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ON THE CAUCHY PROBLEM FOR THE SYSTEM OF FUNDAMENTAL EQUATIONS DESCRIBING THE MOVEMENT OF COMPRESSIBLE VISCOUS FLUID

BY NOBUTOSHI ITAYA

Since the preceding century, a great many studies, classical and modern, have been made on the movement of non-viscous fluid, whether compressible or incompressible, (cf. Lichtenstein [29], Bers [2], Bergmann [1, bibliography], Goldstein [15]). As seen from a methematical point of view, the boundary layer theory by Prandtl [34], [35] is only a variation of the above-mentioned studies.

In early thirties of the present century, Leray [26], [27], [28] began strictly mathematical discussions on the non-stationary problem for incompressible viscous fluid, inventing a concept "solution turbulente", which corresponds to the concept "weak solution" in the present day. About twenty years later, Hopf [16], Kiselev-Ladyzhenskaya [22], and Lions [30], [31] proved the existence of a "generalized" or "weak" solution of this problem, each independently. (Kiselev-Ladyzhenskaya and Lions proved the uniqueness of the solution in a space of functions). In Japan, Ito [17] showed the existence and uniqueness of a classical solution of this problem, and Kato-Fujita [20], [12] improved these results above by the method of functional analysis, utilizing the theories of fractional powers and semi-groups of operators by Glushko-Krein [14], Krasnosel'sky-Sobolevsky [23] (also, cf. [36], 37]), Cattabriga [23], and Kato [18], [19]. They proved the existence and uniqueness of a solution in a certain space of functions. The stationary problem for incompressible viscous fluid was treated by Finn [7] [8], Fujita [11], etc.

As for the stationary and non-stationary problems for compressible viscous fluid, so far as the author knows, there have been very few studies on them, probably, because of the complexities enjoyed by the system of equations describing the movement of this kind of fluid.

In view of these circumstance, we dare to find a way of solving this problem firstly from a classical point of view, restricting the spatial domain to R^3 only. The whole discussion consists of two parts: §1~§4 and §5~§6. The former introduces the problem to be discussed and treats linear equations connected with it. The latter discusses on the original problem, trying to demonstrate the existence and uniqueness of a solution. In the last stage, Tikhonov's fixed point theorem [39] plays a very important role. The proof of the uniqueness of the solution requires very delicate techniques and considerably lengthy calculations.

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CONTENTS

§ 1. Introduction.

1. 1. **Introduction and main theorems.** The movement of fluid is described by a system of five differential equations corresponding to the laws of conservation of mass, momenta, and energy.

i) Conservation of mass.

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho v = 0, \qquad (\rho, \text{ density}; \ v, \text{ velocity}). \tag{1. 1}$$

ii) Conservation of momenta.

$$\rho \frac{Dv}{Dt} = \rho f + \operatorname{Div} P, \tag{1. 2}$$

$$\left(\frac{D}{Dt} = \frac{\partial}{\partial t} + (v \cdot \nabla); \ f, \text{ outer force}; \ P, \text{ stress tensor}; \ \operatorname{Div} P = \frac{\partial}{\partial x_i} P_{ik}\right).$$

In isotropic Newtonian fluid, P is expressed in the following way:

$$P_{ik}=-\left(p+\frac{2}{3}\mu\Theta\right)\delta_{ik}+\mu e_{ik}, \qquad (p, \text{ pressure}; \ \mu, \text{ viscosity}), \tag{1. 3}$$

where

$$e_{ik}=\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}, \qquad \Theta=\frac{1}{2}e_{kk}=\operatorname{div} v. \tag{1. 3$'$}$$

From (1. 2), by (1. 3), we obtain the so-called Navier-Stokes equation:

$$\rho\frac{Dv}{Dt}=\rho f-\operatorname{grad}\left(p+\frac{2}{3}\mu\operatorname{div} v\right)+(\nabla\cdot\mu\nabla)v+\operatorname{Div}(\mu\operatorname{grad} v). \tag{1. 4}$$

a) If μ is a constant, (1. 4) has the form:

$$\rho\frac{Dv}{Dt}=\rho f-\nabla p+\frac{\mu}{3}\operatorname{div} v+\mu\Delta v. \tag{1. 4$'$}$$

b) If μ and ρ are constants, (1. 1) and (1. 4) have the forms:

$$\operatorname{div} v=0, \tag{1. 1$'$}$$

$$\rho\frac{Dv}{Dt}=\rho f-\nabla p+\mu\Delta v. \tag{1. 4$''$}$$

c) If $\mu=0$, (1. 4) has the form:

$$\rho\frac{Dv}{Dt}=\rho f-\nabla p, \qquad \text{(Euler's equation of perfect gas).} \tag{1. 4$'''$}$$

iii) Conservation of energy.

$$\rho\frac{DE}{Dt}=\frac{1}{2}P_{ik}e_{ik}+\operatorname{div}(\kappa\operatorname{grad}\theta)=\Psi-p\operatorname{div} v+\operatorname{div}(\kappa\operatorname{grad}\theta). \tag{1. 5}$$

(E, internal energy; κ, heat conductivity; θ, absolute temperature),

where Ψ is the function of dissipation defined by

$$\Psi\equiv p\Theta+\frac{1}{2}P_{ik}e_{ik}=\mu\sum_{i<k}e_{ik}^2+\frac{\mu}{6}\sum_{i<j}(e_{ii}-e_{jj})^2(\geqq 0). \tag{1. 6}$$

$$\frac{DE}{Dt}=\frac{\partial E}{\partial\rho}\frac{D\rho}{Dt}+\frac{\partial E}{\partial\theta}\frac{D\theta}{Dt}=-\frac{1}{\rho^2}\left(\theta\frac{\partial p}{\partial\theta}-p\right)(-\rho\operatorname{div} v)+C_v\left(\frac{\partial\theta}{\partial t}+v\cdot\nabla\theta\right), \tag{1. 7}$$

(C_v, specific heat at constant volume).

