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SOME ALMOST PRODUCT STRUCTURES ON MANIFOLDS WITH LINEAR CONNECTION

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Differentiable manifolds with almost product structure were investigated by G. Walker [9], Willmore [10], Yano [12], [13] and others. The main purpose of the present paper is to study two special linear connections with respect to which an almost product structure given globally is parallel. These special linear connections have been determined in terms of local coordinates by Schouten [6] and Vranceanu [8].

In § 3 and § 4, we treat the case of the tangent bundle, using the theory of prolongation of tensor fields and linear connections to tangent bundles [1], [5], [11].

§ 1. Let M be an n-dimensional differentiable manifold[1]. Let M_x be the tangent space at each point x of the manifold M. A mixed tensor field defines an endomorphism on each tangent space M_x. If there exists a mixed tensor field P which satisfies

$$P^2=I, \tag{1. 1}$$

we say that the field P gives an *almost product structure* to the manifold and we call the manifold an *almost product manifold*. Then the manifold M carries two globally complementary distributions $\mathscr{D}$ and $\mathscr{D}'$. If we denote with V and V' respectively the projection tensor fields corresponding to the two distributions $\mathscr{D}$ and $\mathscr{D}'$, we have

$$VV'=V'=0, \qquad V+V=I, \qquad V^2=V, \qquad V'^2=V'. \tag{1. 2}$$

We say that X is a vector field which belongs to the distribution $\mathscr{D}$ if, for every point x of manifold M, we have $X_x \in \mathscr{D}_x(M)$, where $\mathscr{D}_x(M)=V(M_x)$.

Let ∇ be a linear connection on manifold M. Then we say that the distribution $\mathscr{D}$ is parallel with respect to the connection ∇ if, for every vector field X which belongs to the distribution $\mathscr{D}$, the vector field $\nabla_Y X$ belongs to the distribution $\mathscr{D}$ for every vector field Y of the manifold M.

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1) The manifolds, functions, tensor fields and connections appearing in the discussion will be supposed to be of the differentiability class C^∞.

§ 2. On a manifold M we consider a linear connection ∇ and an almost product structure P. We define two linear connections $\bar{\nabla}$ and $\tilde{\nabla}$ respectively by

$$\bar{\nabla}_X Y = V\nabla_X VY + V'\nabla_X V'Y, \tag{2.1}$$

$$\tilde{\nabla}_X Y = V\nabla_{VX} VY + V'\nabla_{V'X} V'Y + V[V'X, VY] + V'[VX, V'Y]. \tag{2.2}$$

Proposition 2.1. *The distributions of almost product structure P are both parallel with respect to the connections $\bar{\nabla}$ and $\tilde{\nabla}$ defined by* (2.1) *and* (2.2), *for every connection ∇.*

Proof. Let Y be a vector field which belongs to the distribution $\mathcal{D}$ of the almost product structure P. Then we have $V'Y=0$ and hence, by the formulas (2.1) and (2.2),

$$V'\bar{\nabla}_X Y = 0, \qquad V'\tilde{\nabla}_X Y = 0 \tag{2.3}$$

for every vector field X on the manifold M.

Proposition 2.2. *The connection $\bar{\nabla}$ is equal to the connection ∇ if and only if the distributions of almost product structure P are parallel with respect to the connection ∇.*

Proof. If the connections $\bar{\nabla}$ and ∇ are equal, then by virtue of (2.1), we get

$$V\nabla_X V'Y + V'\nabla_X VY = 0 \tag{2.4}$$

and by (1.2)

$$V\nabla_X V'Y = 0, \qquad V'\nabla_X VY = 0. \tag{2.5}$$

Consequently, the distributions of P are parallel with respect to the connection ∇. The converse can be verified immediately.

Proposition 2.3. *If the linear connection ∇ is symmetric and the almost product structure P is integrable, then the connection $\tilde{\nabla}$ is symmetric.*

Proof. Denoting by $\tilde{T}$ the torsion tensor of the connection $\tilde{\nabla}$, we have

$$V\tilde{T}(X, Y) = V\nabla_{VX} VY + V[V'X, VY] - V\nabla_{VY} VX - V[V'Y, VX] - V[X, Y] \tag{2.6}$$

for any two vector fields X, Y in M. Now, taking account that ∇ is symmetric, we get

$$V\tilde{T}(X, Y) = V[VX, VY] + V[V'X, VY] - V[V'Y, VX] - V[X, Y] \tag{2.7}$$

and, as a consequence of (1.2),

$$V\tilde{T}(X, Y) = V[V'Y, V'X] \tag{2.8}$$

for any vector fields X, Y in M. Analogously, we find

(2. 9) $$V'\tilde{T}(X, Y)=V'[VY, VX]$$

for any vector fields X, Y in M. Thus, taking account that P is integrable, we get

(2. 10) $$V\tilde{T}=V'\tilde{T}=0.$$

Therefore the torsion tensor $\tilde{T}$ is null.

