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THE LAW OF THE ITERATED LOGARITHM FOR STATIONARY PROCESSES SATISFYING MIXING CONDITIONS

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0. Summary.

The law of the iterated logarithm for various stochastic sequences has long been studied by many authors. Recently, Iosifescu proved in [5] that the law holds for stationary sequences satisfying the uniformly strong mixing condition and Reznik showed in [8] that the one is also valid for stationary processes satisfying the strong mixing condition. But, the conditions used in [5] and [8] are slightly stringent. The purpose of this paper is to weaken those conditions, that is, to prove the law under as similar as possible requirements to the conditions in [3].

1. Definitions and notations.

Let $\{x_j, -\infty<j<\infty\}$ be processes which are strictly stationary and satisfy one of the following conditions:

$$\text{(I)} \qquad \sup_{A\in\mathcal{M}_{-\infty}^{k}, B\in\mathcal{M}_{k+n}^{\infty}} \frac{1}{P(A)}|P(A\cap B)-P(A)P(B)|=\varphi(n)\to 0 \ (n\to\infty)$$

or

$$\text{(II)} \qquad \sup_{A\in\mathcal{M}_{-\infty}^{k}, B\in\mathcal{M}_{k+n}^{\infty}} |P(A\cap B)-P(A)P(B)|=\alpha(n)\to 0 \ (n\to\infty),$$

where $\mathcal{M}_a^b$ denotes the σ-algebra generated by events of the type

$$\{(x_{i_1}, \cdots, x_{i_k})\in E\}, \qquad a\leqq i_1<\cdots<i_k\leqq b$$

and E is a k-dimensional Borel set. In line with [4], we shall call Condition (I) the uniformly strong mixing (u.s.m.) condition and (II) the strong mixing (s.m.) codition.

In what follows, we assume that all processes $\{x_j\}$ are strictly stationary, $Ex_j=0$ and $Ex_j^2<\infty$. We shall agree to denote by the letter K_i a quantity bounded in absolute value.

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2. A sufficient condition for the validity of the law of the iterated logarithm.

In this and next sections, we write

$$S_n = x_1 + \cdots + x_n, \qquad \sigma_n^2 = \mathrm{var}\,(S_n)$$

and put

$$\sigma^2 = Ex_0^2 + 2\sum_{j=1}^{\infty} Ex_0 x_j$$

if the series converges. We shall use σ^2 only when σ^2 is positive.

THEOREM 1. *Let the strictly stationary process $\{x_j\}$ satisfy the s.m. condition. Suppose that $\sum \alpha(n) < \infty$ and*

(1) $$\sigma_n^2 = n\sigma^2(1+o(1)) \qquad (\sigma^2 > 0).$$

Then, the process $\{x_j\}$ obeys the law of the iterated logarithm, if the following requirements are fulfilled for some $\rho > 0$ and for all sufficiently large n:

(i) $$\sup_{-\infty < z < \infty} |P(S_n < z\sigma\sqrt{n}) - \Phi(z)| = O\left(\frac{1}{(\log n)^{1+\rho}}\right)$$

where

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2}\, dt$$

(ii) $$P(\max_{1 \leqq j \leqq n} |S_j| \geqq b\chi(n)) = O\left(\frac{1}{(\log n)^{1+\rho}}\right)$$

where $b > 1$ is an arbitrary number and

(2) $$\chi(n) = (2\sigma^2 n \log\log n)^{1/2}.$$

Proof. We will use the method of the proof in [7]. The assertion will be proved if we show that for any $\varepsilon > 0$

(3) $$P(|S_n| > (1+\varepsilon)\chi(n) \text{ i.o.}) = 0$$

and

(4) $$P(|S_n| > (1-\varepsilon)\chi(n) \text{ i.o.}) = 1.$$

Firstly, we shall prove (3). For an arbitrarily chosen positive number τ, there exists a non-decreasing sequence of positive integers such that

(5) $$(n_k - 1)\sigma^2 \leqq (1+\tau)^k < n_k\sigma^2$$

for $k = k_0+1,\ k_0+2, \cdots$, where k_0 is a positive integer. So, for all sufficiently large k

$$n_k \sim \frac{1}{\sigma^2}(1+\tau)^k \tag{6}$$

and

$$n_k - n_{k-1} = n_k\left(1-\frac{n_{k-1}}{n_k}\right) \sim n_k \frac{\tau}{1+\tau}. \tag{7}$$

From (ii)

$$P(\max_{1\leqq j\leqq n_k}|S_j|>(1+\gamma)\chi(n_k)) \leqq K(\log n_k)^{-(1+\rho)} < K[k\log(1+\tau)]^{-(1+\gamma_1)}$$

for any $\gamma(>0)$, γ_1 $(0<\gamma_1<\rho)$ and for all k sufficiently large. Thus

$$\sum_k P(\max_{1\leqq j\leqq n_k}|S_j|>(1+\gamma)\chi(n_k))<\infty. \tag{8}$$

We note here that for all sufficiently large k

$$\frac{\chi(n_k)}{\chi(n_{k-1})} < \sqrt{1+2\tau}.$$

For a fixed number $\gamma(0<\gamma<\varepsilon)$, choose a positive constant τ such that

$$\frac{1+\varepsilon}{\sqrt{1+2\tau}} > 1+\gamma.$$

Then, from the Borel-Cantelli lemma and (8), we have

$$\begin{aligned} P(|S_n|>(1+\varepsilon)\chi(n) \text{ i.O.}) &\leqq P(\max_{n_{k-1}\leqq n\leqq n_k}|S_n|>(1+\varepsilon)\chi(n_{k-1}) \text{ i.o.}) \\ &\leqq P(\max_{1\leqq n\leqq n_k}|S_n|>(1+\varepsilon)\chi(n_{k-1}) \text{ i.o.}) \\ &\leqq P\left(\max_{1\leqq n\leqq n_k}|S_n|>\frac{1+\varepsilon}{\sqrt{1+2\tau}}\,\chi(n_{k-1}) \text{ i.o.}\right) \\ &\leqq P(\max_{1\leqq n\leqq n_k}|S_n|>(1+\gamma)\chi(n_k) \text{ i.o.})=0. \end{aligned}$$

Thus, (3) holds.

Now, we turn to a proof of (4). For a sufficiently large number $A>0$ and sufficiently small $\delta>0$, let

$$E_j=\{|S_{A^i}|\leqq(1-\delta)\chi(A^i),\ i<j;\ |S_{A^j}|>(1-\delta)\chi(A^j)\} \qquad (j=1, 2, \cdots).$$

Let γ be a positive number such that for some $\varepsilon'>0$, $2/\sqrt{A}+\gamma+\varepsilon'<\delta$. From the s.m. condition (II)

$$\begin{aligned} &P(\{|S_{A^i}|\leqq(1-\delta)\chi(A^i),\ i<j\}\cap\{|S_{A^j}-S_{A^{j-1}+[A^{j/2}]}|>(1-\gamma)\chi(A^j)\}) \\ &\geqq P(|S_{A^i}|\leqq(1-\delta)\chi(A_i),\ i<j)\cdot P(|S_{A^j}-S_{A^{j-1}+[A^{j/2}]}|>(1-\gamma)\chi(A^j))-\alpha([A^{j/2}]). \end{aligned} \tag{9}$$

While, from (i)

$$P(|S_n|>b\chi(n))\geqq\frac{K_0}{(\log n)(\log\log n)}$$

holds for any $b>1$ and for all n sufficiently large. So, noting that $A^j-(A^{j-1}+[A^{j/2}])>A^j/2$ for all sufficiently large A, we have

$$\begin{aligned} v_j &= P(|S_{A^j}-S_{A^{j-1}+[A^{j/2}]}|>(1-\gamma)\chi(A^j)) \\ &\geqq P\left(|S_{A^j-A^{j-1}-[A^{j/2}]}|>2(1-\gamma)\chi\left(\left[\frac{A^j}{2}\right]\right)\right) \\ &\geqq P(|S_{A^j-A^{j-1}-[A^{j/2}]}|>2(1-\gamma)\chi(A^j-A^{j-1}-[A^{j/2}]))\geqq\frac{K_1}{j\log j} \end{aligned}\tag{10}$$

and, moreover, from Chebyshev's inequality

$$P(|S_{A^{j-1}+[A^{j/2}]}-S_{A^{j-1}}|\geqq\varepsilon'\chi(A^j))\leqq K_2A^{-j/2}.\tag{11}$$

So, using the method of the proof of Theorem 1. 1 in [8], we have

$$P(|S_{A^i}|>(1-\delta)\chi(A^i)\text{ for some }i,\ 1\leqq i\leqq k)\to 1\ (k\to\infty),$$

which implies (4). Hence, the proof is completed.

REMARK 1. For the process $\{x_j\}$, satisfying the s.m. condition, the requirement (ii) is fulfilled if (i) holds and there exists a function $r=r(n)$ such that $r(n)\to\infty$ and

$$\max\left(\frac{n}{r}P(|x_1|+\cdots+|x_r|\geqq\varepsilon\chi(n),\ \frac{n}{r}\alpha(r)\right)=O\left(\frac{1}{(\log n)^{1+\rho}}\right)\tag{12}$$

for any ε $(0<\varepsilon<(b-1)/b)$ where $b>1$ is an arbitrarily fixed number.

