$$\Delta(a_1, f) + \sum_{j=2}^{4} \delta(a_j, f) > 3.$$

Further any two of $\{F(z, a_j)\}$ are not proportional. Then one of $\{a_j\}$ is a Picard exceptional value of f

THEOREM 3. *Let $f(z)$ be the same as in the above theorem* 2. *Let a_1, a_2, a_3, a_4 and a_5 be five different finite numbers satisfying*

$$\Delta(a_1, f) + \delta(a_2, f) + \delta(a_3, f) + \delta(a_4, f) > 3,$$

$$\Delta(a_1, f) + \delta(a_2, f) + \delta(a_3, f) + \delta(a_5, f) > 3.$$

Then at least two of $\{a_j\}$ are Picard exceptional values of f or more precisely it occurs either

(a) $\delta(a_1, f) = \delta(a_2, f) = 1$ *and* $\delta(a_3, f) = \delta(a_4, f) = \delta(a_5, f) > \frac{1}{2}$ *or*

(b) $\delta(a_1, f) = \delta(a_4, f) = 1$ *and* $\delta(a_2, f) = \delta(a_3, f) = \delta(a_5, f) > \frac{1}{2}$ *or*

(c) $\delta(a_2, f) = \delta(a_4, f) = 1$ *and* $\Delta(a_1, f) = \Delta(a_3, f) = \Delta(a_5, f) > \frac{1}{2}$ *or*

(d) $\delta(a_2, f) = \delta(a_3, f) = 1$ *and* $\Delta(a_1, f) = \Delta(a_4, f) = \Delta(a_5, f) > \frac{1}{2}$ *or*

(e) $\delta(a_4, f) = \delta(a_5, f) = 1$ *and* $\Delta(a_1, f) = \Delta(a_2, f) = \Delta(a_3, f) > \frac{1}{2}$.

Further if there is another deficient value a_6 then

(a)′ $\delta(a_6, f) \leqq 1 - \delta(a_3, f)$ *or*

(b)′ $\delta(a_6, f) \leqq 1 - \Delta(a_1, f)$

corresponding to the cases (a), (b) *or* (c), (d), (e).

We need the following Lemma which is quite analogous to the expository Lemma in [1].

Here the author should like to thank sincerely Prof. M. Ozawa for his kind suggestions.

LEMMA. *Let g_1, g_2 be two transcendental entire functions of finite order satisfying $\alpha g_1+\beta g_2=1$, $\alpha\beta\neq 0$. Then*

$$\Delta(0, g_1)+\delta(0, g_2)\leqq 1.$$

Proof. Suppose that $\Delta(0, g_1)+\delta(0, g_2)>1$. Therefore we have

$$\delta(\infty, g_2)+\Delta\left(\frac{1}{\beta}, g_2\right)+\delta(0, g_2)>2,$$

which contradicts the Nevanlinna defect relation

$$\Delta(c_1, f)+\sum_{j=2}^{q}\delta(c_j, f)\leqq 2,$$

where the c_j are any q ($\geqq 3$) distinct complex numbers.

§2. **Proof of Theorem 1.** Niino-Ozawa's argument does work in our case. We firstly have

$$F(z, a_j)=g_j, \qquad j=1, 2, 3$$

and

$$\sum_{j=1}^{4}\alpha_j g_j=1, \tag{1}$$

where

$$\alpha_1=\frac{1}{(a_1-a_2)(a_1-a_3)}, \quad \alpha_2=-\frac{1}{(a_1-a_2)(a_2-a_3)}, \quad \alpha_3=\frac{1}{(a_1-a_3)(a_2-a_3)}.$$

Now we suppose that all $g_j(z)$, $j=1, 2, 3$, are transcendental. Differentiating both sides of (1) we have

$$\sum_{j=1}^{3}\alpha_j\frac{g_j'}{g_j}g_j=0, \qquad \sum_{j=1}^{3}\alpha_j\frac{g_j''}{g_j}g_j=0. \tag{2}$$