Thus, we have

$$\rho C_v\frac{\partial\theta}{\partial t}=\operatorname{div}(\kappa\operatorname{grad}\theta)+\Psi-\theta\frac{\partial p}{\partial\theta}\operatorname{div} v-C_v\rho v\cdot\nabla\theta. \tag{1. 5$'$}$$

If we assume that p is virially expanded, i.e.,

(1. 8) $\quad p=\theta\sum_{n=1}\hat{a}_n\rho^n, \quad (0<\rho<\rho^*=\text{radius of convergence of the power-series}<+\infty),$

then (1. 5)$'$ has a form,

(1. 5)$''$ $$\rho C_v\frac{\partial\theta}{\partial t}=\operatorname{div}(\kappa\operatorname{grad}\theta)+\Psi-p\operatorname{div}v-C_v\rho v\cdot\nabla\theta.$$

When μ, κ, and C_v are constants and p is virially expanded, the system of equations to be considered is as follows:

$$\begin{cases}\dfrac{\partial\rho}{\partial t}+\operatorname{div}\rho v=0, & (1.\,9)^1\\[2mm] \rho\dfrac{\partial v}{\partial t}=\rho f-\dfrac{\partial p}{\partial\theta}\nabla\theta-\dfrac{\partial p}{\partial\rho}\nabla\rho+\mu\Delta v+\dfrac{\mu}{3}\nabla\cdot\operatorname{div}v-\rho(v\cdot\nabla)v, & (1.\,9)^2\\[2mm] \rho C_v\dfrac{\partial\theta}{\partial t}=\kappa\Delta\theta+\Psi-p\cdot\operatorname{div}v-C_v\rho v\cdot\nabla\theta. & (1.\,9)^3\end{cases}$$

Hereafter, we shall consider a Cauchy problem of (1. 9) in which the initial condition is given by

(1. 10) $\quad v(x,0)=v_0(x), \qquad \theta(x,0)=\theta_0(x)(\geqq 0), \qquad \rho(x,0)=\rho_0(x)(>0).$

As final results for the Cauchy problem (1. 9)–(1. 10), we have the following three main theorems. (As for notations, see 1. 2, 5. 1, 5. 5, and § 6).

THEOREM 1. *For some* $T'\in(0,T]$, *there exists* $(v,\theta,\rho)\in H_{T'}^{2+\alpha}\times H_{T'}^{2+\alpha}\times B_{T'}^{1+\alpha}$ $(\theta\geqq 0$ *and* $+\infty>\rho^*>\rho>0)$ *such that* (v,θ,ρ) *satisfies* (1. 9)–(1. 10), *where* $(v_0,\theta_0,\rho_0)\in H^{2+\alpha}\times H^{2+\alpha}\times H^{1+\alpha}$ $(0<\bar{\rho}_0\leqq\rho_0\leqq\bar{\bar{\rho}}_0<\rho^*$, $\bar{\rho}_0$ *and* $\bar{\bar{\rho}}_0$ *are constants) and* $f\in H_T^{\alpha}$.

THEOREM 2. *For* $(v_0,\theta_0,\rho_0)\in H^{3+L}\times H^{2+\alpha}\times H^{2+L}$ $(0<\bar{\rho}_0\leqq\rho_0\leqq\bar{\bar{\rho}}_0<\rho^*<+\infty)$ *and* $f\in H_T^{\alpha}$, *if there exists a solution* (v,θ,ρ) *of* (1. 9)–(1. 10) *in*

$$\Big[H_T^{2+\alpha}\cap\Big\{u(x,t):D_x^m u(|m|=3)\ \textit{are continuous},\ \sum_{|m|=3}(|D_x^m u|_T^{(0)}+|D^m u|_{x,T}^{(L)})<+\infty\Big\}\Big]\times H_T^{2+\alpha}\times B_T^{1+\alpha},$$

then the solution is unique there. ("L" *indicates that the Hölder exponent* $\alpha=1$).

THEOREM 3. *If* $(v_0,\theta_0,\rho_0)\in H^{4+\alpha}\times H^{3+\alpha}\times H^{2+L}$ $(0<\bar{\rho}_0\leqq\rho_0\leqq\bar{\bar{\rho}}_0<\rho^*<+\infty)$ *and* $f\in H_T^{2+\alpha}$, *then, for some* $T'\in(0,T]$, *there exists a unique solution* (v,θ,ρ) *of* (1. 9)–(1. 10) *in* $H_{T'}^{4+\alpha}\times H_{T'}^{3+\alpha}\times B_{T'}^{1+\alpha}$ $(\theta\geqq 0$ *and* $\rho^*>\rho>0)$.