Proof. We use the method in [6]. For any $b>1$, let

$$E_j=\{|S_i|<b\chi(n),\ i<j\ ;\ |S_j|\geqq b\chi(n)\}\qquad(j=1,\cdots,n)$$

and $k=[n/r]$. It follows from the s.m. condition that for any $a>0$

$$\begin{aligned} &P(\max_{1\leqq j\leqq n}|S_j|\geqq b\chi(n))=P\left(\bigcup_{j=1}^n E_j\right) \\ &\leqq P(|S_n|\geqq b(1-\varepsilon)\chi(n))+\sum_{i=0}^{k-2}P\left(\bigcup_{j=1}^r[E_{ir+j}\cup\{|S_n-S_{ir+j}|\geqq\varepsilon b\chi(n)\}]\right) \\ &\quad+\sum_{l=(k-1)r+1}^n P(E_l\cap\{|S_n-S_l|\geqq\varepsilon\chi(n)\}) \\ &\leqq P(|S_n|\geqq b(1-\varepsilon)\chi(n))+\sum_{i=0}^{k-2}P\left(\left(\bigcup_{j=1}^r E_{ir+j}\right)\cap\left\{|S_n-S_{(i+2)r}|\geqq\frac{\varepsilon}{2}\chi(n)\right\}\right) \end{aligned}\tag{13}$$

$$+\sum_{i=0}^{k-2} P\left(\bigcup_{j=1}^{r}\left[E_{ir+j} \cup \left\{|S_{(i+2)r}-S_{ir+j}| \geqq \frac{\varepsilon}{2}\chi(n)\right\}\right]\right)$$

$$+\sum_{l=(k-1)r+1}^{n} P(E_l \cap \{|S_n-S_l| \geqq \varepsilon\chi(n)\})$$

$$\leqq P(|S_n| \geqq b(1-\varepsilon)\chi(n)) + \sum_{i=0}^{k-2} P\left(\bigcup_{j=1}^{r} E_{ir+j}\right) P\left(|S_n - S_{(i+2)r}| \geqq \frac{\varepsilon}{2}\chi(n)\right)$$

$$+k\alpha(r)+(k+1)P\left(|x_1|+\cdots+|x_{2r}| \geqq \frac{\varepsilon}{2}\chi(n)\right).$$

Since for any i $(0 \leqq i \leqq k-1)$

$$P(|S_n-S_{(i+2)r}| \geqq \varepsilon\chi(n)) = P(|S_{n-(i+2)r}| \geqq \varepsilon\chi(n)) \leqq \frac{\sigma^2_{n-(i+2)r}}{\varepsilon^2\{\chi(n)\}^2} \to 0,$$

so for sufficiently large n

$$P(|S_n-S_{(i+2)r}| \geqq \varepsilon\chi(n)) \leqq \frac{1}{2}. \tag{14}$$

Thus, from (12), (13) and (14)

$$P(\max_{1 \leqq j \leqq n} |S_j| \geqq b\chi(n))$$

$$\leqq P(|S_n| \geqq b(1-\varepsilon)\chi(n)) + \frac{1}{2}P(\max_{1 \leqq j \leqq n} |S_j| \geqq b\chi(n)) + O\left(\frac{1}{(\log n)^{1+\rho}}\right).$$

Hence, from (i) we have

$$P(\max_{1 \leqq j \leqq n} |S_j| \geqq b\chi(n)) = O\left(\frac{1}{(\log n)^{1+\rho_1}}\right)$$

where ρ_1 is a positive constant.

REMARK 2. For the process $\{x_j\}$, satisfying the u.s.m. condition (I), the requirement (ii) is satisfied if (i) holds and there exists a function $r=r(n)$ such that $r(n) \to \infty$ and

$$\frac{n}{r} P(|x_1|+\cdots+|x_r| \geqq \varepsilon\chi(n)) = O\left(\frac{1}{(\log n)^{1+\rho}}\right) \tag{15}$$

for any $\varepsilon(0<\varepsilon<(b-1)/b)$ where $b>1$ is an arbitrarily fixed number.

3. The law of the iterated logarithm for the process $\{x_j\}$ satisfying one of the conditions (I) or (II).

THEOREM 1. 1 in [8] may be generalized in two ways:

(a) One way is to weaken the requirement $E|x_0|^{2+\delta}<\infty$ retaining the condition $\sum \{\varphi(n)\}^{1/2}<\infty$, (Therem 2);

(b) The other is to weaken the requirement $\sum \{\varphi(n)\}^{1/2}<\infty$ retaining the condition $E|x_0|^{2+\delta}<\infty$, (Theorem 3).

THEOREM 2. *Let the process $\{x_j\}$ satisfying the u.s.m. condition have the following properties*:

1°. *For all sufficiently large N*

$$\int_{|x|>N} x^2 dP = O\left(\frac{1}{(\log N)^5}\right) \tag{16}$$

2°.
$$\sum_{j=1}^{\infty} \{\varphi(j)\}^{1/2} < \infty.$$

Then the law of the iterated logarithm is applicable to the process $\{x_j\}$.

Proof. We remark first that from 2°

$$\sigma_n^2 = n\sigma^2(1+o(1))$$

(cf. [3] and [4]).

Let

$$f_N(x) = \begin{cases} x & (|x| \leqq N), \\ 0 & (|x| > N) \end{cases}$$

and $\bar{f}_N = x - \bar{f}_N(x)$. Furthermore, let $r(n) = [n^{1/3}]$ and $N = [n^{1/6}]$. Then forany $\lambda > 0$

$$\begin{aligned}
&P(|x_1| + \cdots + |x_r| \geqq 2\lambda\chi(n)) \\
&\leqq P(|\bar{f}_N(x_1)| + \cdots + |\bar{f}_N(x_r)| \geqq \lambda\chi(n)) \\
&\quad + P(|f_N(x_1)| + \cdots + |f_N(x_r)| \geqq \lambda\chi(n)) \\
&= P(|\bar{f}_N(x_1)| + \cdots + |\bar{f}_N(x_r)| \geqq \lambda\chi(n)) \\
&\leqq \frac{1}{\lambda^2\{\chi(n)\}^2} E\left(\sum_{j=1}^{r} |\bar{f}_N(x_j)|\right)^2 \\
&\leqq \frac{r}{\lambda^2\{\chi(n)\}^2} \left\{ E|\bar{f}_N(x_0)|^2 + 2\sum_{j=1}^{r-1} E|\bar{f}_N(x_0)| \cdot |\bar{f}_N(x_j)| \right\} \\
&\leqq \frac{r}{\lambda^2\{\chi(n)\}^2} \left\{ E|\bar{f}_N(x_0)|^2 + 2r(E|\bar{f}_N(x_0)|)^2 + 4(E|\bar{f}_N(x_0)|^2) \sum_{j=1}^{r-1} \{\varphi(j)\}^{1/2} \right\} \\
&\leqq \frac{r}{\lambda^2\{\chi(n)\}^2} E|\bar{f}_N(x_0)|^2 \left\{ 1 + 2r \cdot \frac{1}{N^2} E|\bar{f}_N(x_0)|^2 + 4\sum_{j=1}^{\infty} \{\varphi(j)\}^{1/2} \right\} \\
&\leqq K \frac{r}{\{\chi(n)\}^2} \cdot \frac{1}{(\log n)^5}
\end{aligned}$$

and so

$$\frac{n}{r} P(|x_1| + \cdots + |x_r| \geqq 2\lambda\chi(n)) = O\left(\frac{1}{(\log n)^5}\right).$$

Thus, (15) holds.

Next, we shall prove that (i) in Theorem 1 is satisfied. Define

$$S_n'=\frac{1}{\sigma\sqrt{n}}\sum_{j=1}^{n}(f_N(x_j)-Ef_N(x_j))$$

and

$$S_n''=\frac{1}{\sigma\sqrt{n}}\sum_{j=1}^{n}(\bar f_N(x_j)-E\bar f_N(x_j)).$$

For a small $\alpha(0<\alpha<1/2)$, put

$$p(n)=[n^{1/2+\alpha}],\quad q(n)=[n^{1/2-\alpha}],\quad k=\left[\frac{n}{p+q}\right]$$

and set

$$T_n'=\sum_{i=0}^{k-1}\sum_{j=1}^{p}\frac{1}{\sigma\sqrt{n}}(f_N(x_{i(p+q)+j})-Ef_N(x_{i(p+q)+j})),\quad T_n''=\sum_{i=0}^{k}\zeta_i$$

where

$$\zeta_i=\sum_{j=1}^{q}\frac{1}{\sigma\sqrt{n}}(f_N(x_{i(p+q)+p+j})-Ef_N(x_{i(p+q)+p+j}))\qquad (i=0,1,\cdots,k-1),$$

$$\zeta_k=\sum_{j=k(p+q)+1}^{n}\frac{1}{\sigma\sqrt{n}}(f_N(x_j)-Ef_N(x_j)).$$

Then

$$\begin{aligned}ES_n''^2&=\frac{1}{\sigma^2}\Big\{E(\bar f_N(x_0)-E\bar f_N(x_0))^2\\&\quad+2\sum_{j=1}^{n-1}\Big(1-\frac{j}{n}\Big)E(\bar f_N(x_0)-E\bar f_N(x_0))(\bar f_N(x_j)-E\bar f_N(x_j))\Big\}\\&\leqq\frac{2}{\sigma^2}E|\bar f_N(x_0)|^2\Big\{1+2\sum_{j=1}^{n-1}\{\varphi(j)\}^{1/2}\Big\}=O\Big(\frac{1}{(\log n)^5}\Big)\end{aligned}\tag{17}$$

