KŌDAI MATH. SEM. REP.
23 (1971), 480—485

ON THE UNIQUENESS OF THE EXTREMAL FUNCTION OF HARMONIC LENGTH PROBLEM AND ITS APPLICATION

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1. Landau and Osserman [4] introduced the notion of harmonic length as follows: Let R be an arbitrary Riemann surface, we denote by U_R the family of all functions u harmonic on R satisfying $0<u<1$ on R. Let γ be an arbitrary cycle on R. We call

$$h_R(\gamma)=\sup_{u\in U_R}\int_\gamma *du$$

the harmonic length of γ.

They showed the following

THEOREM A. *Let D be a Dirichlet region on a Riemann surface R and let u be a harmonic measure in D. If γ is homologous in D to a level locus of u, then u is the unique extremal function in determining $h_D(\gamma)$.*

Here a Dirichlet region means a relatively compact region on a Riemann surface whose boundary is regular for the Dirichlet problem, and which has at least two boundary components.

They applied this theorem to problems of conformal rigidity of plane Dirichlet regions.

Recently, Suita and the author [7] have shown the following improvement of theorem A. An English reference will be made to Suita [6].

THEOREM B. *Let R be an arbitrary Riemann surface and let γ be a dividing cycle relative to a regular partition (A, B) on R. If $h_R(\gamma)>0$, the function satisfying*

$$h_R(\gamma)=\int_\gamma *du_0, \qquad u_0\in U_R$$

is unique and coincides with the harmonic measure of B.

In the present paper we shall show the uniqueness of the extremal function in determining the harmonic length of any cycle on a finite Riemann surface. Using the uniqueness, we shall give an elementary proof of a theorem of Huber in the case of finite Riemann surfaces [3]. The author expresses his heartiest thanks to Professor N. Suita for his kind suggestion in preparing this paper.

Received November 16, 1970.

2. Surfaces with which we are concerned are open Riemann surfaces, denoted by F, of finite genus, say $g \geqq 0$, and with a finite number of boundary components, say $m > 0$, each of which consists of an analytic Jordan curve. We assume that F is not conformally equivalent to a disk. Throughout this paper, we shall denote by F such a Riemann surface.

Let $G(p, q)$ be Green's function of F, and let γ be a cycle on F. Then we have

LEMMA 1. [5] *γ is homologous to zero if and only if*

$$d \int_\gamma \frac{\partial}{\partial n_p} G(p, q)\, ds_p \equiv 0$$

where $\partial/\partial n_p$ denotes the inner normal derivative with respect to γ.

THEOREM 1. *If γ is a cycle on F which is not homologous to zero, then there exists the unique extremal function in determining $h_F(\gamma)$, and the extremal function is a harmonic measure on F.*

Proof. We may assume that γ is a closed analytic curve. Let ∂F be the boundary of F. For any function u in U_F, there exists a measurable function $\hat{u}$ defined at almost all points on ∂F, satisfying $0 \leqq \hat{u} \leqq 1$ and

$$u(p) = \frac{1}{2\pi} \int_{\partial F} \hat{u}(q) \frac{\partial}{\partial n_q} G(p, q)\, ds_q.$$

Let F_0 be a compact subregion of F which contains γ, and such that each component of $F - \bar{F}_0$ is noncompact. We have

$$\begin{aligned} \int_\gamma *du &= \frac{1}{2\pi} \int_\gamma \frac{\partial}{\partial n_p} \int_{\partial F} \hat{u}(q) \frac{\partial}{\partial n_q} G(p, q)\, ds_q ds_p \\ &= \frac{1}{2\pi} \int_{\partial F} \hat{u}(q) \frac{\partial}{\partial n_q} \int_\gamma \frac{\partial}{\partial n_p} G(p, q)\, ds_p ds_q. \end{aligned}$$

Since

$$d \int_\gamma \frac{\partial}{\partial n_p} G(p, q)\, ds_p$$

is a harmonic differential on $F - \gamma$, it may be assumed that

$$W(q) = \int_\gamma \frac{\partial}{\partial n_p} G(p, q)\, ds_p$$

is a function harmonic on each component of $F - \bar{F}_0$, and that W is equal to a constant on ∂F. Indeed, if q is in ∂F, then $G(p, q) = G(q, p) = 0$ for all p in F_0. Therefore we have a harmonic continuation of W across ∂F.

By lemma 1 dW has at most a finite number of zeros on ∂F, therefore $(\partial/\partial n_q) W(q)$ has at most a finite number of zeros on ∂F, since W is a constant on each component of ∂F.