1. 2. **Preliminary surveys.** In connection with (1. 9)^2, we consider the following linear problem:

$$(1.11)\qquad \begin{cases} \dfrac{\partial v}{\partial t}=\sigma(x,t)\left(\Delta+\dfrac{1}{3}\nabla\,\mathrm{div}\right)v+f, & ((x,t)\in R_T^3\equiv R^3\times(0,T)),\\ v(x,0)=v_0(x), & (x\in R^3),\end{cases}$$

where

$$(1.11)\qquad \sigma, f\in H_T^\alpha,\quad 0<\sigma_0\leqq\sigma(x,t)\leqq\sigma_1\equiv|\sigma|_T^{(0)}<+\infty,\quad v_0(x)\in H^{2+\alpha},\quad (\alpha\in(0,1)).$$

For $\alpha\in(0,1)$, $H_T^{n+\alpha}$ (n, integer $\geqq 0$) is a Banach space of real functions $g(x,t)$ defined on $\overline{R_T^3}$ such that

$$(1.12)\qquad \|g\|_T^{(n+\alpha)}\equiv\sum_{2r+|m|=0}^{n}|D_t^rD_x^mg|_T^{(0)}+\sum_{2r+|m|=n-1}^{n}|D_t^rD_x^mg|_{t,T}^{(\alpha/2)}+\sum_{2r+|m|=n}|D_t^rD_x^mg|_{x,T}^{(\alpha)}<+\infty,$$

where, for $h(x,t)$,

$$(1.12)'\qquad \begin{cases} |h|_T^{(0)}\equiv\sup\limits_{(x,t)\in R_T^3}|h(x,t)|, & |h|_{t,T}^{(\alpha/2)}\equiv\sup\limits_{\substack{t,t',x\\(t\neq t')}}\dfrac{|h(x,t)-h(x,t')|}{|t-t'|^{\alpha/2}},\\ |h|_{x,T}^{(\alpha)}\equiv\sup\limits_{\substack{x,x',t\\(x\neq x')}}\dfrac{|h(x,t)-h(x',t)|}{|x-x'|^{\alpha}}, & |h|_T^{(\alpha)}\equiv|h|_{x,T}^{(\alpha)}+|h|_{t,T}^{(\alpha/2)}.\end{cases}$$

and, for $m=(m_i)$ ($m_i\geqq 0$),

$$(1.12)''\qquad |m|\equiv\sum_i m_i,\qquad D_x^m\equiv\frac{\partial^{|m|}}{\partial x_1^{m_1}\partial x_2^{m_2}\partial x_3^{m_3}}.$$

For a vector function $g(x,t)=(g_i)$,

$$(1.13)\qquad |g|_T^{(0)}\equiv\sum_i|g_i|_T^{(0)},\qquad |g|_T^{(\alpha)}\equiv\sum_i|g_i|_T^{(\alpha)},\qquad \|g\|_T^{(n+\alpha)}\equiv\sum_i\|g_i\|_T^{(n+\alpha)},\quad \text{etc.}$$

$H^{n+\alpha}$ is a Banach space of functions $u(x)$ defined on R^3 such that

$$(1.13)'\qquad \|u\|^{n+\alpha}\equiv\sum_{|m|=0}^{n}|D_x^mu|^{(0)}+\sum_{|m|=n}|D_x^mu|^{(\alpha)}<+\infty,$$

where

$$(1.13)''\qquad |u|^{(0)}\equiv\sup_{x\in R^3}|u(x)|,\qquad |u|^{(\alpha)}\equiv\sup_{\substack{x,x'\\(x\neq x')}}\frac{|u(x)-u(x')|}{|x-x'|^{\alpha}}.$$

We put

$$(1.14)\qquad P_0(D_x)\equiv\Delta+\frac{1}{3}\nabla\,\mathrm{div}.$$

LEMMA 1.1. *The system* (1.11) *is uniformly parabolic in Petrowsky's sense, i.e., there exists a number* $\delta>0$ *such that*

(1.15) $$\max_j \sup_{|\xi|=1} \mathrm{Re}\,\{\lambda_j(\xi; x, t)\} \leqq -\delta, \qquad ({}^\forall(x, t) \in R^3_T),$$

(λ_j's are the roots of $\det[\sigma(x, t)P_0(i\xi)-I\lambda]$).

Proof. For $|\xi|=1$, λ_j $(j=1, 2, 3)$ satisfies

(1.16) $$\det[\sigma(x, t)P_0(i\xi)-I\lambda]=\lambda^3+\frac{4}{3}\sigma^2\lambda+\sigma^3\left(\frac{4}{3}-\frac{2}{27}\xi_1^2\xi_2^2\xi_3^2\right)\equiv D(\lambda; \sigma; \xi)=0.$$

(1.16)′ $$\begin{cases} \dfrac{d}{d\lambda}D(\lambda; \sigma; \xi)=3\lambda^2+\dfrac{8}{3}\sigma\lambda+\dfrac{11}{3}\sigma^2=3\left(\lambda+\dfrac{4}{9}\sigma\right)^2+\dfrac{7}{3}\sigma^2\geqq\dfrac{7}{3}\sigma^2, \quad (\text{for } \lambda\in R^1), \\ \dfrac{d}{d\lambda}D(0; \sigma; \xi)=\dfrac{11}{3}\sigma^2, \\ D(0; \sigma; \xi)=\sigma^3\left(\dfrac{4}{3}-\dfrac{2}{27}\xi_1^2\xi_2^2\xi_3^2\right)\geqq\dfrac{35}{27}\sigma^3>0. \end{cases}$$