and

$$\begin{aligned}ET_n''^2&=E\Big(\sum_{i=0}^{k-1}\zeta_i\Big)^2\\&\leqq\frac{1}{\sigma^2 n}\Big\{(k-1)E\zeta_0^2+2k\sum_{i=0}^{k-1}|E\zeta_0\zeta_i|+E\zeta_k^2+2\sum_{i=0}^{k-1}|E\zeta_i\zeta_k|\Big\}\\&\leqq\frac{1}{\sigma^2 n}\Big\{kE\zeta_0^2+4kE\zeta_0^2\cdot\sum_{i=1}^{k-1}\{\varphi(i(p+q))\}^{1/2}+E\zeta_k^2\\&\quad+4\sum_{i=0}^{k-2}\sqrt{E\zeta_0^2}\sqrt{E\zeta_k^2}\{\varphi((k-i)(p+q))\}^{1/2}+2\sqrt{E\zeta_{k-1}^2}\cdot\sqrt{E\zeta_k^2}\Big\}\end{aligned}\tag{18}$$

$$\leqq \frac{1}{\sigma^2 n}\left\{k\sigma_q^2+\sigma_{p+q}^2+4k\sigma_q^2\sum_{i=0}^{\infty}\{\varphi(i(p+q))\}^{1/2}\right.$$

$$\left.+4k\sigma_q\sigma_{p+q}\left(\frac{1}{2k}+\sum_{i=1}^{\infty}\{\varphi(i(p+q))\}^{1/2}\right)\right\}$$

$$=O(n^{-\gamma_2})$$

for some $\gamma_2>0$. Since

$$|Ee^{itS_n/\sigma\sqrt{n}}-Ee^{itT_n'}|$$

$$\leqq |Ee^{itS_n/\sigma\sqrt{n}}-Ee^{itS_n'}|+|Ee^{itS_n'}-Ee^{itT_n'}|$$

$$\leqq Ee|^{itS_n''}-1|+E|e^{itT_n''}-1|$$

$$\leqq |t|\cdot E|S_n''|+|t|\cdot E|T_n''|\leqq |t|\{\sqrt{E|S_n''|^2}+\sqrt{E|T_n''|^2}\},$$

so, from (17) and (18)

$$I_1=\int_{-(\log n)^{5/4}}^{(\log n)^{5/4}}\left|\frac{Ee^{itS_n/\sigma\sqrt{n}}-Ee^{itT_n'}}{t}\right|dt$$

(19)

$$\leqq\int_{-(\log n)^{5/4}}^{(\log n)^{5/4}}\{\sqrt{E|S_n''|^2}+\sqrt{E|T_n''|^2}\}dt=O\left(\frac{1}{(\log n)^{5/4}}\right).$$

Furthermore, let $\eta_0, \eta_1, \cdots, \eta_{k-1}$ be independent random variables distributed in the same way as the corresponding

$$\frac{1}{\sigma\sqrt{n}}\sum_{j=1}^{p}(f_N(x_{i(p+q)+j})-Ef_N(x_{i(p+q)+j}))\qquad (j=0, 1, \cdots, k-1).$$

From Condition (I)

$$\left|Ee^{itT_n'}-\prod_{j=0}^{k-1}Ee^{it\eta_j}\right|\leqq k\varphi(q)=k\cdot O(q^{-2})=O(n^{-\gamma_3})$$

for some $\gamma_3>0$ and for all n sufficiently large. On the other hand

$$\left|Ee^{itT_n'}-\prod_{j=0}^{k-1}Ee^{it\eta_j}\right|\leqq\frac{t^2}{2}(E|T_n'|^2+kE\eta_0^2)$$

for all sufficiently small $|t|$. So

$$I_2=\int_{-(\log n)^{5/4}}^{(\log n)^{5/4}}\left|\frac{Ee^{itT_n'}-Ee^{it\Sigma_{j=0}^{k-1}\eta_j}}{t}\right|dt$$

$$\leqq\int_{-n^{-1/4}}^{n^{-1/4}}\left|\frac{Ee^{itT_n'}-Ee^{it\Sigma_{j=0}^{k-1}\eta_j}}{t}\right|dt+\int_{n^{-1/4}\leqq|t|\leqq(\log n)^{5/4}}\left|\frac{Ee^{itT_n'}-Ee^{it\Sigma_{j=0}^{k-1}\eta_j}}{t}\right|dt$$

(20)

$$\leqq \frac{1}{2}(E|T_n'|^2+kE\eta_0^2)\int_{-n^{-1/4}}^{n^{-1/4}}|t|dt+O(n^{-\gamma_3})\int_{n^{-1/4}\leqq|t|\leqq(\log n)^{5/4}}\frac{dt}{|t|}$$

$$=O\left(\frac{1}{(\log n)^{5/4}}\right)$$

Next, let

$$\eta_j'=\frac{\sigma\sqrt{n}}{\sqrt{kE\eta_0^2}}\eta_j \qquad (j=0, 1, \cdots, k-1).$$

Then, by the analogous argument, we have

$$I_3=\int_{-(\log n)^{5/4}}^{(\log n)^{5/4}}\left|\frac{Ee^{it\Sigma_{j=0}^{k-1}\eta_j}-Ee^{it\Sigma_{j=0}^{k-1}\eta_j'}}{t}\right|dt=O\left(\frac{1}{(\log n)^{5/4}}\right) \tag{21}$$

for all sufficiently large n.

Finally, by applying Esseen's lemma to the sum $\sum_{j=0}^{k-1}\eta_j'$, we obtain

$$\left|\frac{Ee^{it\Sigma_{j=0}^{k-1}\eta_j'}-e^{-t^2/2}}{t}\right|\leqq\frac{KE|\eta_0'|^{2+\delta}}{\sigma_p^{2+\delta}k^{\delta/2}}|t|^{1+\delta}e^{-t^2/4}\leqq Kk^{-\delta/2}|t|^{1+\delta}e^{-t^2/4}$$

for all t such that

$$|t|\leqq\frac{\sqrt{n}\{E|\eta_0'|^2\}^{(2+\delta)/2}}{24E|\eta_0'|^{2+\delta}}\leqq K_2\sqrt{n}$$

(cf. Lemma 1. 9 in [3]). So

$$I_4=\int_{-(\log n)^{5/4}}^{(\log n)^{5/4}}\left|\frac{Ee^{it\Sigma_{j=0}^{k-1}\eta_j}-e^{-t^2/2}}{t}\right|dt=O(k^{-\delta/2}). \tag{22}$$

Combining (19)-(22), we have from Esseen's theorem

$$\sup_{-\infty<z<\infty}|F(S_n<z\sigma\sqrt{n})-\Phi(z)|$$

$$\leqq K_1\int_{-(\log n)^{5/4}}^{(\log n)^{5/4}}\left|\frac{Ee^{itS_n/\sqrt{n}\sigma}-e^{-t^2/2}}{t}\right|dt+\frac{K_2}{(\log n)^{5/4}} \tag{23}$$

$$\leqq K_1(I_1+I_2+I_3+I_4)+\frac{K_2}{(\log n)^{5/4}}=O\left(\frac{1}{(\log n)^{5/4}}\right).$$

Thus, from Theorem 1 and Remark 2, we have the theorem.

THEOREM 3. *The process $\{x_j\}$, satisfying the u.s.m. condition, obeys the law of the iterated logarithm, if the following requirements are fulfilled*:

1°. $E|x_j|^{2+\delta}<\infty$ *for some* $\delta>0$;

2°. $\varphi(n)=O(1/n^{1+\varepsilon})$ *for some* $\varepsilon>1/(1+\delta)$.

Proof. Without loss of generality, we may assume that $\varepsilon\leqq 1$. Let

$$\rho=\frac{(1+\varepsilon)(1+\delta)-(2+\delta)}{2\delta(2+\delta)}=\frac{\varepsilon(1+\delta)-1}{2\delta(2+\delta)}>0. \tag{24}$$

We define $f_N(x)$ and $\bar{f}_N(x)$ as before. For any positive integer j, put $N_j=j^{\rho}$. Then from the inequalities in [3]

$$\begin{aligned}|Ex_0x_j|&\leqq|Ex_0(f_{N_j}(x_j)-Ef_{N_j}(x_j))|+|Ex_0(\bar{f}_{N_j}(x_j)-E\bar{f}_{N_j}(x_j))|\\&\leqq 4N_jE|x_0|\varphi(j)+2(E|x_0|^{2+\delta})^{1/(2+\delta)}(E|\bar{f}_{N_j}(x_j)\\&\quad-E\bar{f}_{N_j}(x_j)|^{(2+\delta)/(1+\delta)})^{(1+\delta)/(2+\delta)}\{\varphi(j)\}^{(1+\delta)/(2+\delta)}\\&\leqq 4N_jE|x_0|\varphi(j)+4N_j^{-\delta}E|x_0|^{2+\delta}\{\varphi(j)\}^{(1+\delta)/(2+\delta)}\\&\leqq 4E|x_0|\frac{1}{j^{1+(\varepsilon-\delta)}}+4E|x_0|^{2+\delta}\frac{1}{j^{(1+\varepsilon)(1+\delta)/(2+\delta)+\rho\delta}}.\end{aligned}$$

Since $\varepsilon-\rho>0$ and

$$\left\{(1+\varepsilon)\frac{1+\delta}{2+\delta}+\rho\delta\right\}-1=\frac{3\{\varepsilon(1+\delta)-1\}}{2(2+\delta)}>0$$

so

$$\sum_{j=1}^{\infty}|Ex_0x_j|\leqq\sum_{j=1}^{\infty}\left\{4E|x_0|\cdot\frac{1}{j^{1+(\varepsilon-\rho)}}+4E|x_0|^{2+\delta}\frac{1}{j^{(1+\varepsilon)(1+\delta)/(2+\delta)+\rho\delta}}\right\}<\infty. \tag{25}$$

Thus, the series

$$\sigma^2=Ex_0^2+2\sum_{j=1}^{\infty}Ex_0x_j$$

converges absolutely.