Let $\hat{u}_0$ be a function on ∂F, such that $\hat{u}_0(q)=1$, if $(\partial/\partial n_q)W(q)\geqq 0$ and $\hat{u}_0(q)=0$, if $(\partial/\partial n_q)W(q)<0$. We set

$$u_0(p)=\frac{1}{2\pi}\int_{\partial F}\hat{u}_0(q)\frac{\partial}{\partial n_q}G(p, q)\,ds_q.$$

It is to be noted that u_0 does not degenerate to a constant because of

$$\int_{\partial F}\frac{\partial}{\partial n_q}W(q)\,ds_q=0.$$

Evidently we have that u_0 is in U_F.

For any u in U_F, let $\hat{u}$ be a boundary function associated with u. We have

$$\begin{aligned}\int_\gamma *du_0 &= \frac{1}{2\pi}\int_{\partial F}\hat{u}_0(q)\frac{\partial}{\partial n_q}W(q)\,ds_q \\ &\geqq \frac{1}{2\pi}\int_{\partial F}\hat{u}(q)\frac{\partial}{\partial n_q}W(q)\,ds_q=\int_\gamma *du.\end{aligned}$$

This shows that u_0 is an extremal function.

If there is another extremal function, i.e. if there is u_1 in U_F such that $\int_\gamma *du_1=\int_\gamma *du_0$, then we have

$$\begin{aligned}0&=\int_\gamma *d(u_0-u_1) \\ &=\frac{1}{2\pi}\int_{\partial F}(\hat{u}_0-\hat{u}_1)\frac{\partial}{\partial n_q}W(q)\,ds_q.\end{aligned}$$

By the definition of $\hat{u}_0$ we have $\hat{u}_0\equiv\hat{u}_1$ almost everywhere on ∂F. Then we have established $u_0\equiv u_1$ in F. From the construction of u_0, we have assured that u_0 is a harmonic measure of F. This completes the proof.

3. In this section we shall give an alternative proof of the following theorem due to Huber [3].

THEOREM 2. *Let f be an analytic mapping of F into itself. Then either f is an automorphism of F, or else f takes a cycle, which is not homologous to zero, to a cycle which is homologous to zero.*

To do this we prepare two lemmas.

LEMMA 2. *Let f be an analytic mapping of F into itself. If there is a cycle γ on F which is not homologous to zero satisfying $h_F(\gamma)=h_F(f(\gamma))$, then f is an automorphism of F.*

Proof. By theorem 1 there are uniquely determined harmonic measures u and ω satisfying

$$h_F(\gamma)=\int_\gamma *du$$

and

$$h_F(f(\gamma))=\int_{f(\gamma)} *d\omega=\int_\gamma *d\omega\circ f.$$

By virtue of theorem 1 we have

$$u\equiv\omega\circ f.$$

We show that f is a boundary preserving mapping. Indeed, let $\{p_n\}$ be a sequence of points in F tending to a point on ∂F, say p_0. If there exists $\lim_{n\to\infty} u(p_n)$ which equals either 0 or 1, then $\lim_{n\to\infty}\omega\circ f(p_n)$ equals $\lim_{n\to\infty} u(p_n)$. This shows that $f(p_n)$ tends to the boundary. Thus it is to be observed only the case when $u(p_n)$ tends to neither 0 nor 1. From the context of the proof of theorem 1, the set of all accumulation points of such sequences $\{p_n\}$ is a finite set. Therefore, we may choose a suitable uniformizer $A(z)$ for the double of F with domain $\Delta=\{|z|<1\}$, which takes 0 into p_0, real axis into ∂F and $\Delta_1=\Delta\cap\{\operatorname{Im} z>0\}$ into F. Furthermore we may assume $u\circ A(x)=0$ for $x>0$ and $u\circ A(x)=1$ for $x<0$. If it is not the case we shall take $1-u$ instead of u.

We introduce an auxiliary function on Δ_1

$$B(z)=u\circ A(z)-\frac{\arg z}{\pi}.$$

Evidently, B is a bounded harmonic function on Δ_1 and $B=0$ on $\Delta\cap\{\operatorname{Im} z=0\}-\{0\}$. Let $v\circ A$ be a conjugate harmonic function of $u\circ A$ in Δ_1 then

$$v\circ A(z)=B^*(z)+\frac{\log|z|}{\pi}$$

where B^* is a conjugate harmonic function of B. Then we have $\lim_{\Delta_1\ni z\to 0} v\circ A(z)=-\infty$, that is, $\lim_{n\to\infty} v(p_n)=-\infty$.