By (1.16)′, $D(\lambda; \sigma; \xi)$ has one negative root λ_1 and two complex conjugate roots λ_2 and $\lambda_3=\bar{\lambda}_2$. λ_1, λ_2, and λ_3 satisfy

$$\lambda_1+\lambda_2+\lambda_3=-\frac{4}{3}\sigma.$$

$D_\lambda(\lambda; \xi; \sigma)\equiv(d/d\lambda)D(\lambda; \xi; \sigma)$ increases monotonically in $[-4\sigma/9, +\infty)$, and

$$D\left(-\frac{4}{9}\sigma; \sigma; \xi\right)=\frac{4}{3}\left(1-\frac{361}{243}\right)\sigma^3-\frac{2}{27}\sigma^3(\xi_1\xi_2\xi_3)^2\leqq-\frac{4}{3}\cdot\frac{118}{243}\sigma^3<0.$$

Therefore,

$$-\frac{4}{9}\sigma\leqq\lambda_1\leqq-\{D_\lambda(0; \sigma; \xi)\}^{-1}\cdot D(0; \sigma; \xi)\leqq-\frac{3}{11}\sigma\left(\frac{4}{3}-\frac{2}{27}\xi_1^2\xi_2^2\xi_3^2\right)$$

(1.17) $$\leqq-\frac{3}{11}\sigma\left(\frac{4}{3}-\frac{2}{27^2}\right)\leqq-\frac{35}{99}\sigma\leqq-\frac{35}{99}\sigma_0<0,$$

$$\mathrm{Re}\,\lambda_2=\mathrm{Re}\,\lambda_3=\frac{1}{2}\left\{-\frac{4}{3}\sigma-\lambda_1\right\}\leqq\frac{1}{2}\left\{-\frac{4}{3}\sigma+\frac{4}{9}\sigma\right\}=-\frac{4}{9}\sigma\leqq-\frac{4}{9}\sigma_0<0.$$

Thus,

(1.18) $$\max_j \sup_{|\xi|=1} \mathrm{Re}\,\lambda_j(\xi; x, t)\leqq-\frac{35}{99}\sigma_0.$$

This shows that the system (1.11) is uniformly parabolic in Petrowsky's sense.
Q.E.D.

In the same way, we can relate $(1.9)^3$ with the following linear equation:

$$\text{(1. 19)}\qquad \begin{cases} \dfrac{\partial\theta}{\partial t}=\tilde{\sigma}(x,t)\Delta\theta+g, & (\tilde{\sigma},g\in H_T^{\alpha,\alpha/2},\ 0<\tilde{\sigma}_0\leqq\tilde{\sigma}\leqq\tilde{\sigma}_1<+\infty), \\ \theta(x,0)=\theta_0(x). \end{cases}$$

It is obvious that this equation is uniformly parabolic in Petrowsky's sense. So, we can treat (1. 19) in parallel with (1. 11).

§ 2. The fundamental solution of a linear problem.

2. 1. **Parametix** $Z(x-\xi,t;y,\tau)$. In connection with (1. 11), we consider the following system of ordinary differential equations in t:

$$\text{(2. 1)}\qquad \begin{cases} \dfrac{dV}{dt}=\sigma(y,\tau)P_0(i\zeta)V(\zeta,t;y,\tau), & (\zeta\in C^3), \\ V(\zeta,t;y,\tau)|_{t=\tau}=I & \text{(unit matrix)}. \end{cases}$$

V can be solved directly from (2. 1), i.e.,

$$\text{(2. 2)}\qquad V(\zeta,t;y,\tau)=\exp\{(t-\tau)\sigma(y,\tau)P_0(i\zeta)\}.$$

We define the parametrix $Z=(Z^{ij}(x-\xi,t;y,\tau))$ of (2. 1) in the following way:

$$\text{(2. 3)}\qquad Z^{ij}(x-\xi,t;y,\tau)\equiv\frac{1}{(2\pi)^3}\int_{R^3}e^{i\xi_0(x-\xi)}V^{ij}(\xi_0,t;y,\tau)d\xi_0,\qquad (i,j=1,2,3).$$

We note that the following relation holds between the roots λ_j $(j=1,2,3)$ of $\det[\sigma(x,t)P_0(i\zeta)-\lambda I]$ and the roots $\lambda_j^{(0)}$ of $\det[P_0(i\zeta)-\lambda I]$:

$$\text{(2. 4)}\qquad \lambda_j(\zeta;x,t)=\sigma(x,t)\lambda_j^{(0)}(\zeta),\qquad (j=1,2,3).$$

For $\lambda_j^{(0)}$ $(i=1,2,3)$, we have

$$\text{(2. 5)}\qquad \max_j\sup_{|\xi|=1}\operatorname{Re}\lambda_j^{(0)}(\xi)\leqq-\frac{35}{99},$$

$$\text{(2. 5)}'\qquad \lambda_j^{(0)}(\nu\zeta)=\nu^2\lambda_j^{(0)}(\zeta),\qquad (\nu\geqq 0,\ \zeta\in C^3).$$

The following two lemmas are taken from Friedman ([9], [10]) with some modifications.