Next, we shall show that for some $\gamma>0$

$$\sigma_n^2=n\sigma^2(1+O(n^{-\gamma})). \tag{26}$$

It follows from (25) that

$$\begin{aligned}\left|\sigma^2-\frac{1}{n}ES_n^2\right|&\leqq 2\sum_{j=n}^{\infty}|Ex_0x_j|+\frac{2}{n}\sum_{j=1}^{n-1}j|Ex_0x_j|\\&\leqq 8\left\{E|x_0|\sum_{j=n}^{\infty}\frac{1}{j^{1+(\varepsilon-\rho)}}+E|x_0|^{2+\delta}\sum_{j=n}^{\infty}\frac{1}{j^{(1+\varepsilon)(1+\delta)/(2+\delta)+\rho\delta}}\right\}\\&\quad+\frac{8}{n}\left\{E|x_0|\sum_{j=1}^{n-1}\frac{1}{j^{(\varepsilon-\rho)}}+E|x_0|^{2+\delta}\sum_{j=1}^{n-1}\frac{1}{j^{3\{\varepsilon(1+\delta)-1\}/2(2+\delta)}}\right\}\end{aligned}$$

and so we have (26).

Now, we define p, q and k by the formulas

$$p=[n^{1/2+\alpha}],\quad q=[n^{1/2-\alpha}],\quad k=\left[\frac{n}{p+q}\right]\quad(\alpha>0)$$

and set

$$\xi_i=\sum_{j=i(p+q)+1}^{(i+1)p+iq} x_j, \qquad i=0, 1, \cdots, k-1;$$

$$\eta_i=\sum_{j=(i+1)p+iq+1}^{(i+1)(p+q)} x_j, \qquad i=0, 1, \cdots, k-1; \qquad \eta_k=\sum_{j=k(p+q)+1}^{n} x_j.$$

Then, it follows from (26) that for some $\gamma>0$

$$\left|\frac{D\left(\sum_{i=0}^{k-1}\xi_i'\right)}{n\sigma^2}-1\right|\leqq Cn^{-\gamma}$$

and

$$\left|E\exp\left(it\frac{1}{\sigma\sqrt{n}}\sum_{j=0}^{k-1}\xi_j\right)-\prod_{j=0}^{k-1}E\exp\left(it\frac{\xi_j}{\sigma\sqrt{n}}\right)\right|\leqq 4k\varphi(q)\leqq Cn^{-\gamma}$$

where $\xi_0', \xi_1', \cdots, \xi_{k-1}'$ are independent random variables distributed in the same way as the corresponding ξ_i. Thus, the method of the proof of Lemma 1 in [8] can be completely carried over to this case, and we obtain the theorem.

Two theorems below are concerned with the processes satisfying the s.m. condition.

THEOREM 4. *The process* $\{x_j\}$, *satisfying the s.m. condition, obeys the law of the iterated logarithm if the following requirements are fulfilled*:

1. $|x_j|<c$ *with probability one*;
2. $\alpha(n)=O(1/n^{1+\varepsilon})$ *for some* $\varepsilon>0$.

Proof. Define p, q, k and r by

$$p(n)=[n^{1/2}\log^3 n], \quad q(n)=r(n)=[n^{1/2}\log^{-3} n], \quad k(n)=\left[\frac{n}{p+q}\right].$$

Then, for any $b>0$

$$\frac{n}{r}P(|x_1|+\cdots+|x_r|\geqq b\chi(n))=0$$

and for some $\gamma_1>0$

$$\frac{n}{r}\alpha(r)\leqq K_1 n^{1/2}(\log n)^3\frac{1}{(n^{1/2}(\log n)^{-3})^{1+\varepsilon}}=O(n^{-\gamma_1}).$$

So, (12) holds. Thus, from Remark 1 to Theorem 1, it is enough to prove Condition (i) in Theorem 1. Put $\xi_0, \cdots, \xi_{k-1}, \xi_0', \cdots, \xi_{k-1}', \eta_0, \cdots, \eta_k, S_n', S_n''$ as the same ones in the proof of Theorem 3.

Since from Condition (II)

$$\left|E\left(\exp it\frac{\xi_0+\cdots+\xi_{k-1}}{\sqrt{n}\,\sigma}\right)-\prod_{j=0}^{k-1}E\left(\exp it\frac{\xi_j}{n}\right)\right|\leqq k\alpha(q)=O(n^{-\gamma_1}),$$

so from Esseen's lemma

$$\int_{-(\log n)^{3/2}}^{(\log n)^{3/2}}\left|\frac{Ee^{it\sum_{j=0}^{k-1}\xi_j/\sqrt{n}\,\sigma}-Ee^{it\sum_{j=0}^{k-1}\xi_j/\sqrt{n}\,\sigma}}{t}\right|dt=O(n^{-\gamma_2}) \tag{27}$$

for some $\gamma_2>0$.

Secondly, from the proof of Lemma 18. 5.2 in [4]

$$E\left(\sum_{j=1}^{n}x_j\right)^4=O\left(n^2\sum_{j=1}^{n}j\alpha(j)\right)=O(n^{3-\varepsilon}).$$

So, if we choose a positive number δ such that $0<\delta<2$ and $\delta(1+\varepsilon)>2$, then from Esseen's lemma

$$\begin{aligned}&\int_{-(\log n)^{3/2}}^{(\log n)^{3/2}}\left|\frac{Ee^{it\sum_{j=0}^{k-1}\xi_j'/\sqrt{kE\xi_0^2}}-e^{-t^2/2}}{t}\right|dt\\ &\leqq\frac{K_1E|\xi_0|^{2+\delta}}{k^{\delta/2}\sigma_p^{2+\delta}}\leqq\frac{K_1\{E\xi_0^4\}^{(2+\delta)/4}}{k^{\delta/2}\sigma_p^{2+\delta}}=\frac{K_1(p^{3-\varepsilon})^{(2+\delta)/4}}{k^{\delta/2}(p(1+o(1))}=O(n^{-\varepsilon/4}).\end{aligned} \tag{28}$$

Finally,

$$\begin{aligned}\left|E\left(\sum_{i=0}^{k-1}\xi_i\right)-kE\xi_0^2\right|&\leqq 2k\sum_{j=1}^{k-1}|E\xi_0\xi_j|\\ &\leqq 2k\sum_{j=1}^{k-1}\sum_{i=1}^{p}\sum_{l=1}^{p}|Ex_ix_{j(p+q)+l}|\\ &\leqq K_1kp^2\sum_{j=2}^{k-1}\alpha((j-1)(p+q))+K_2kp\sum_{l=1}^{p}\alpha(q+l)\\ &\leqq K_3\frac{kp}{(p+q)^{\varepsilon}}+K_4\frac{kp}{q^{1+\varepsilon}}\end{aligned}$$

and

$$\begin{aligned}E\left(\sum_{j=0}^{k}\eta_j\right)^2&=(k-1)E\eta_0^2+2k\sum_{j=1}^{k-1}\left(1-\frac{j}{k}\right)E\eta_0\eta_j+E\eta_k^2+2\sum_{j=1}^{k-1}E\eta_j\eta_k\\ &\leqq(k-1)E\eta_0^2+2k\sum_{i=1}^{k-2}\sum_{i=1}^{q}\sum_{l=1}^{q}|Ex_{p+i}x_{j(p+q)+l}|+\sigma_{p+q}^2+2\sum_{j=1}^{k-1}\sum_{i=1}^{q}\sum_{l=1}^{n-k(p+q)}|Ex_{j(p+q)+i}x_{k(p+q)+l}|\\ &\leqq k\sigma_q^2+K_5\frac{kq}{(p+q)^{\varepsilon}}+K_6\frac{kq^2}{p^{1+\varepsilon}}+\sigma_{p+q}^2.\end{aligned}$$

Hence, we have

$$\left|\frac{kE\xi_0^2}{\sigma_n^2}-1\right|\leqq K_7\left\{\left|\frac{kE\xi_0^2}{n\sigma^2}-ES_n'^2\right|+2\sqrt{ES_n'^2ES_n''^2}+ES_n''^2\right\}=O\left(\frac{1}{(\log n)^{3/2}}\right) \tag{29}$$

On the other hand,

$$\left|\frac{\sigma_n^2}{n\sigma^2}-1\right|=\frac{1}{\sigma^2}\left\{2\sum_{j=1}^{n-1}\frac{j}{n}|Ex_0x_j|+2\sum_{j=n}^{\infty}|Ex_0x_j|\right\}$$

(30)

$$\leqq\frac{1}{\sigma^2}\left\{\frac{K_8}{n}\sum_{j=1}^{n-1}j\alpha(j)+K_9\sum_{j=n}^{\infty}\alpha(j)\right\}=O(n^{-\varepsilon/2}).$$

Combining (29) and (30) and using Esseen's lemma, we have

$$\int_{-(\log n)^{3/2}}^{(\log n)^{3/2}}\left|\frac{Ee^{it\Sigma_{j=0}^{k-1}\xi_j'/\sqrt{n}\sigma}-Ee^{it\Sigma_{j=0}^{k-1}\xi_j'/\sqrt{kE\xi_0^2}}}{t}\right|dt=O\left(\frac{1}{(\log n)^{3/2}}\right). \tag{31}$$

Thus, from (27), (28) and (31), Condition (i) in Theorem 1 follows, and the proof is completed.