Assuming that g_1, g_2, g_3 are linearly independent, we have

$$g_1=\frac{\Delta_1}{\alpha_1\Delta}, \qquad g_2=\frac{\Delta_2}{\alpha_2\Delta},$$

where

$$\Delta=\begin{vmatrix} 1 & 1 & 1 \\ \dfrac{g_1'}{g_1} & \dfrac{g_2'}{g_2} & \dfrac{g_3'}{g_3} \\ \dfrac{g_1''}{g_1} & \dfrac{g_2''}{g_2} & \dfrac{g_3''}{g_3} \end{vmatrix} \neq 0,$$

$$\Delta_1=\frac{g_2'}{g_2}\frac{g_3''}{g_3}-\frac{g_3'}{g_3}\frac{g_2''}{g_2}, \qquad \Delta_2=\frac{g_1''}{g_1}\frac{g_3'}{g_3}-\frac{g_1'}{g_1}\frac{g_3''}{g_3}.$$

Let

$$2\mu(r, A)=\frac{1}{2\pi}\int_0^{2\pi}\log A\,d\theta, \qquad A=\max(1, |A_1|, |A_2|).$$

By Valiron's theorem [3]

$$|T(r, f)-\mu(r, A)|=O(1).$$

Further we have

$$2\mu(r, A)=2\mu(r, g)+O(1), \qquad g=\max(1, |g_1|, |g_2|)$$

Hence

$$\log g=\log\max\left(1, \frac{|\Delta_1|}{|\alpha_1\Delta|}, \frac{|\Delta_2|}{|\alpha_2\Delta|}\right)$$

$$\leqq \log^+|\Delta_1|+\log^+|\Delta_2|+\log^+\frac{1}{|\Delta|}+O(1).$$

Thus

$$2\mu(r, g)\leqq m(r, \Delta_1)+m(r, \Delta_2)+m\left(r, \frac{1}{\Delta}\right)+O(1)$$

$$\leqq \sum_{j=1}^{3}N(r; 0, g_j)+o\left(\sum_{j=1}^{3}m(r, g_j)\right)$$

without exceptional set. Further for $j=1, 2$

$$m(r, g_j)\leqq m(r, g)=2\mu(r, g)$$

and

$$m(r, g_3)\leqq m(r, g_1)+m(r, g_2)+O(1)$$

$$\leqq 4\mu(r, g)+O(1).$$

Hence

$$\sum_{j=1}^{3}m(r, g_j)\leqq 8\mu(r, g)+O(1).$$

Then we have

$$2\mu(r, g) \leqq \sum_{j=1}^{3} N(r; 0, g_j) + o(\mu(r, g))$$

i.e.

$$1 \leqq \sum_{j=1}^{3} \frac{N(r; 0, g_j)}{2\mu(r, g)} + o(1)$$

without exceptional set.

By the definition of Valiron deficiency

$$\varliminf_{r\to\infty} \frac{N(r; 0, g_1)}{2\mu(r, g)} = \varliminf_{r\to\infty} \frac{N(r; a_1, f)}{\mu(r, g)} = \varliminf_{r\to\infty} \frac{N(r; a_1, f)}{T(r, f)}$$
$$= 1 - \Delta(a_1, f),$$

we have

$$1 \leqq \varliminf_{r\to\infty} \frac{N(r; 0, g_1)}{2\mu(r, g)} + \sum_{j=2}^{3} \varlimsup_{r\to\infty} \frac{N(r; 0, g_j)}{2\mu(r, g)}$$
$$= 1 - \Delta(a_1, f) + \sum_{j=2}^{3} (1 - \delta(a_j, f)),$$