LEMMA 2. 1. *If $f(z)$ is a continuous real-valued function on C^n, satisfying*

$$\text{(2. 6)}\qquad \begin{cases} f(\nu z)=\nu^{2p}f(z), & (\nu\geqq 0,\ z\in C^n;\ p,\ \text{positive integer}), \\ f(x)\leqq-\delta_0|x|^{2p}, & (x\in R^n,\ \delta_0>0), \end{cases}$$

then, there exists a number $a_0>0$ such that

(2.6)′ $f(x+iy) \leqq -\frac{\delta_0}{2}|x|^{2p}+a_0|y|^{2p}$, $(x, y \in R^3)$, (*cf.* [9], *CH. IX*, § 2).

LEMMA 2.2. *For the matrix* $e^{t\sigma(y,\tau)}P_0(i\zeta)$,

$$e^{t\sigma(y,\tau)P_0(i\zeta)} \leqq 2\sqrt{3}\{1+\sigma(y,\tau)|P_0(i\zeta)|\, t+\sigma(y,\tau)^2|P_0(i\zeta)|^2t^2\} \exp\left\{t \max_i \operatorname{Re}\{\sigma(y,\tau)\lambda_i^{(0)}(\zeta)\}\right\},$$

(*cf.* [10], *CH.* 7, 2),

where, for the matrix $A=(a_{ij})$,

(2.7)′ $$|A| \equiv \sum_{i,j}|a_{ij}|.$$

$$\left(N.B.: \frac{|A|}{2\sqrt{3}} \leqq \|A\| \text{ (norm as an operature)} \leqq (\sum|a_{ij}|^2)^{1/2} \leqq |A|\right).$$

By Lemma 2.1, we have:

LEMMA 2.3. *There exist* δ *and* $a>0$ *such that*

(2.8) $$\operatorname{Re}\{\sigma(y,\tau)\lambda_i^{(0)}(\zeta=\xi+i\eta)\} \leqq \sigma(y,\tau)\left\{-\frac{\delta}{2}|\xi|^2+a|\eta|^2\right\}, \quad (i=1,2,3).$$

Proof. We have only to put,

(2.8)′ $$f_i(\zeta)=\operatorname{Re}\{\sigma(y,\tau)\lambda_i^{(0)}(\zeta)\}, \qquad \delta_0=\delta\equiv\frac{35}{99}.$$ Q.E.D.

If we define for $\zeta \in C^3$

(2.9) $$|\zeta|^2 \equiv |\zeta_1|^2+|\zeta_2|^2+|\zeta_3|^2,$$

then

(2.9)′ $$|P_0(i\zeta)| \leqq 4|\zeta|^2.$$

By the lemmas 2.2 and 2.3, we have:

LEMMA 2.4.

(2.10) $$\begin{aligned}|V(\zeta,t;y,\tau)| &= |e^{(t-\tau)\sigma(y,\tau)P_0(i\zeta)}| \\ &\leqq C_0 \exp\left\{(t-\tau)\left\{-\frac{\delta}{4}\sigma_0|\xi|^2+\left(\frac{a}{4}+\delta\right)\sigma_1|\eta|^2\right\}\right\} \equiv C_0e^{(t-\tau)Q_0(\xi,\eta)}.\end{aligned}$$

Proof.

$$-\frac{\delta}{2}|\xi|^2+a|\eta|^2=-\frac{\delta}{4}|\zeta|^2-\frac{\delta}{4}|\xi|^2+\left(\frac{\delta}{4}+a\right)|\eta|^2.$$

$$[\sigma(y,\tau)(t-\tau)|\zeta|^2]^2 \exp\left\{-\frac{\delta}{4}(t-\tau)\sigma(y,\tau)|\zeta|^2\right\} \leqq \left(\frac{4}{\delta}\right)^2 K(2,0), \qquad \left(K(\alpha,b)\equiv\left(\frac{\alpha}{e(1-b)}\right)^\alpha\right).$$

Hence,

$$|V(\zeta,t;y,\tau)| \leqq C_0\cdot\exp\left\{(t-\tau)\sigma(y,\tau)-\frac{\delta}{4}|\xi|^2+\left(a+\frac{\delta}{4}\right)|\eta|^2\right\} \leqq C_0 e^{(t-\tau)Q_0(\xi,\eta)}. \quad \text{Q.E.D.}$$

LEMMA 2.5. *For the index-vector* m $(|m|\geqq 0)$,

$$(2.11) \qquad D_x^m Z^{ij}(x-\xi,t;y,\tau) \leqq C_1^{(|m|)}(t-\tau)^{-(|m|+3)/2}\exp\left\{-\frac{|x-\xi|^2}{24a_1\sigma_1(t-\tau)}\right\}, \quad \left(a_1\equiv a+\frac{\delta}{4}\right).$$

Proof. By Cauchy's theorem, for $\eta\in R^3$,

$$(2.12) \qquad Z^{ij}(x-\xi,t;y,\tau)=\frac{1}{(2\pi)^3}\int_{R^3} e^{i(\xi_0+i\eta)(x-\xi)}V^{ij}(\xi_0+i\eta,t;y,\tau)d\xi_0.$$

$$(2.13) \qquad D_x^m Z^{ij}(x-\xi,t;y,\tau)=\frac{1}{(2\pi)^3}\int_{R^3}(i(\xi_0+i\eta))^m e^{i(\xi_0+i\eta)\cdot(x-\xi)}V^{ij}(\xi_0+i\eta,t;y,\tau)d\xi_0.$$

$$|(\xi_0+i\eta)^m| \leqq (|\xi_0|+|\eta|)^{|m|} \leqq 2^{|m|-1}(|\xi_0|^{|m|}+|\eta|^{|m|}).$$