THEOREM 5. *The process $\{x_j\}$, satisfying s.m. condition, obeys the law of the iterated logarithm if the following requirements are fulfilled for some δ and δ' such that* $0<\delta'<\delta$:

1°. $E|x_j|^{2+\delta}<\infty$;

2°. $\sum_{n=1}^{\infty}\{\alpha(n)\}^{\delta'/(2+\delta')}<\infty$.

Proof. Define $f_N(x)$ and $\bar{f}_N(x)$ as before. Let

$$N=n^{1/2(1+\delta')}(\log n)^{-3}$$

and

$$r(n)=[n^{\delta'/2(1+\delta')}(\log n)^3].$$

Then, for any $b>0$

$$\frac{n}{r}P(|x_1|+\cdots+|x_r|\geqq b\chi(n))=\frac{n}{r}P(|\bar{f}_N(x_1)|+\cdots+|\bar{f}_N(x_r)|\geqq b\chi(n))$$

$$\leqq\frac{n}{b^2\{\chi(n)\}^2r}E\left(\sum_{j=1}^{r}|\bar{f}_N(x_j)|\right)^2\leqq\frac{n}{b^2\{\chi(n)\}^2}\left\{E|\bar{f}_N(x_0)|^2+2\sum_{j=1}^{r-1}E|\bar{f}_N(x_0)|\cdot|\bar{f}_N(x_j)|\right\}$$

$$\leqq\frac{n}{b^2\{\chi(n)\}^2}\left[E|\bar{f}_N(x_0)|^2+2\sum_{i=1}^{r-1}\left\{(E|\bar{f}_N(x_0)|)^2+8(E|\bar{f}_N(x_0)|^{2+\delta'})^{2/(2+\delta')}(\alpha(i))^{\delta'/(2+\delta')}\right\}\right]$$

$$\leqq\frac{n}{b^2\{\chi(n)\}^2}\left\{\frac{1}{N^\delta}E|\bar{f}_N(x_0|^{2+\delta}+\frac{2r}{N^{2(1+\delta)}}(E|\bar{f}_N(x_0)|^{2+\delta})^2+\frac{K_1}{N^{2(\delta-\delta')}}(E|\bar{f}_N(x_0)|^{2+\delta})^{1/(2+\delta')}\right\}$$

$$=O(n^{-\gamma})$$

holds for some $\gamma>0$ and

$$\frac{n}{r}\alpha(r)=\frac{n}{r}O(r^{-(2+\delta')/\delta'})=O\left(\frac{1}{(\log n)^3}\right).$$

Hence, Remark 1 to Theorem 1 it suffices to show

$$\sup_{-\infty<z<\infty}\left|P\left(\sum_{i=1}^{n} x_i<\sqrt{n}\,\sigma z\right)-\Phi(z)\right|=O\left(\frac{1}{(\log n)^3}\right).$$

Define p, q and k by

$$p(n)=[n^{1/2+\alpha}],\quad q(n)=[n^{1/2-\alpha}]\quad\text{and}\quad k(n)=\left[\frac{n}{p+q}\right]$$

where α is a small positive number. Let $N'=n^{\delta'/16(1+\delta')}$ if $0<\delta\leqq 2$ and $N'=n^{1/16(1+\delta')}$ if $\delta>2$. Put

$$S_n'=\frac{1}{\sigma\sqrt{n}}\sum_{j=1}^{n}(f_{N'}(x_j)-Ef_{N'}(x_j)),\quad S_n''=\frac{1}{\sigma\sqrt{n}}\sum_{j=1}^{n}(\bar{f}_{N'}(x_j)-E\bar{f}_{N'}(x_j)),$$

$$\zeta_i=\sum_{j=1}^{p}(f_{N'}(x_{(i-1)(p+q)+j})-Ef_{N'}(x_{(i-1)(p+q)+j}))\qquad(i=1,2,\cdots,k),$$

$$T_n'=\frac{1}{\sqrt{n}\,\sigma}\sum_{i=1}^{k}\zeta_i,\ T_n''=S_n'-T_n'.$$

Then, it is easily proved that for some $\gamma>0$

$$\begin{aligned}&E|S_n''|^2\leqq\frac{1}{\sigma^2}\left\{\frac{1}{N'^{\delta}}E|\bar{f}_{N'}(x_0)|^{2+\delta}+\frac{K_1}{N'^{2(\delta-\delta')/(2+\delta')}}(E|\bar{f}_{N'}(x_0)|^{2+\delta})^{2/(2+\delta')}\right\}=O(n^{-\gamma}),\\ &E|T_n''|^2=O(n^{-\gamma})\qquad\text{and}\qquad|ET_n'^2-1|=O(n^{-\gamma})\end{aligned}\tag{32}$$

Now, let $f_n(t)$ be the characteristic function of $S_n/\sqrt{n}\,\sigma$. Then

$$\begin{aligned}|f_n(t)-e^{-t^2/2}|&\leqq|f_n(t)-Ee^{itS_n'}|+|Ee^{itS_n'}-Ee^{itT_n'}|\\&\quad+\left|Ee^{itT_n'}-\prod_{j=1}^{k}Ee^{it\zeta_j/\sqrt{kE\zeta_0^2}}\right|+\left|e^{-t^2/2}-\prod_{j=1}^{k}Ee^{it\zeta_j/\sqrt{kE\zeta_0^2}}\right|\\&\leqq|t|E|S_n''|+|t|E|T_n''|+\left|Ee^{itT_n'}-\prod_{j=1}^{k}Ee^{it\zeta_j/\sqrt{kE\zeta_0^2}}\right|\\&\quad+\left|e^{-t^2/2}-\prod_{j=1}^{k}Ee^{it\zeta_j/\sqrt{kE\zeta_0^2}}\right|.\end{aligned}\tag{33}$$

From Esseen's lemma

$$\left|e^{-t^2/2}-\prod_{j=1}^{k}Ee^{it\zeta_j/\sqrt{kE\zeta_0^2}}\right|\leqq K\frac{E|\zeta_0|^{2+\delta}}{k^{2/\delta}(E\zeta_0^2)^{(2+\delta)/2}}|t|^{2+\delta}e^{-t^2/4}\tag{34}$$

holds for all t such that

$$|t|\leqq\sqrt{n}\Big/24\frac{E|\zeta_0|^{2+\delta}}{(E\zeta_0^2)^{(2+\delta)/2}}.$$

Since

$$E\zeta_0^4 \leqq K_1(N')^4 p^2 \sum_{j=1}^{p} j\alpha(j) \leqq K_1(N')^4 p^2 \sum_{j=1}^{p} j^{-2/\delta'}$$
$$\leqq K_2(N')^4 p^2 \max(1, p^{-2/\delta'})$$

and

$$E\zeta_0^2 = p\sigma^2(1+o(1))$$

for all sufficiently large n, so

$$\frac{E|\zeta_0|^{2+\delta}}{k^{\rho/2}(E\zeta_0^2)^{(2+\rho)/2}} \leqq \frac{(E\zeta_0^4)^{(2+\delta)/4}}{k^{\rho/2}(E\zeta_0^2)^{(2+\rho)/2}} = O(n^{-\gamma})$$

holds for all sufficiently large n where $\rho=\min(2, \delta)$ and γ is a positive number.

Consequently, from (34)

$$\left| e^{-t^2/2} - \prod_{j=1}^{k} E e^{it\zeta_j/\sqrt{kE\zeta_0^2}} \right| \leqq Kn^{-\gamma}|t|^{2+\delta}e^{-t^2/4} \tag{35}$$

holds for all sufficiently large n and for all t such that $|t| \leqq \sqrt{n}$. From Condition (II)

$$\left| Ee^{itT_n'} - \prod_{j=1}^{k} E e^{it\zeta_j/\sqrt{kE\zeta_0^2}} \right| \leqq k\alpha(q) = n^{1/2-\alpha}\cdot o(\{n^{1/2-\alpha}\}^{-(2+\delta')/\delta'}). \tag{36}$$

Using (31)-(36), we have

$$\begin{aligned}
&|P(x_1+\cdots+x_n<z\sigma\sqrt{n})-\Phi(z)| \\
&\leqq \int_{-(\log n)^3}^{(\log n)^3} \left| \frac{f_n(t)-e^{-t^2/2}}{t} \right| dt + \frac{C}{(\log n)^3} \\
&\leqq \int_{-(\log n)^3}^{(\log n)^3} Kn^{-\gamma}|t|^{1+\delta}dt + \int_{-(\log n)^3}^{(\log n)^3} \{E|S_n''| + E|T_n''|\}dt \\
&\quad + C_2\left\{ \int_{0\leqq|t|\leqq n^{-1/4}} dt + \int_{n^{-1/4}\leqq|t|\leqq(\log n)^3} \frac{k\alpha(q)}{|t|}dt \right\} + \frac{C_1}{(\log n)^3} \\
&= O\left(\frac{1}{(\log n)^3}\right).
\end{aligned} \tag{37}$$

Hence, from Theorem 1, we have the theorem.