which implies

$$\Delta(a_1, f) + \sum_{j=2}^{3} \delta(a_j, f) \leqq 2.$$

This is absurd. Therefore g_1, g_2, g_3 are linearly dependent. Thus we have

$$\beta_1 g_1 + \beta_2 g_2 + \beta_3 g_3 = 0.$$

The above equation together with (1) gives

(A) $$\gamma_1 g_1 + \gamma_2 g_2 = 1, \qquad \gamma_1 \gamma_2 \neq 0,$$

(B) $$\gamma_2 g_2 + \gamma_3 g_3 = 1, \qquad \gamma_2 \gamma_3 \neq 0.$$

In the first place we consider the case (A). Now

$$m(r, g_1) \leqq m(r, g) = 2\mu(r, g)$$
$$\leqq \frac{1}{2\pi} \int_0^{2\pi} \log^+ \max(|\gamma_1 g_1|, 1 + |\gamma_1 g_1|) d\theta + O(1)$$
$$\leqq \frac{1}{2\pi} \int_0^{2\pi} \log^+ |g_1| d\theta + O(1)$$
$$= m(r, g_1) + O(1).$$

Hence

$$|m(r, g_1)-2\mu(r, g)|=O(1).$$

Further evidently

$$|m(r, g_1)-m(r, g_2)|=O(1).$$

Therefore

$$\varlimsup_{r\to\infty}\frac{N(r; a_1, f)}{T(r, f)}=\varlimsup_{r\to\infty}\frac{N(r; 0, g_1)}{2\mu(r, g)}=\varlimsup_{r\to\infty}\frac{N(r; 0, g_1)}{m(r, g_1)}.$$

This implies

$$\Delta(a_1, f)=\Delta(0, g_1).$$

Similarly we get

$$\delta(a_2, f)=\delta(0, g_2).$$

Hence we have

$$\Delta(0, g_1)+\delta(0, g_2)=\Delta(a_1, f)+\delta(a_2, f)>1.$$

By virtue of Lemma we have a contradiction.

Secondly we consider the case (B). Similarly with the case (A), we have

$$\delta(a_j, f)=\delta(0, g_j), \qquad j=2, 3.$$

Hence

$$\delta(0, g_2)+\delta(0, g_3)=\delta(a_2, f)+\delta(a_3, f)>1,$$

which gives similarly a contradiction.

These contradictions give that one of $\{g_j\}_{j=1}^{3}$ must be a polynomial: i.e. (1A) g_1 is a polynomial, or (1B) g_2 is a polynomial.

Consider the case (1A). Assume that $\alpha_1 g_1 \not\equiv 1$.

From

$$\alpha_2 g_2+\alpha_3 g_3=1-\alpha_1 g_1,$$

we get

$$\alpha_2\frac{g_2'}{g_2}g_2+\alpha_3\frac{g_3'}{g_3}g_3=-\alpha_1 g_1'.$$

Thus, noticing that g_1 is a polynomial

$$m(r, g_2)\leqq N(r; 0, g_2)+N(r; 0, g_3)+o(\mu(r, g)).$$

Then

$$1 \leqq \varlimsup_{r\to\infty} \sum_{j=2}^{3} \frac{N(r; 0, g_j)}{m(r, g_2)} \leqq \sum_{j=2}^{3} (1-\delta(a_j, f))$$

by virtue of $|m(r, g_2)-2\mu(r, g)|=O(\log r)$. Hence we have

$$\delta(a_2, f)+\delta(a_3, f)\leqq 1,$$

which is a contradiction. Hence g_2 and g_3 are linearly dependent. Consequently we have that g_2 and g_3 are polynomials respectively. Thus A_1 and A_0 are polynomials, which is absurd. This leads us to the following fact: $\alpha_1 g_1 \equiv 1$. Hence

$$\delta(a_2, f)=\delta(a_3, f)>\frac{1}{2},$$

and $\Delta(a_1, f)=\delta(a_1, f)=1$. This is the desired result (a). Next consider the case (1B). Assume that $\alpha_2 g_2 \not\equiv 1$. Then we can obtain similarly to the case (1A)

$$m(r, g_1)\leqq N(r; 0, g_1)+N(r; 0, g_3)+o(\mu(r, g))$$

without exceptional set. Then we have

$$\Delta(a_1, f)+\delta(a_3, f)\leqq 1,$$

which is a contradiction. Consequently we get $\alpha_2 g_2 \equiv 1$. Hence we have

$$\delta(a_1, f)=\delta(a_3, f) \quad \text{and} \quad \Delta(a_1, f)=\Delta(a_3, f)>\frac{1}{2},$$

which is the desired result (b).