THEOREM 2. 4. *Let a manifold M be with the almost product structure P and a linear connection ∇. Then the formulas* (2. 1) *and* (2. 2) *define respectively linear connections $\bar{\nabla}$ and $\tilde{\nabla}$. If one of them is symmetric, then the almost product structure P is integrable.*

Proof. The structure P is integrable if the distributions $\mathcal{D}$ and $\mathcal{D}'$ are involutive, that is, if

(2. 11) $$V'[VX, VY]=0, \qquad V[V'X, V'Y]=0$$

for any two vector fields X, Y on the manifold M. If the connection $\bar{\nabla}$ is symmetric, we have

(2. 12) $$\bar{\nabla}_X Y-\bar{\nabla}_Y X=[X, Y].$$

Then, the first relation in (2. 11) is equivalent to the condition

(2. 13) $$V'\bar{\nabla}_{VX}VY-V'\bar{\nabla}_{VY}VX=0.$$

This condition is verified by the Proposition 2. 1.

§3. Let $TM=\cup_x M_x$ be the tangent bundle of the manifold M, π the natural projection and π_* its differenaial. Let Y be a vector field on M and $X_x \in M_x$. If we consider Y as a map Y: $M\to TM$, we get the image $Y_*(X_x)$ of X_x by its differential Y_*: $TM\to TTM$. The connection map K: $TTM\to TM$ is determined by

(3. 1) $$KY_*(X_x)=\nabla_{X_x}Y$$

for any $X_x \in M_x$ and any vector field Y on M (for the details see [1]).

For a vector field X on M, there are two vector fields on TM, one called the vertical lift and the other horizontal lift of X and denoted by X^v and X^h respectively, such that

(3. 2) $$KX^v=X, \qquad \pi_*X^v=0,$$
$$KX^h=0, \qquad \pi_*X^h=X.$$

The almost product structure P of TM defined in [3] and [4] is as follows:

(3. 3) $$\pi_*P=K, \qquad KP=\pi_*.$$

Yano and Ishihara [11] defined the horizontal lift ∇^h of a linear connection ∇ by

$$\nabla^h_{X^h} Y^h = (\nabla_X Y)^h, \qquad \nabla^h_{X^h} Y^v = (\nabla_X Y)^v,$$
$$\tag{3.4}$$
$$\nabla^h_{X^v} Y^h = 0, \qquad \nabla^h_{X^v} Y^v = 0.$$

THEOREM 3.1. *The distributions of the almost product structure P defined by* (3.3) *are parallel with respect to the horizontal lift ∇^h of the connection ∇.*

Proof. The projection tensor fields corresponding to the two distributions of the almost product structure P are

$$\bar{P} = \frac{1}{2}(I+P), \qquad \bar{\bar{P}} = \frac{1}{2}(I-P) \tag{3.5}$$

(See [13]). We have, by (3.2) and (3.3),

$$PX^h = X^v, \qquad PX^v = X^h. \tag{3.6}$$

Now, the theorem is immediate by (3.4), (3.5) and (3.6).

It is known that the connection ∇^h is symmetric if and only if ∇ is locally flat [11]. Thus we get

COROLLARY. *The almost product structure P is integrable if and only if the connection ∇ on M is locally flat.*

PROPOSITION 3.2. *Let ∇ be a linear connection in M and P the almost product structure of tangent bundle TM defined by* (3.3). *Then the linear connection $\bar{\nabla}$ associated with the almost product structure P and the horizontal lift ∇^h of the linear connection ∇ coincide to each other.*

Proof. We have to demonstrate that there are satisfied the relations

$$\bar{\nabla}_{X^h} Y^h = (\nabla_X Y)^h, \qquad \bar{\nabla}_{X^h} Y^v = (\nabla_X Y)^v,$$
$$\tag{3.7}$$
$$\bar{\nabla}_{X^v} Y^h = 0, \qquad \bar{\nabla}_{X^v} Y^v = 0.$$

For every vector field X on M, we have

$$\bar{P}X^h = \frac{1}{2}(X^h + X^v), \qquad \bar{\bar{P}}X^h = \frac{1}{2}(X^h - X^v) \tag{3.8}$$

directly from (3.5) and (3.6). From (2.1) and (3.8), we find

$$\bar{\nabla}_{X^h} Y^h = \frac{1}{2}(\nabla^h_{X^h} Y^h + \nabla^h_{X^h} Y^v) + \frac{1}{2}\bar{\bar{P}}(\nabla^h_{X^h} Y^h - \nabla^h_{X^h} Y^v)$$

and consequently, by the definition of the horizontal lift ∇^h and (3.8), we get

$$\bar{\nabla}_{X^h} Y^h = \nabla^h_{X^h} Y^h.$$

Similarly, we can prove the other relations.