4. Functions of processes.

Let $\{x_j, j=0, \pm1, \pm2, \cdots\}$ be strictly stationary and satisfy one of the requirements (I) or (II). Let f be a measurable mapping from the space of doubly infinitely sequences $(\cdots, \alpha_{-1}, \alpha_0, \alpha_1, \cdots)$ of real numbers into the real line. Define random variables

$$f_j=f(\cdots, x_{j-1}, x_j, x_{j+1}, \cdots) \tag{39}$$

where x_j occupies the 0th place in the argument of f. It is obvious that $\{f_j\}$ is a strictly stationary process. We shall prove theorems establishing the law of the iterated logarithm for the process $\{f_j\}$ (see [3] and [4]).

Let

$$S_i=f_1+\cdots+f_i \tag{40}$$

and

$$\sigma^2=Ef_0^2+2\sum_{j=1}^{\infty} Ef_0f_j \tag{41}$$

if the series converges. In what follows, we use σ^2 only when σ^2 is positive.

The following theorem is a generalization of Theorem 1. 2 in [8].

THEOREM 6. *Let the stationary process $\{x_j\}$ satisfy the u.s.m. condition and let the process $\{f_j\}$ be obtained by the method indicated above. Further, let the following requirements be fulfilled:*

1. $Ef=0$ *and* $E|f|^{2+\delta}<\infty$ *for some* $\delta>0$;
2. $\varphi(n)=O\left(\frac{1}{n^{1+\varepsilon}}\right)$ *for some* $\varepsilon>\frac{1}{1+\delta}$;
3. $E\{|f-E\{f|\mathcal{M}_{-k}^{k}\}|^2\}=\psi(k)=O(n^{-2-\delta_1})$ *for some* $\delta_1>0$.

Then, the processs $\{f_j\}$ obeys the law of the iterated logarithm.

Proof. The series in (41) converges under the conditions of Theorem 6. (cf. [3]). In fact, as in [8] (cf. [3] and [4]), let

$$\xi_j^{(s)}=E\{f_j|\mathcal{M}_{s-j}^{s+j}\}$$

and

$$\eta_j^{(s)}=f_j-\xi_j^{(s)}.$$

Then the stationary process $\{\xi_j^{(s)}\}$ satisfies Condition (I) with the function $\varphi_\xi(n)=1$ for $n\leqq 2s$, $\varphi_\xi(n)=\varphi(n-2s)$ for $n>2s$. Since

$$E|\xi_j^{(s)}|^{2+\delta}=E\{|E\{f_j|\mathcal{M}_{s-j}^{s+j}\}|^{2+\delta}\}\leqq E\{E\{|f_j|^{2+\delta}|\mathcal{M}_{s-j}^{s+j}\}\}=E|f|^{2+\delta}<\infty$$

the stationary process $\{\xi_j^{(s)}\}$ satisfies all the conditions of Theorem 3. Furthermore, as before,

$$\begin{aligned}
|Ef_0f_j|&=|E(\xi_0^{([j/3])}+\eta_0^{([j/3])})(\xi_j^{([j/3])}+\eta_j^{([j/3])})|\\
&\leqq|E\xi_0^{([j/3])}\xi_j^{([j/3])}|+2\{E|\xi_0^{([j/3])}|^2\}^{1/2}\{E|\eta_0^{([j/3])}|^2\}^{1/2}+\{E|\eta_0^{([j/3])}|^2\}\\
&\leqq 4E|\xi_0^{([j/3])}|\left(\frac{3}{j}\right)^{1+(\varepsilon-\rho)}+4E|\xi_0^{([j/3])}|^{2+\delta}\cdot\left(\frac{3}{j}\right)^{(1+\varepsilon)(1+\delta)/(2+\delta)+\rho\delta} \qquad (42)\\
&\quad+2\{E|\xi_0^{([j/3])}|^2\}^{1/2}\left\{\psi\left(\left[\frac{j}{3}\right]\right)\right\}^{1/2}+\psi\left(\left[\frac{j}{3}\right]\right)
\end{aligned}$$

$$\leqq K\left[\left(\frac{3}{j}\right)^{1+(\varepsilon-\rho)}+\left(\frac{3}{j}\right)^{(1+\varepsilon)(1+\delta)/(2+\delta)+\rho\delta}+\left\{\psi\left(\left[\frac{j}{3}\right]\right)\right\}^{1/2}\right]$$

where

$$\rho=\frac{\varepsilon(1+\delta)-1}{2\delta(2+\delta)}>0.$$

It follows from (42) that the series in (41) converges.

Moreover, from (42) we easily obtain that

$$\sigma_n^2=n\sigma^2(1+o(1)).$$

Next, we shall prove that

$$(43)\qquad P(\max_{1\leqq j\leqq n}|S_j|\geqq 6a\chi(n))\leqq 2P(|S_n|\geqq a\chi(n))+O\left(\frac{1}{(\log n)^3}\right)$$

holds for all sufficiently large n. Let

$$r(n)=n^{\delta/2(2+\delta)}\cdot(\log n)^{-3}$$

and

$$g_j(N)=\begin{cases} f_j & (|f_j|\leqq N),\\ 0 & (|f_j|>N);\end{cases}\qquad \bar{g}_j(N)=f_j-g_j(N)\qquad (j=0,1,2,\cdots)$$

where $N=n^{1/(2+\delta)}$. Then

$$(44)\qquad \begin{aligned}&\frac{n}{r}P(|f_1|+\cdots+|f_r|\geqq b\chi(n))\\ &\leqq\frac{n}{r}\left\{P(|\bar{g}_1(N)|+\cdots+|\bar{g}_r(N)|\geqq\frac{b}{2}\chi(n))\right.\\ &\qquad\left.+P(|g_1(N)|+\cdots+|g_r(N)|\geqq\frac{b}{2}\chi(n))\right\}\\ &\leqq\frac{n}{r}\frac{4}{b^2n\sigma^2}E(|\bar{g}_1(N)|+\cdots+|\bar{g}_r(N)|)^2\\ &\leqq\frac{n}{r}\frac{4r}{b^2n\sigma^2}E|\bar{g}_0(N)|^2(1+2r)\\ &\leqq\frac{K}{N^\delta}(1+2r)=O(n^{-\delta/2(2+\delta)}).\end{aligned}$$

Now, as in [1], define

$$U_i=E\{S_{i-2r}|\mathcal{M}_{-\infty}^{i-r}\}$$

and

$$V_i=E\{S_n-S_{i+2n}|\mathcal{M}_{i+r}^{\infty}\}.$$

Here, we adopt the conventions that $S_{i-2r}=0$ if $i<2r$ and $S_n-S_{i+2r}=0$ if $i+2r>n$.

If we put

$$\mu(r)=\sum_{k=r}^{\infty}\{\varphi(k)\}^{1/2} \tag{45}$$

then $\mu(r)=O(n^{-\gamma})$ for some $\gamma>0$, and

$$\begin{aligned} &E|S_k-E\{S_k|\mathcal{M}_{-\infty}^{k+r}\}|^2\leqq\{\mu(r)\}^2,\\ &E|U_i-S_i|^2\leqq 2ES_{2r}^2+2\{\mu(r)\}^2,\\ &E|V_i-(S_n-S_i)|^2\leqq 2ES_{2r}^2+2\{\mu(r)\}^2 \end{aligned} \tag{46}$$

for all k and i. Thus

$$\begin{aligned} &P(|S_i-U_i|\geqq a\chi(n))\\ &\leqq P(|S_{i-2r}-E\{S_{i-2r}|\mathcal{M}_{-\infty}^{i-r}\}|\geqq a\chi(n)-b\sigma\sqrt{n})\\ &\quad+P(|f_1|+\cdots+|f_{2r}|\geqq b\sigma\sqrt{n})\\ &\leqq\frac{4\{\mu(r)\}^2}{(a\chi(n)-b\sigma\sqrt{n})}+\frac{r}{n}\cdot O(n^{-\delta/2(2+\delta)})=O\left(\frac{1}{n(\log n)^3}\right) \end{aligned} \tag{47}$$

and similarly

$$\begin{aligned} &P(|V_i-(S_n-S_i)|\geqq a\chi(n))\\ &\leqq\frac{4\{\mu(r)\}^2}{(a\chi(n)-b\sigma\sqrt{n})^2}+\frac{r}{n}\cdot O(n^{-\delta/2(2+\delta)})=O\left(\frac{1}{n(\log n)^3}\right). \end{aligned} \tag{48}$$