Assume that there is another deficiency $\delta(a_4, f)$ satisfying

$$\Delta(a_1, f)+\delta(a_2, f)+\delta(a_4, f)>2.$$

Then we have

$$(a_2-a_4)g_1+(a_4-a_1)g_2+(a_1-a_2)g_4=-(a_2-a_4)(a_4-a_1)(a_1-a_2).$$

By the above discussion we have in the case (a)

$$(a_2-a_3)g_1=-(a_2-a_3)(a_3-a_1)(a_1-a_2),$$

which shows

$$(a_4-a_1)g_2+(a_1-a_2)g_4=-(a_2-a_4)(a_4-a_3)(a_1-a_2)\neq 0.$$

This implies a contradiction. Hence

$$\delta(a_4, f)\leqq 2-\Delta(a_1, f)-\delta(a_2, f)$$
$$=1-\delta(a_2, f).$$

In the case (b) we have

$$(a_3-a_1)g_2=-(a_3-a_1)(a_1-a_2)(a_2-a_3),$$

then

$$\delta(a_4,f)\leqq 2-\varDelta(a_1,f)-\delta(a_2,f)=1-\varDelta(a_1,f).$$

§3. **Proof of Theorem 2.** We put

$$g_j=F(z,a_j),\qquad j=1,2,3,4,$$

and assume that all g_j, $j=1,2,3,4$, are transcendental. Now we have

$$\sum_{j=1}^{4}\alpha_j g_j=1, \tag{3}$$

where

$$\alpha_j=1\Big/\prod_{\substack{k=1\\k\neq j}}^{4}(a_j-a_k),\qquad j=1,2,3,4.$$

Assume that g_1, g_2, g_3 and g_4 are linearly independent. Then the Wronskian does not vanish identically. By differentiating (3) we have

$$\sum_{j=1}^{4}\alpha_j\frac{g_j^{(\mu)}}{g_j}g_j=0,\qquad \mu=1,2,3. \tag{4}$$

We can solve (3) and (4). Then we get

$$g_j=\frac{\varDelta_j}{\alpha_j\varDelta},\qquad j=1,2,3,4,$$

where

$$\varDelta=\frac{1}{\prod_{j=1}^{4}g_j}\begin{vmatrix} g_1 & g_2 & g_3 & g_4\\ g_1' & g_2' & g_3' & g_4'\\ g_1'' & g_2'' & g_3'' & g_4''\\ g_1''' & g_2''' & g_3''' & g_4''' \end{vmatrix}$$

and $\varDelta_j$ is a polynomial of

$$\frac{g_1^{(\mu)}}{g_1},\ \cdots,\ \frac{g_{j-1}^{(\mu)}}{g_{j-1}},\ \frac{g_{j+1}^{(\mu)}}{g_{j+1}},\ \cdots,\ \frac{g_4^{(\mu)}}{g_4},\qquad \mu=1,2,3.$$

Then

$$\log g = \log \max(1, |g_1|, |g_2|, |g_3|)$$

$$\leqq \log^+ \frac{1}{|\varDelta|} + \sum_{j=1}^{3} \log^+ |\varDelta_j| + O(1).$$

Hence

$$3\mu(r, g) = \frac{1}{2\pi}\int_0^{2\pi} \log g\, d\theta = m(r, g)$$

$$\leqq m\left(r, \frac{1}{\varDelta}\right) + \sum_{j=1}^{3} m(r, \varDelta_j) + O(1)$$

$$\leqq \sum_{j=1}^{4} N(r; 0, g_j) + o\left(\sum_{j=1}^{4} m(r, g_j)\right)$$

without exceptional set. Further we have

$$\sum_{j=1}^{4} m(r, g_j) \leqq 6m(r, g) + O(1)$$

and

$$\mu(r, A) = \mu(r, g) + O(1),$$

where

$$A = \max(1, |A_0|, |A_1|, |A_2|).$$

Therefore

$$1 \leqq \varliminf_{r\to\infty} \frac{N(r; 0, g_1)}{3\mu(r, g)} + \sum_{j=2}^{4} \varlimsup_{r\to\infty} \frac{N(r; 0, g_j)}{3\mu(r, g)}$$