Hence,

$$|D_x^m Z^{ij}(x-\xi,t;y,\tau)| \leqq \frac{C_0 2^{|m|-1}}{(2\pi)^3}\cdot e^{-\eta\cdot(x-\xi)}\cdot e^{a_1\sigma(y,\tau)}|\eta|^2(t-\tau)$$
$$\times\int_{R^3}(|\xi_0|^{|m|}+|\eta|^{|m|})e^{-(t-\tau)(\delta/4)\sigma(y,\tau)|\xi_0|^2}d\xi_0$$

We choose η such that

$$(2.14) \qquad \begin{cases} \eta_i=|\eta_1|\,\mathrm{sgn}\,(x_i-\xi_i), & (i=1,2,3),\\ |\eta|=\dfrac{|x-\xi|}{\sqrt{3}\cdot 2a_1\sigma(y,\tau)(t-\tau)}. \end{cases}$$

$$-\eta\cdot(x-\xi) \leqq -|\eta|\cdot\frac{|x-\xi|}{\sqrt{3}},$$

$$-\eta\cdot(x-\xi)+a_1\sigma(y,\tau)(t-\tau)|\eta|^2$$
$$\leqq -|\eta|\cdot\frac{|x-\xi|}{\sqrt{3}}+(t-\tau)a_1\sigma(y,\tau)|\eta|^2=-\frac{|x-\xi|^2}{12a_1\sigma(y,\tau)(t-\tau)}$$

Thus,

$$
(2.15)\quad
\begin{aligned}
&|D_x^m Z^{ij}(x-\xi, t; y, \tau)| \\
&\leqq C_0 \frac{2^{|m|-1}}{(2\pi)^3} \cdot \sigma(y, \tau)^{-(|m|+3)/2} \left\{ \left(\frac{4}{\delta}\right)^{(|m|+3)/2} \cdot I(|m|+2) \right. \\
&\left. + K\left(\frac{|m|}{2}, \frac{1}{2}\right)(2\sqrt{3})^{|m|}(12a_1)^{-|m|/2}\left(\frac{4\pi}{\delta}\right)^{3/2}\right\}(t-\tau)^{-(|m|+3)/2} \cdot \exp\left\{\frac{|x-\xi|^2}{24a_1\sigma_1(t-\tau)}\right\} \\
&\leqq C_1^{(|m|)}(C_0, \sigma_0, a_1)(t-\sigma)^{-(|m|+3)/2} \exp\left\{-\frac{|x-\xi|^2}{24a_1\sigma_1(t-\tau)}\right\},
\end{aligned}
$$

$$
(2.15)' \qquad \left(I(\mu) \equiv \int_0^\infty r^\mu e^{-r^2} dr\right). \qquad \text{Q.E.D.}
$$

LEMMA 2. 6. *For m such that* $|m| \geqq 1$,

$$
(2.16) \qquad \int_{R^3} D_x^m Z^{ij}(x-\xi, t; y, \tau) dx = 0, \qquad (t > \tau).
$$

Proof. Using the preceding lemma, we can prove this by means of direct integration; e.g.,

$$
(2.17)\quad
\begin{aligned}
\int_{R^3} D_{x_1} Z^{ij}(x-\xi, t; y, \tau) dx &= \lim_{N\to+\infty} \int_{R^2} dx_2 dx_3 \int_{-N}^{+N} D_{x_1} Z^{ij}(x-\xi, t; y, \tau) dx_1 \\
&= \int_{R^2} dx_2 dx_3 \lim_{N\to+\infty} \{Z^{ij}(x-\xi, t; y, \tau)|_{x_1=-N}^{N}\} = 0. \qquad \text{Q.E.D.}
\end{aligned}
$$

COROLLARY of Lemma 2. 6.

$$
(2.18) \qquad \int_{R^3} D_t Z^{ij}(x-\xi, t; y, \tau) dx = 0.
$$

Proof. This is obvious from the fact that Z satisfies

$$
(2.18)' \qquad D_t Z = \sigma(y, \tau) P_0(D_x) Z. \qquad \text{Q.E.D.}
$$

2. 2. **Some calculations.** The difference

$$
(2.19) \qquad W(\zeta, t; y, \tau; h) \equiv V(\zeta, t; y, \tau) - V(\zeta, t; y+h, \tau)
$$

satisfies

$$
(2.20) \qquad
\begin{cases}
\dfrac{dW}{dt} = \sigma(y, \tau) P_0(i\zeta) W + \{\sigma(y, \tau) - \sigma(y+h, \tau)\} P_0(i\zeta) V(\zeta, t; y+h, \tau), \\
W|_{t=\tau} = 0, \qquad (t > \tau \geqq 0).
\end{cases}
$$

Hence

$$W(\zeta, t; y, \tau; h)=\int_{\tau}^{t} e^{(t-\tau_0)\sigma(y,\tau)P_0(i\zeta)}\{\sigma(y, \tau)-\sigma(y+h; \tau)\}P_0(i\zeta)V(\zeta, \tau_0; y+h, \tau)d\tau_0. \tag{2. 20$'$}$$

From (2. 20)$'$, we have:

LEMMA 2. 7.