If there is a subsequence $\{p_{n_j}\}$ of $\{p_n\}$ such that $\lim_{j\to\infty} f(p_{n_j})=q$ is in F, we define a conjugate harmonic function μ of ω in a neighborhood of q. Then we have

$$\lim_{j\to\infty}\mu\circ f(p_{n_j})=\mu(q).$$

But $\mu(q)\neq-\infty$ in any choice of μ. This contradicts the fact $\lim_{n\to\infty} v(p)=-\infty$. This shows that $f(p_n)$ tends to the boundary.

Since f is a boundary preserving mapping there exists a positive integer k such that f is a k-to-one analytic mapping of F onto itself [2]. Therefore, we may understand that f is the restriction of an analytic mapping of the double of F onto itself to F. From the Riemann-Hurwitz relation, we conclude necessarily $k=1$. This completes the proof.

LEMMA 3. *For any fixed positive number M, there are at most a finite number* *of homology classes on F whose harmonic lengths are less than M.*

Proof. Let Γ_j, $j=1, \cdots, 2g+m-1$, be a homology basis of cycles on F. By lemma 1, there exist functions v_i, $i=1, \cdots, 2g+m-1$ bounded and harmonic on F satisfying

$$\int_{\Gamma_j} *dv_i = \begin{cases} 1 & (i=j), \\ 0 & (i \neq j). \end{cases}$$

We set

$$u_i = \frac{v_i - \inf_F v_i}{\sup_F v_i - \inf_F v_i}.$$

Then u_i is in U_F. Set

$$p_i = \int_{\Gamma_i} *du_i.$$

On the other hand, for any cycle γ on F there is a uniquely determined system $\{c_i\}_{i=1}^{2g+m-1}$ such that γ is homologous to $\sum_{i=1}^{2g+m-1} c_i \Gamma_i$. If there is an index j such that $p_j |c_j| \geqq M$, then we have

$$h_F(\gamma) \geqq \left| \int_{\Sigma c_i \Gamma_i} *du_j \right| = p_j |c_j| \geqq M.$$

This completes the proof.

Proof of Theorem 2. In the present case $h_F(\gamma)=0$ if and only if γ is homologous to zero by lemma 1. By lemma 3 there is a cycle γ_0 which is not homologous to zero such that

$$h_F(\gamma) \geqq h_F(\gamma_0)$$

for each cycle γ which is not homologous to zero.

We note

$$h_F(f(\gamma)) \leqq h_F(\gamma)$$

for any cycle γ [4]. Hence, if f takes any cycle, which is not homologous to zero, to a cycle which is not homologous to zero, then we have

$$h_F(f(\gamma_0)) = h_F(\gamma_0).$$

Then, from lemma 2 we have concluded the desired assertion.

4. Remarks.

REMARK 1. In lemma 2, if F has the non-commutative fundamental group then by virtue of the classical theorem of Schwarz the group of automorphisms of the double of F has finite order. Since an automorphism of F is the restriction of an automorphism of the double of F to F, we have concluded that the group of automorphisms of F has finite order.

REMARK 2. In theorem 2, if f is not an automorphism then for any cycle γ there is an integer n depending on γ such that $f_n(\gamma)$ is homologous to zero, where f_n is the n-th iteration of f. Indeed, if we set $M=h_F(\gamma)$, then there is a number n such that $f_n(\gamma)$ is homologous to zero by lemma 3 and the hypothesis on f. Huber proved, however, that n is determined independently on γ and it is the Betti number of F [3].

REMARK 3. We have used homology in sections 3 and 4 while Huber used homotopy. But it is not intrinsic in our situation. For, in the hypothesis in lemma 2, if we replace $h_F(\gamma)=h_F(f(\gamma))$ by the fact that γ is homologous to $f(\gamma)$ we can prove it by use of a result of Heins [1] cited below. Using this we can establish that f, which is not an automorphism of F, takes a cycle which is not homotopic to a point to a cycle which is homotopic to a point. The replacement of the hypothesis in lemma 2 does not cause any trouble in proving theorem 2. But it is not so elementary as our method and slightly complicated in proving the theorem. For this reason we have abandoned this method.

The result of Heins [1]: Let R be a noncompact Riemann surface whose fundamental group is non-commutative, let f be an analytic mapping of R into itself and let K_1 and K_2 be given compact subsets of R. If f neither possesses a fixed point nor has finite period, then $f_n(K_1)$ lies in one and the same component of $R-K_2$ for n sufficiently large.

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