$$= 1 - \varDelta(a_1, f) + \sum_{j=2}^{4} (1 - \delta(a_j, f))$$

by virtue of Valiron's theorem. Thus

$$\varDelta(a_1, f) + \sum_{j=2}^{4} \delta(a_j, f) \leqq 3,$$

which is a contradiction. Hence we have the linear dependency of g_1, g_2, g_3 and g_4, that is,

$$\sum_{j=1}^{4} \beta_j g_j = 0$$

with constant $\{\beta_j\}$ not all zero. Here at least two of $\{\beta_j\}$ are not zero. Hence we may assume that

(I) $\beta_3\beta_4\neq 0$ and $\beta_4=\alpha_4$, or (II) $\beta_1\beta_2\neq 0$ and $\beta_1=\alpha_1$.

We divide the cases (I), (II) into several subcases as follows:

(I)

Case 1) $\beta_1\beta_2\neq 0$.

(i) $\alpha_1\neq\beta_1$, $\alpha_2\neq\beta_2$, $\alpha_3\neq\beta_3$,

(ii) $\alpha_1\neq\beta_1$, $\alpha_2\neq\beta_2$, $\alpha_3=\beta_3$, $\alpha_1\beta_2\neq\alpha_2\beta_1$,

(iii) $\alpha_1\neq\beta_1$, $\alpha_2\neq\beta_2$, $\alpha_3=\beta_3$, $\alpha_1\beta_2=\alpha_2\beta_1$,

(iv) $\alpha_1\neq\beta_1$, $\alpha_2=\beta_2$, $\alpha_3=\beta_3$.

Case 2) $\beta_1=0$, $\beta_2\neq 0$.

(i) $\alpha_2\neq\beta_2$, $\alpha_3\neq\beta_3$,

(ii) $\alpha_2\neq\beta_2$, $\alpha_3=\beta_3$,

(iii) $\alpha_2=\beta_2$, $\alpha_3=\beta_3$.

Case 3) $\beta_1=\beta_2=0$.

(i) $\alpha_3\neq\beta_3$,

(ii) $\alpha_3=\beta_3$.

(II)

Case 4) $\beta_3\beta_4\neq 0$.

(i) $\alpha_2\neq\beta_2$, $\alpha_3\neq\beta_3$, $\alpha_4\neq\beta_4$,

(ii) $\alpha_2=\beta_2$, $\alpha_3\neq\beta_3$, $\alpha_4\neq\beta_4$, $\alpha_3\beta_4\neq\alpha_4\beta_3$,

(iii) $\alpha_2=\beta_2$, $\alpha_3\neq\beta_3$, $\alpha_4\neq\beta_4$, $\alpha_3\beta_4=\alpha_4\beta_3$,

(iv) $\alpha_2=\beta_2$, $\alpha_3\neq\beta_3$, $\alpha_4=\beta_4$.

Case 5) $\beta_3=0$, $\beta_4\neq 0$.

(i) $\alpha_2\neq\beta_2$, $\alpha_4\neq\beta_4$,

(ii) $\alpha_2=\beta_2$, $\alpha_4\neq\beta_4$,

(iii) $\alpha_2=\beta_2$, $\alpha_4=\beta_4$.

Case 6) $\beta_3=\beta_4=0$.

(i) $\alpha_2\neq\beta_2$,

(ii) $\alpha_2=\beta_2$.

The cases 1), (iv); 2), (iii); 4), (iv) and 5), (iii) give trivially the desired result.