$$\begin{aligned}|W(\zeta, t; y, \tau; h)| &\leqq |h|^{\alpha}C_2 \exp\left\{(t-\tau)-\frac{\delta}{8}\sigma_0|\xi|^2+\left(a+\frac{3}{8}\delta\right)\sigma_1|\eta|^2\right\}\\ &\equiv |h|^{\alpha}C_2e^{(t-\tau)Q_1(\xi,\eta)}.\end{aligned} \tag{2. 21}$$

Proof.

$$\begin{aligned}|W(\zeta; t; y, \tau; h)| &\leqq \int_{\tau}^{t} C_0^2\cdot\exp\{(t-\tau_0)Q_0(\xi, \eta)\}\cdot|\sigma|_T^{(\alpha)}|h|^{\alpha}\cdot 4|\zeta|^2\cdot\exp\{(\tau_0-\tau)Q_0(\xi, \eta)\}d\tau_0\\ &\leqq |h|^{\alpha}\cdot(C_0)^2\cdot 4\frac{4}{\delta\sigma_0}K(1, 0)\exp\left\{(t-\tau)\left\{-\frac{\delta}{8}\sigma_0|\xi|^2+\left(a+\frac{3}{8}\delta\right)\sigma_1|\eta|^2\right\}\right\}\\ &\equiv |h|^{\alpha}C_2e^{(t-\tau)Q_1(\xi,\eta)}.\end{aligned} \tag{2. 22}$$

Q.E.D.

LEMMA 2. 8. *For m ($|m|\geqq 0$),*

$$\begin{aligned}&|D_x^mZ^{ij}(x-\xi, t; y, \tau)-D_x^mZ^{ij}(x-\xi, t; y+h, \tau)|\\ &\leqq |h|^{\alpha}\bar{C}_1^{(|m|)}\exp\left\{-\frac{1}{24a_2\sigma_1}\cdot\frac{|x-\xi|^2}{t-\tau}\right\}(t-\tau)^{-(|m|+3)/2}, \quad \left(a_2=a+\frac{3}{8}\delta\right)\end{aligned} \tag{2. 23}$$

Proof.

$$\begin{aligned}&D_x^mZ^{ij}(x-\xi, t; y, \tau)-D_x^mZ^{ij}(x-\xi, t; y+h, \tau)\\ &=(2\pi)^{-3}\int_{R^3}(i(\xi_0+i\eta))^m e^{i(\xi_0+i\eta)(x-\xi)}W^{ij}(\xi_0+i\eta, t; y, \tau; h)d\xi_0.\end{aligned} \tag{2. 23}$$

From this equality we obtain (2. 23) as we did (2. 11) from (2. 13). Q.E.D.

In the same way we have the following lemmas.

LEMMA 2. 9.

$$\begin{aligned}&|D_x^mZ^{ij}(x-\xi, t; y, \tau)-D_{x'}^mZ^{ij}(x'-\xi, t; y, \tau)|\\ &\leqq C_1^{(|m|+1)}(t-\tau)^{-(|m|+4)/2}\cdot\exp\left\{-\frac{1}{24a_1\sigma_1}\cdot\frac{|\hat{x}-\xi|^2}{t-\tau}\right\}|x-x'|,\end{aligned} \tag{2. 24}$$

($\hat{x}\in\overline{xx'}$ = *convex hull of* $\{x, x'\}$).

LEMMA 2. 10. *For $t>t'>\tau$*

$$
\begin{aligned}
&|D_x^m Z^{ij}(x-\xi, t; y, \tau)-D_x^m Z^{ij}(x-\xi, t'; y, \tau)| \\
&\leqq 4\cdot\sigma_1 C_1^{(|m|+2)}(t-t')(t'-\tau)^{-(|m|+5)/2}\cdot\exp\left\{-\frac{1}{24a_1\sigma_1}\cdot\frac{|x-\xi|^2}{t-\tau}\right\}.
\end{aligned}
\tag{2.25}
$$

LEMMA 2.11.

$$
\begin{aligned}
&|D_t D_x^m Z^{ij}(x-\xi, t; y, \tau)-D_t D_x^m Z^{ij}(x-\xi, t; y+h, \tau)| \\
&\leqq |h|^\alpha C_3^{(|m|)}(t-\tau)^{-(|m|+5)/2}\exp\left\{-\frac{1}{24a_2\sigma_1}\cdot\frac{|x-\xi|^2}{t-\tau}\right\}.
\end{aligned}
\tag{2.26}
$$

LEMMA 2.12.

$$
\begin{aligned}
&|D_t Z^{ij}(x-\xi, t; y, \tau)-D_t Z^{ij}(x-\xi, t; y', \tau)| \\
&\leqq 2\{C_1^{(2)}|\sigma|_T^{(\alpha)}+\sigma_1\bar{C}_1^{(2)}\}|y-y'|^\alpha(t-\tau)^{-5/2}\exp\left\{-\frac{|x-\xi|^2}{24a_2\sigma_1(t-\tau)}\right\}.
\end{aligned}
\tag{2.27}
$$

2. 3. **Construction of the fundamental solution.** It is known that the system (1. 11) has a unique solution in $H_T^{2+\alpha}$, (cf. [24], [6]). [More generally, (1. 11) has a unique regular solution (cf. [6], Ch. 2 and 3)].