§4. Let ξ be a vector field and η a 1-form on the manifold such that

$$\eta(\xi)=1. \tag{4.1}$$

Now we define a mixed tensor field P on TM by

$$\begin{aligned} KP\tilde{X} &= \pi_*\tilde{X} - \eta(\pi_*\tilde{X})\xi - \eta(K\tilde{X})\xi, \\ \pi_*P\tilde{X} &= K\tilde{X} - \eta(\pi_*\tilde{X})\xi - \eta(K\tilde{X})\xi, \end{aligned} \tag{4.2}$$

where $\tilde{X}$ is an arbitrary vector field on TM.

PROPOSITION 4.1. *The mixed field P defined by* (4.2) *is an almost product structure on TM.*

Proof. From (4.2), we get by direct calculation

$$KPP\tilde{X} = K\tilde{X}, \qquad \pi_*PP\tilde{X} = \pi_*\tilde{X}.$$

For the vertical lift X^v and horizontal lift X^h of an arbitrary vector field X, the almost product structure P is characterized by

$$\begin{aligned} PX^v &= X^h - \eta(X)\xi^v - \eta(X)\xi^h, \\ PX^h &= X^v - \eta(X)\xi^v - \eta(X)\xi^h. \end{aligned} \tag{4.3}$$

THEOREM 4.2. *Let M be a manifold and ∇ a linear connection on M. The almost product structure P on TM defined by* (4.2) *is parallel with respect to the horizontal lift ∇^h of ∇ if and only if the mixed tensor field $\eta\otimes\xi$ is parallel with respect to the connection ∇.*

Proof. If P is parallel with respect to the connection ∇^h, we have

$$\nabla^h_{\tilde{X}} P\tilde{Y} = P\nabla^h_{\tilde{X}}\tilde{Y} \tag{4.4}$$

for any vector fields $\tilde{X}$ and $\tilde{Y}$ on TM. If $\tilde{X}=X^h$, $\tilde{Y}=Y^h$, we get

$$\begin{aligned} \nabla^h_{X^h} Y^h &= \nabla^h_{X^h} Y^v - \eta(Y)\nabla^h_{X^h}\xi^h - \eta(Y)\nabla^h_{X^h}\xi^v \\ &\quad - X^h(\eta(Y))\xi^h - X^h(\eta(Y))\xi^v, \\ P\nabla^h_{X^h} Y^h &= \nabla^h_{X^h} Y^v - \eta(\nabla_X Y)\xi^h - \eta(\nabla_X Y)\xi^v \end{aligned} \tag{4.5}$$

for any vector X, Y on M. From (4.4) and (4.5), we have

$$X^h(\eta(Y))\xi^h + \eta(Y)\nabla^h_{X^h}\xi^h = \eta(\nabla_X Y)\xi^h,$$

(4. 6)

$$X^h(\eta(Y))\xi^v + \eta(Y)\nabla^h_{X^h}\xi^v = \eta(\nabla_X Y)\xi^v$$

for any vector fields X, Y on M. Therefore we get

$$(\nabla_X\eta)(Y)\xi + \eta(Y)\nabla_X\xi = 0 \tag{4. 7}$$

for any vector fields X, Y on M. Consequently, the mixed tensor field $\eta\otimes\xi$ is parallel with respect to the linear connection ∇.

References

[1] Dombrovski, P., On the geometry of the tangent bundle. J. reine und augew. Math. **210** (1962), 73–88.

[2] Gray, A., Pseudo-Riemannien almost product manifolds and submersions. J. Math. Mech. **16** (1967), 715–737.

[3] Ianus, S., and C. Udriste, On the tangent bundle of a differentiable manifold. St. Cercet. Mat. **22** (1970), 599–611.

[4] Tamnou, T., Géométrie différentielle du fibré tangent-connexion de type finslérien. Thèse de 3e cycle Université de Grenoble (1969).

[5] Sasaki, S., On the differential geometry of tangent bundles of Riemannian manifolds. Tôhoku Math. J. **10** (1958), 338–354.

[6] Schouten, J., and R. van Kampen, Zur Einbettungs-und Krummungstheorie nichtholonomer Gebilde. Math. Ann. **103** (1930), 752–783.

[7] Tanno, S., An almost complex structure of the tangent bundle of an almost contact manifold. Tôhoku Math. J. **17** (1965), 7–15.

[8] Vranceanu, G., Leçons de Géométrie Différentielle, II. Gauthier-Villars, Paris (1964).

[9] Walker, A. G., Connexions for parallel distribution in the large. Quart. J. Math. (Oxford) (2) **6** (1955), 301–308.

[10] Willmore, T. J., Parallel distributions on manifolds. Proc. London Math. Soc. (3) **6** (1956), 191–204.

[11] Yano, K., and S. Ishihara, Horizontal lift of tensor fields and connections to tangent bundles. J. Math. Mech. **16** (1967), 1015–1029.

[12] Yano, K., and St. Petrescu, Sur les espaces metrique non holonomes complementaires. Disquis. Math. et Phys., Tom I, fas. 1 (1940), 191–246.

[13] Yano, K., Differential geometry on complex and almost, complex spaces. Pergamon Press (1965).

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