The cases 1), (i), (ii); 2), (i), (ii) and 3), (i) lead to an identity of the following type:

(A) $$\gamma_1 g_1+\gamma_2 g_2+\gamma_3 g_3=1, \qquad \gamma_1\gamma_2\gamma_3\neq 0.$$

The cases 4), (i), (ii); 5), (i), (ii) and 6), (i) also lead to an identity of the following type:

(A)′ $$\gamma_2 g_2+\gamma_3 g_3+\gamma_4 g_4=1, \qquad \gamma_2\gamma_3\gamma_4\neq 0.$$

The cases 1), (iii) and 4), (iii) lead to

(B) $$\gamma_1 g_1+\gamma_2 g_2=1, \quad \gamma_3 g_3+\gamma_4 g_4=1, \qquad \gamma_1\gamma_2\gamma_3\gamma_4\neq 0.$$

The cases 3), (ii) and 6), (ii) lead to

(C) $$\alpha_1 g_1+\alpha_2 g_2=1, \quad \alpha_3 g_3+\alpha_4 g_4=0$$

and

(C)′ $$\alpha_1 g_1+\alpha_2 g_2=0, \quad \alpha_3 g_3+\alpha_4 g_4=1$$

respectively.

By our assumption the cases (C) and (C)′ may be omitted.

In the first place we suppose that (A) occurs. Assuming the linear independency of g_1, g_2, g_3, we can apply the same method as in the above and then we arrive at a contradiction. Hence g_1, g_2, g_3 are linearly dependent. This and (A) imply

(a) $$\delta_1 g_1+\delta_2 g_2=1, \quad \delta_1\delta_2\neq 0, \quad \text{or}$$

(b) $$\delta_2 g_2+\delta_3 g_3=1, \quad \delta_2\delta_3\neq 0.$$

Considering the cases (a) or (b), we arrive at a contradiction in either case by the Lemma. Hence we can say that one of $\{g_j\}_{j=1,2,3}$ is a polynomial.

Similarly consider the case (A)′, we can obtain that g_2, g_3, g_4 are linearly dependent. This and (A)′ imply for example

$$\delta_3 g_3+\delta_4 g_4=1, \quad \delta_3\delta_4\neq 0.$$

In this case we have the same desired result.

Secondly we suppose that (B) occurs. Let $G=\max(1, |g_1|, |g_3|)$. Then

$$m(r, g)\leqq m(r, G)+O(1)\leqq m(r, g)+O(1).$$

Further

$$m(r, g_j)\leqq m(r, G)+O(1), \qquad j=2, 4.$$

Hence

$$m(r, G)\leqq m(r, g_1)+m(r, g_3)$$
$$\leqq \sum_{j=1}^{4} N(r; 0, g_j)+o(m(r, G))$$

without exceptional set. This leads to the following contradictory inequality

$$\Delta(a_1, f)+\sum_{j=2}^{4}\delta(a_j, f)\leqq 3.$$

Thus either g_1, g_2 or g_3, g_4 are proportional, which is absurd.

§4. **Proof of Theorem 3.** We set also

$$g_j=F(z, a_j), \qquad j=1, \cdots, 5,$$

and

(5) $$\beta_1 g_1+\beta_2 g_2+\beta_3 g_3+\beta_5 g_5=1$$

and suppose that all $g_j(z)$, $j=1, \cdots, 5$, are transcendental. Therefore the reasoning in the proof of Theorem 2 leads to the following cases:

(i) (C) and (5), (ii) (C)′ and (5).

Since the case (ii) can be handled quite similarly, we only consider the case (i). Since $\alpha_1\beta_2 \neq \beta_1\alpha_2$, we have

$$(\alpha_1\beta_2-\alpha_2\beta_1)g_2+\alpha_1\beta_3 g_3+\alpha_1\beta_5 g_5=\alpha_1-\beta_1 \neq 0$$

or

$$(\alpha_2\beta_1-\alpha_1\beta_2)g_1+\alpha_2\beta_3 g_3+\alpha_2\beta_5 g_5=\alpha_2-\beta_2 \neq 0.$$

Thus we obtain a desired contradiction in either case. Hence one of $\{g_j\}$ is a polynomial.

Consequently we have the following fact: At least one of $\{g_j\}_{j=1}^5$ must be a polynomial, that is

(3A) g_1 is a polynomial, or (3B) g_2 is a polynomial, or (3C) g_4 is a polynomial.