Because of uniform integrability of S_n^2/n, (cf. the proof of Theorem 21. 1 in [1]) there exists a $\lambda>1$ such that

$$P(|S_j|\geqq\lambda\sigma\sqrt{j})\leqq\frac{\varepsilon}{\lambda^2} \tag{49}$$

for all j, where $\varepsilon>0$ is arbitrarily small. Let

$$E_i=\{\max_{j<i}|U_j|<5a\chi(n)\leqq|U_i|\}.$$

As $E_i\in\mathcal{M}_{-\infty}^{i-r}$ and V_{i+2r} is measurable $\mathcal{M}_{i+3r}^{\infty}$, so from (44), (48) and (49)

$$\begin{aligned} &P\left(\bigcup_{j=1}^{n-1}[E_j\cap\{|V_j|\geqq 2a\chi(n)\}]\right)\\ &\leqq\sum_{i=0}^{k-2}P\left(\bigcup_{j=1}^{r}[E_{ir+j}\cap\{|V_{(i+2)p}|\geqq a\chi(n)\}]\right)+\frac{n}{r}P(|f_1|+\cdots+|f_{2r}|\geqq a\chi(n))\\ &\leqq\sum_{i=0}^{k-2}P\left(\left[\bigcup_{j=1}^{r}E_j\right]\cap\{|V_{(i+2)p}|\geqq a\chi(n))\}\right)+O(n^{-\delta/2(2+\delta)}) \end{aligned}$$

$$\leqq \sum_{i=0}^{k-2} P\Big(\bigcup_{j=1}^{r} E_{ir+j}\Big)\{P(|V_{(i+2)p}| \geqq a\chi(n)) + \varphi(r)\} + O(n^{-\delta/2(2+\delta)})$$

$$\leqq \sum_{i=0}^{k-2} P\Big(\bigcup_{j=1}^{r} E_{ir+j}\Big)\{P(|S_{n-(i+2)r}| \geqq \lambda\sigma\sqrt{n-(i+2)r}) + O(n^{-\gamma})\} + O(n^{-\gamma})$$

where $\gamma>0$ is a positive number. Thus, for all n sufficiently large

$$P\Big(\bigcup_{j=1}^{n-1}[E_j \cap \{|V_j| \geqq 2a\chi(n)\}\Big) \leqq \frac{1}{2} P(\max_{1\leqq j\leqq n} |U_j| \geqq 5a\chi(n)) + O(n^{-\gamma}) \tag{50}$$

and so from (47), (48) and (50)

$$\begin{aligned}
&P(\max_{1\leqq i\leqq n} |U_i| \geqq 5a\chi(n)) \\
&\leqq P(|S_n| \geqq a\chi(n)) + P\Big(\bigcup_{j=1}^{n-1}[E_j \cap \{|S_n - U_j| \geqq 4a\chi(n)\}]\Big) \\
&\leqq P(|S_n| \geqq a\chi(n)) + \sum_{j=1}^{n-1} P(|S_n - S_i - V_i| \geqq a\chi(n)) \\
&\quad + P\Big(\bigcup_{j=1}^{n-1}[E_j \cup \{|V_j| \geqq 2a\chi(n)\}]\Big) + \sum_{j=1}^{n-1} P(|S_i - U_i| \geqq a\chi(n)) \\
&\leqq P(|S_n| \geqq a\chi(n)) + \frac{4n\{\mu(r)\}^2}{a^2\{\chi(n)\}^2} \\
&\quad + \Big\{\frac{1}{2} P(\max_{1\leqq j\leqq n} |U_j| \geqq 5a\chi(n)) + O(n^{-\gamma})\Big\} + \frac{4n\{\mu(r)\}^2}{a^2\{\chi(n)\}^2}.
\end{aligned} \tag{51}$$

Consequently, we have

$$P(\max_{1\leqq j\leqq n} |U_j| \geqq 5a\chi(n)) \leqq 2P(|S_n| \geqq a\chi(n)) + O(n^{-\gamma_1}) \tag{52}$$

for some $\gamma_1(>0)$. Combining (52) and (47), we obtain

$$\begin{aligned}
&P(\max_{1\leqq j\leqq n} |S_j| \geqq 6a\chi(n)) \leqq P(\max_{1\leqq j\leqq n} |U_j| \geqq 5a\chi(n)) + O\Big(\frac{1}{(\log n)^3}\Big) \\
&\leqq 2P(|S_n| \geqq a\chi(n)) + O\Big(\frac{1}{(\log n)^3}\Big).
\end{aligned}$$

Next, we shall prove that

$$\sup_{-\infty<z<\infty} |P(S_n < z\sigma\sqrt{n}) - \Phi(z)| = O\Big(\frac{1}{(\log n)^3}\Big).$$

By the same method of estimation of (42)

$$|E\eta_0^{(s)}\eta_j^{(s)}| \leqq K_1\Big[\Big(\frac{3}{j}\Big)^{1+\varepsilon-\rho} + \Big(\frac{3}{j}\Big)^{(1+\varepsilon)(1+\delta)/(2+\delta)+\rho\delta} + \Big\{\Big(\psi\Big(\Big[\frac{j}{3}\Big]\Big)\Big)\Big\}^{1/2}\Big] \tag{53}$$

Taking into account of (53), we have

$$E|S_n''|^2=\frac{1}{\sigma^2 n}\left(nE|\eta_0^{(s)}|^2+2\sum_{j=1}^{n-1}(n-j)E\eta_0^{(s)}\eta_j^{(s)}\right)$$
$$\leqq\frac{1}{\sigma^2}(2N+1)\psi(s)+K_2\sum_{j=N}^{n}\left[\left(\frac{3}{j}\right)^{1+\varepsilon-\rho}+\left(\frac{3}{j}\right)^{(1+\varepsilon)(1+\delta)/(2+\delta)+\rho\delta}+\left\{\psi\left(\left[\frac{j}{3}\right]\right)\right\}^{1/2}\right]$$

where

$$S_n''=\frac{1}{\sigma\sqrt{n}}\sum_{j=1}^{n}\eta_j^{(s)}.$$

Putting $N=n^{1/2-2\varepsilon_1}$ and $r=n^{1/2-\varepsilon_1}$, where $\varepsilon_1>0$ is a sufficiently small number, we obtain that

(54) $$E|S_n''|^2=O(n^{-\gamma})$$

for some $\gamma>0$. Furthermore, with the same s, let

$$S_n'=\frac{1}{\sigma\sqrt{n}}\sum_{j=1}^{n}\xi_j^{(s)}$$

and

$$d_n^2=E|\xi_0^{(s)}|^2+2\sum_{j=1}^{\infty}E\xi_0^{(s)}\xi_j^{(s)}.$$

Then

(55) $$\left|\frac{d_n^2}{\sigma^2}-1\right|=O(n^{-\gamma})$$

for some $\gamma>0$. Thus, noting that

$$|Ee^{itS_n/\sigma\sqrt{n}}-e^{-t^2/2}|\leqq|t|\{E|S_n''|^2\}^{1/2}$$
$$+|Ee^{itS_n'}-e^{-(t^2/2)(d_n{}^2/\sigma^2)}|+|e^{-(t^2/2)(d_n/\sigma^2)}-e^{-t^2/2}|$$

and using the method of the proof of Theorem 3, we have

(56) $$\sup_{-\infty<z<\infty}|P(S_n<z\sigma\sqrt{n})-\Phi(z)|=O\left(\frac{1}{(\log n)^3}\right).$$

Hence, we obtain

$$P(|S_n|>(1+\delta_0)\chi(n)\text{ i.o.})=0.$$

Now, we shall prove that

$$P(|S_n|>(1-\delta_0)\chi(n)\text{ i.o.})=1.$$

We proceed as the proof of Theorem 1. 2 in [8]. Let $A>0$ be sufficiently large and $\varepsilon>0$, $\delta_1>0$ sufficiently small. We write

$$\sigma_n^2(s)=E(\xi_1^{(s)}+\cdots+\xi_n^{(s)})^2$$

and

$$\chi'(n)=(2\sigma_n^2(s)\log\log\sigma_n^2(s))^{1/2}$$

where $s=n^{1/2-\varepsilon_1}$ (ε_1 being the same defined above). Then, it follows from (55) that

$$\left|1-\frac{\chi'(n)}{\chi(n)}\right|=O(n^{-\gamma})$$

for some $\gamma>0$. Put $s_i=A^{i/2-\varepsilon_1}$ and for some positive numbers $\delta_2<\delta_1<\delta_0$

$$E_k=\left\{\left|\sum_{j=1}^{A^i}\xi_j^{(s_i)}\right|>(1-\delta_2)\chi'(A^i),\ i<k;\ \left|\sum_{j=1}^{A^k}\xi_j^{(s_k)}\right|>(1-\delta_2)\chi'(A^k)\right\},$$

$$C=\bigcap_{i=1}^{m}\left\{\left|\sum_{j=1}^{A^i}\eta_j^{(s_i)}\right|<\frac{1}{2}(\delta_1-\delta_2)\chi'(A^i)\right\}$$

and

$$\bar{U}_k=P\left(\left|\sum_{j=1}^{A^i}\xi_j^{(s_i)}\right|>(1-\delta_2)\chi'(A^i)\quad\text{for at least one}\quad i,\ 1\leqq i\leqq k\right)$$

$$=\sum_{j=1}^{k}P(E_j)$$

Then, from Chebyshev's inequality and (54)

$$P(C)=1-P\left(\bigcup_{j=1}^{m}\left[\left|\sum_{j=1}^{A^i}\eta_j^{(s_i)}\right|\geqq\frac{1}{2}(\delta_1-\varrho_2)\chi'(A^i)\right]\right)$$