Firstly we consider the case (3A). Further assume that the other g_j are transcendental. If $\alpha_1 g_1 \not\equiv 1$, then the identity (3) implies

$$\alpha_2 g_2+\alpha_3 g_3+\alpha_4 g_4=1-\alpha_1 g_1.$$

By the same method as in the proof of Theorem 1, this case gives a contradiction. Thus we have the existence of a polynomial among g_2, g_3, g_4. In this case we get

(a) $\delta(a_1, f)=\delta(a_2, f)=1$ for example or

(b) $\delta(a_1, f)=\delta(a_4, f)=1.$

The case (a) leads to

$$\alpha_3 g_3+\alpha_4 g_4=1-\alpha_1 g_1-\alpha_2 g_2,$$

$$\beta_3 g_3+\beta_5 g_5=1-\beta_1 g_1-\beta_2 g_2.$$

By virtue of the argument in the case (1A) of Theorem 1, we have the linear dependency of g_3 and g_4, g_3 and g_5 respectively, that is,

$$\delta(a_3, f)=\delta(a_4, f)=\delta(a_5, f)>\frac{1}{2}.$$

The case (b) leads to

$$\alpha_2 g_2+\alpha_3 g_3=1-\alpha_1 g_1-\alpha_4 g_4=0,$$

$$\beta_2 g_2+\beta_3 g_3+\beta_5 g_5=1-\beta_1 g_1,$$

which yeilds also

$$\delta(a_2, f)=\delta(a_3, f)=\delta(a_5, f)>\frac{1}{2}$$

by virtue of our standard method. If $\alpha_1 g_1 \equiv 1$, then we have

$$\beta_2 g_2+\beta_3 g_3+\beta_5 g_5=1-\frac{\beta_1}{\alpha_1}$$

by (5), where

$$\beta_1=\frac{1}{(a_1-a_2)(a_1-a_3)(a_1-a_5)}.$$

Hence

$$1-\frac{\beta_1}{\alpha_1}=\frac{a_4-a_5}{a_1-a_5}\neq 0.$$

Therefore we can prove the existence of another polynomial among g_2, g_3, g_5. Also we have

(a)′ $\delta(a_1, f)=\delta(a_2, f)=1$ for example or

(b)′ $\delta(a_1, f)=\delta(a_5, f)=1.$

The case (a)′ leads to

$$\alpha_3 g_3+\alpha_4 g_4=-\alpha_2 g_2,$$

$$\beta_3 g_3+\beta_5 g_5=1-\frac{\beta_1}{\alpha_1}-\beta_2 g_2=0,$$

which is absurd.

The case (b)′ also leads to

$$\alpha_2 g_2+\alpha_3 g_3+\alpha_4 g_4=0,$$

$$\beta_2 g_2+\beta_3 g_3=1-\frac{\beta_1}{\alpha_1}-\beta_5 g_5=0.$$

In this case we obtain a part of the desired result:

$$\delta(a_2, f)=\delta(a_3, f)=\delta(a_4, f)>\frac{1}{2}.$$

Secondly we consider the case (3B). Further assume that the other g_j are transcendental. If $\alpha_2 g_2 \not\equiv 1$, then we have the existence of a polynomial among g_1, g_3, g_4 from the identity

$$\alpha_1 g_1+\alpha_3 g_3+\alpha_4 g_4=1-\alpha_2 g_2$$

by the standard method. Thus we get

(c) $$\delta(a_2, f)=\delta(a_4, f)=1 \quad \text{or}$$

(d) $$\delta(a_2, f)=\delta(a_3, f)=1 \quad \text{or} \quad \text{(a)}.$$

The case (c) leads to

$$\alpha_1 g_1+\alpha_3 g_3=1-\alpha_2 g_2-\alpha_4 g_4=0,$$

$$\beta_1 g_1+\beta_3 g_3+\beta_5 g_5=1-\beta_2 g_2,$$

which yeilds

$$\delta(a_1, f)=\delta(a_3, f)=\delta(a_5, f) \quad \text{or}$$

$$\Delta(a_1, f)=\Delta(a_3, f)=\Delta(a_5, f)>\frac{1}{2}.$$

The case (d) leads to

$$\alpha_1 g_1+\alpha_4 g_4=1-\alpha_2 g_2-\alpha_3 g_3=0,$$

$$\beta_1 g_1+\beta_5 g_5=1-\beta_2 g_2-\beta_3 g_3=0,$$

which provides

$$\delta(a_1, f)=\delta(a_4, f)=\delta(a_5, f) \quad \text{or}$$

$$\Delta(a_1, f)=\Delta(a_4, f)=\Delta(a_5, f)>\frac{1}{2}.$$

If $\alpha_2 g_2\equiv 1$, then we have only the following case, that is,

(c)′ $$\delta(a_2, f)=\delta(a_5, f)=1$$

by virtue of the argument in the above case: $\alpha_1 g_1\equiv 1$. In this case we get

$$\delta(a_1, f)=\delta(a_3, f)=\delta(a_4, f) \quad \text{or}$$

$$\Delta(a_1, f)=\Delta(a_3, f)=\Delta(a_4, f)>\frac{1}{2}.$$

Finally we consider the case (3C). If $\alpha_4 g_4\not\equiv 1$, then the reasoning in the above cases leads to the following cases:

(b) or (c).

If $\alpha_4 g_4\equiv 1$, we have

$$\alpha_1 g_1+\alpha_2 g_2+\alpha_3 g_3=0,$$

$$\beta_1 g_1+\beta_2 g_2+\beta_3 g_3+\beta_5 g_5=1.$$

Hence we get

$$\gamma_1 g_1+\gamma_3 g_3+\gamma_5 g_5=1, \qquad \gamma_1\gamma_3\gamma_5\neq 0 \quad \text{or}$$

$$\gamma_2 g_2+\gamma_3 g_3+\gamma_5 g_5=1, \qquad \gamma_2\gamma_3\gamma_5\neq 0.$$

Thus by the standard method we have the existence of a polynomial among g_1, g_3, g_5 or g_2, g_3, g_5 respectively. These cases give

(b) or (c)′ $\qquad \delta(a_3, f)=\delta(a_4, f)=1$ or

(e) $\qquad \delta(a_4, f)=\delta(a_5, f)=1.$

For example the case (e) leads to

$$\gamma_1 g_1+\gamma_3 g_3=1-\gamma_5 g_5=0,$$

$$\alpha_1 g_1+\alpha_2 g_2+\alpha_3 g_3=0,$$

which yields

$$\delta(a_1, f)=\delta(a_2, f)=\delta(a_3, f) \quad \text{or}$$

$$\Delta(a_1, f)=\Delta(a_2, f)=\Delta(a_3, f)>\frac{1}{2}.$$

Thus the proof of Theorem 3 is complete.

§ 5. Applying the method in the proof of Theorem 3, we have the following

THEOREM 4. *Let $f(z)$ be the same as in the theorem* 2. *Let a_1, a_2, a_3, a_4 and a_5 be five different finite numbers satisfying*

$$\delta(a_1, f)+\delta(a_2, f)+\delta(a_3, f)+\Delta(a_4, f)>3,$$

$$\delta(a_1, f)+\delta(a_2, f)+\delta(a_3, f)+\Delta(a_5, f)>3.$$

Then at least two of a_j are Picard exceptional values of f or more precisely it occurs either

(a) $\qquad \delta(a_1, f)=\delta(a_2, f)=1$ *for example and*

$\qquad \Delta(a_3, f)=\Delta(a_4, f)=\Delta(a_5, f)>\frac{1}{2}$ *or*

(b) $\qquad \delta(a_1, f)=\delta(a_4, f)=1$ *for example and*

$\qquad \delta(a_2, f)=\delta(a_3, f)=\delta(a_5, f)>\frac{1}{2}$ *or*

(c) $\qquad \delta(a_4, f)=\delta(a_5, f)=1$ *and*

$\qquad \delta(a_1, f)=\delta(a_2, f)=\delta(a_3, f)>\frac{1}{2}.$

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