$$\geqq 1-\sum_{i=1}^{m}\frac{E\left(\sum_{j=1}^{A^i}\eta_j^{(s_i)}\right)^2}{\left(\frac{1}{2}(\delta_1-\delta_2)\chi'(A^i)\right)^2} \tag{58}$$

$$\geqq 1-K\sum_{i=1}^{m}(A^i)^{-\gamma}(1-A^{-\gamma})^{-1}.$$

Thus, from (58) we obtain that

$$U_m=P\left(\bigcup_{i=1}^{m}[|S_{A^i}|>(1-\delta_0)\chi(A^i)]\right)$$

$$\geqq P\left(\bigcup_{i=1}^{m}[|S_{A^i}|>(1-\delta_1)\chi'(A^i)]\right)$$

$$\geqq P\Bigl(\bigcup_{i=1}^{m}\Bigl[\Bigl\{\Bigl|\sum_{j=1}^{A^i}\xi_j^{(s_i)}\Bigr|>(1-\delta_2)\chi'(A^i)\Bigr\}\cap\Bigl\{\Bigl|\sum_{j=1}^{A^i}\eta_j^{(s_i)}\Bigr|\leqq\frac{1}{2}(\delta_1-\delta_2)\chi'(A^i)\Bigr\}\Bigr]\Bigr) \tag{59}$$

$$\geqq P\Bigl(\bigcup_{i=1}^{m}\Bigl[\Bigl|\sum_{j=1}^{A^i}\xi_j^{(s_i)}\Bigr|>(1-\delta_2)\chi'(A^i)\Bigr]\cap C\Bigr)$$

$$\geqq -1+\bar{U}_m+P(C)\geqq\bar{U}_m-KA^{-\tau}(1-A^{-\tau})^{-1}.$$

Next, let $c_k=A^{k/2}$ and choose $\delta_3>0$ such that for some $\varepsilon'>0$, $2/\sqrt{A}+\delta_3+\varepsilon'<\delta_2$. Then

$$P\Bigl(\Bigl[\Bigl|\sum_{j=1}^{A^i}\xi_j^{(s_i)}\Bigr|\leqq(1-\delta_2)\chi'(A^i),\ i<k\Bigr]\cap\Bigl[\Bigl|\sum_{j=A^{k-1}+c_k+1}^{A^k}\xi_j^{(s_k)}\Bigr|>(1-\delta_3)\chi'(A^k)\Bigr]\Bigr)$$

$$\geqq P\Bigl(\Bigl|\sum_{j=1}^{A^i}\xi_j^{(s_i)}\Bigr|\leqq(1-\delta_2)\chi'(A^i),\ i<k\Bigr)P\Bigl(\Bigl|\sum_{j=A^{k-1}+c_k+1}^{A^k}\xi_j^{(s_k)}\Bigr|>(1-\delta_3)\chi'(A^k)\Bigr)-\varphi(c_k-2s_k).$$

Since from (56)

$$v_k=P\Bigl(\Bigl\{\Bigl|\sum_{j=A^{k-1}+c_k+1}^{A^k}\xi_j^{(s_k)}\Bigr|>(1-\delta_3)\chi'(A^k)\Bigr\}\geqq(\log\sigma^2_{A^k-A^{k-1}-c_k}(s_k)\Bigr)^{-(1+\varepsilon)(1-\delta_4)^2}$$

for some $\delta_4>0$ and $\sigma_n^2(s)=n\sigma^2(1+o(1))$ for all sufficiently large n, so

$$v_k\geqq K_1k^{-(1-\lambda_1)}$$

where $\lambda_1>0$ and does not depend on k. Noting that from (42)

$$E\Bigl(\sum_{j=A^{k-1}+1}^{A^{k-1}+c_k}f_j\Bigr)^2\leqq Kc_k$$

and from (53)

$$E\Bigl(\sum_{j=1}^{A^{k-1}+c_k}\eta_j^{(s_k)}\Bigr)^2\leqq(A^{k-1}+c_k)\Bigl\{E|\eta_0^{(s_k)}|^2+2\sum_{j=1}^{A^{k-1}+c_k}|E\eta_0^{(s_k)}\eta_j^{(s_k)}|\Bigr\}$$

$$\leqq(A^{k-1}+c)\Bigl\{(2N+1)\psi(s_k)+K_2\sum_{j=1}^{A^{k-1}+c_k}\Bigl(\Bigl(\frac{3}{j}\Bigr)^{1+\varepsilon-\rho}+\Bigl(\frac{3}{j}\Bigr)^{(1+\varepsilon)(1+\delta)/(2+\delta)+\rho\delta}\Bigl\{\varphi\Bigl(\Bigl[\frac{j}{3}\Bigr]\Bigr)\Bigr\}^{1/2}\Bigr)\Bigr\}$$

$$\leqq K_3A^{k-1-k\gamma}$$

for some γ $(0<\gamma<1)$, where $N=A^{k/2-2\varepsilon_1}$, we obtain

$$E\Bigl(\sum_{j=1}^{A^{k-1}+c_k}\xi_j^{(s_k)}-\sum_{j=1}^{A^{k-1}}\xi_j^{(s_{k-1})}\Bigr)^2$$

$$=E\Bigl(\sum_{j=A^k+1}^{A^{k-1}+c_k}f_j-\sum_{j=1}^{A^{k-1}+c_k}\eta_j^{(s_{k-1})}+\sum_{j=1}^{A^{k-1}}\eta_j^{(s_{k-1})}\Bigr)^2$$

$$\leqq 3\left[E\left(\sum_{j=A^{k-1}+1}^{A^{k-1}+C_k} f_j\right)^2 + E\left(\sum_{j=1}^{A^{k-1}+C_k} \eta_j^{(s_k)}\right)^2 + E\left(\sum_{j=1}^{A^{k-1}} \eta_j^{(s_{k-1})}\right)^2\right]$$

$$\leqq K_4 A^{k-1-k\gamma}$$

and so from Chebyshev's inequality

$$P\left(\left|\sum_{j=1}^{A^{k-1}+C_k} \xi_j^{(s_k)} - \sum_{j=1}^{A^{k-1}} \xi_j^{(s_{k-1})}\right| \geqq \varepsilon' \chi'(A^k)\right) \leqq K_5 A^{-1-k-\gamma}.$$

Hence, as in [8], we have $\bar{U}_k \to 1$ as $k \to \infty$ and consequently $U_k \to 1$ as $k \to \infty$.

The proofs of the following two theorems are carried out by the method of that of Theorem 6. (cf. [3], [4] and [8])

THEOREM 7. *Let $\{x_j\}$ be a stationary process satisfying Condition* (II), *f a random variable which is measurable with respect to $\mathcal{M}_{-\infty}^{\infty}$, and assume that the process $\{f_j\}$ is obtained from f by the method stated above. Let $\{f_j\}$ have the following properties*:

1. *$Ef_j=0$ and $|f_j|<C$ with probability* 1;
2. *$\alpha(n) \leqq Cn^{-(1+\delta_1)}$, where $\delta_1>0$*;
3. *$E\{|f-E\{f|\mathcal{M}_{-k}^{k}\}|^2\}=O(k^{-(2+\delta_2)})$, where $\delta_2>0$.*

Then the law of the iterated logarithm is applicable to the sequence $\{f_j\}$.

THEOREM 8. *Let the stationary process $\{x_j\}$ satisfy Condition* (II), *let f be measurable with respect to $\mathcal{M}_{-\infty}^{\infty}$, and let the process $\{f_j\}$ be obtained from f in the same way stated above. Moreover, suppose that*

1. *$Ef=0$ and for some $\delta>0$, $E|f|^{2+\delta}<\infty$,*
2. *$E\{|f-E\{f|\mathcal{M}_{-k}^{k}\}|^2\}=O(k^{-2-\delta_1})$ $\quad(\delta_1>0)$,*
3. *$\sum_{j=1}^{\infty}\{\alpha(j)\}^{\delta'/(2+\delta')}<\infty$ for some $0<\delta'<\delta$.*

Then the law of the iterated logarithm is applicable to the sequence $\{f_j\}$.

REFERENCES

(1) BILLINGSLEY, P., Convergence of probability measures. Wiley, New York (1968).

(2) ESSEEN, C. G., Fourier analysis of distribution functions. A mathematical study of the Laplace-Gaussian law. Acta Math. **77** (1945), 1–125.

(3) IBRAGIMOV, I. A., Some limit theorems for stationary processes. Theory Prob. Appl. **7** (1962), 349–382.

(4) IBRAGIMOV, I. A., AND YU. V. LINNIK, Independent stationarily related random variables. Iz-vo "Nauka", Moscow (1965). (In Russian)

(5) IOSIFESCU, M., The law of the iterated logarithm for a class of dependent random variables. Theory Prob. Appl. **13** (1968), 304–313.

(6) OODAIRA, H., AND K. YOSHIHARA, Functional central limit theorems for strictly stationary processes satisfying the strong mixing condition. To appear.

(7) PETROV, V. V., On a relation between an estimate of the remainder in the central limit theorem and the law of the iterated logarithm. Theory Prob. Appl. **11** (1966), 454–458.

(8) REZNIK, M. Kh. The law of the iterated logarithm for some classes of stationary processes. Theory Prob. Appl. **13** (1968), 606–621.

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