As shown in § 4, $v(x, t)$ is expressed as

$$
v(x, t)=v_0(x)+\int_0^t d\tau\int_{R^3}\Gamma(x, t; \xi, \tau)\{f(\xi, \tau)+\sigma(\xi, \tau)\}P_0(D_\xi)v_0(\xi)d\xi,
\tag{2.28}
$$

where $\Gamma(x, t; \xi, \tau)$ is defined by

$$
\Gamma(x, t; \xi, \tau)\equiv Z(x-\xi, t; \xi, \tau)+\int_\tau^t d\tau_0\int_{R^3} Z(x-y, t; y, \tau_0)\Phi(y, \tau_0; \xi, \tau)dy, \qquad (t>\tau\geqq 0)
\tag{2.29}
$$

and Φ satisfies a Volterra-type integral equation,

$$
\Phi(x, t; \xi, \tau)=K(x, t; \xi, \tau)+\int_\tau^t d\tau_0\int_{R^3} K(x, t; y, \tau_0)\Phi(y, \tau_0; \xi, \tau)dy, \qquad (t>\tau\geqq 0)
\tag{2.30}
$$

$$
\begin{aligned}
(K(x, t; \xi, \tau)&\equiv\{\sigma(x, t)P_0(D_x)-ID_t\}Z(x-\xi, t; \xi, \tau) \\
&=\{\sigma(x, t)-\sigma(\xi, \tau)\}P_0(D_x)Z(x-\xi, t; \xi, \tau)).
\end{aligned}
\tag{2.31}
$$

$\Gamma(x, t; \xi, \tau)$ is the unique fundamental solution of (1. 11) ($f=0$), so far as solutions in $H_T^{2+\alpha}$ (more generally, bounded regular solutions) are concerned. It will be known from the Hölder-continuity of Φ in x to be shown in § 3 that $\Gamma(x, t; \xi, \tau)$ satisfies

$$
\frac{\partial\Gamma}{\partial t}(x, t; \xi, \tau)=\sigma(x, t)P_0(D_x)\Gamma(x, t; \xi, \tau), \qquad (t>\tau),
\tag{2.32}
$$

and it is obvious that

$$\lim_{t \searrow 0} \int_{R^3} \Gamma(x, t; \xi, 0) v_0(\xi) d\xi = v_0(x). \tag{2.33}$$

Thus, Γ has all properties as a fundamental solution.

We give some formulas here.

LEMMA 2.13.

$$r^{\mu} e^{-r} \leqq K(\mu, b) e^{-br}, \qquad \left(K(\mu, b) \equiv \left\{ \frac{\mu}{e(1-b)} \right\}^{\mu},\ \mu \geqq 0,\ b \in [0, 1) \right). \tag{2.34}$$

LEMMA 2.14.

$$\begin{aligned} &\int_{R^n} \exp\left\{ -\bar{a} \frac{|x-y|^2}{t-\tau_0} \right\} \cdot \exp\left\{ -\bar{a} \frac{|y-\xi|^2}{\tau_0-\tau} \right\} dy \\ &= \left(\frac{\pi}{\bar{a}} \right)^{n/2} \left\{ \frac{(t-\tau_0)(\tau_0-\tau)}{t-\tau} \right\}^{n/2} \exp\left\{ -\bar{a} \frac{|x-\xi|^2}{t-\tau} \right\}, \qquad (\bar{a}>0,\ t>\tau_0>\tau), \end{aligned} \tag{2.35}$$

$$\int_{\tau}^{t} (t-\tau_0)^{\alpha-1} (\tau_0-\tau)^{\beta-1} d\tau_0 = (t-\tau)^{\alpha+\beta-1} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \qquad (\alpha, \beta>0). \tag{2.35$'$}$$

LEMMA 2.15.

$$\begin{aligned} \int_{\tau}^{t} (t-\tau_0)^{-\mu/2} \exp\left\{ -h \frac{|x-\xi|^2}{t-\tau_0} \right\} d\tau_0 &\leqq \int_0^{\infty} 2h^{-(\mu-2)/2} \cdot |x-\xi|^{-\mu+2} \zeta^{\mu-3} e^{-\zeta^2} d\zeta \\ &= 2h^{-(\mu-2)/2} |x-\xi|^2 I(\mu-3), \qquad (\mu \geqq 3,\ h>0). \end{aligned} \tag{2.36}$$

Φ is given in the form,

$$\Phi(x, t; \xi, \tau) = \sum_{m=0}^{\infty} K_m(x, t; \xi, \tau), \tag{2.37}$$

where

$$\begin{cases} K_m(x, t; \xi, \tau) \equiv \displaystyle\int_{\tau}^{t} d\tau_0 \int_{R^3} K_0(x, t; y, \tau_0) K_{m-1}(y, \tau_0; \tau) dy, \\ K_0 \equiv K. \end{cases} \tag{2.37$'$}$$

$$\begin{aligned} |K(x, t; \xi, \tau)| &= |\sigma(x, t) - \sigma(\xi, \tau)| \cdot |P_0(D_x) Z(x-\xi, t; \xi, \tau)| \\ &\leqq C_4 (t-\tau)^{-5/2+\alpha/2} \exp\left\{ -h_1 \frac{|x-\xi|^2}{t-\tau} \right\}, \qquad \left(h_1 \equiv \frac{1}{48 a_1 \sigma_1} \right). \end{aligned} \tag{2.38}$$

Inductively,

$$|K_m(x,t;\xi,\tau)|\leqq\left[\left(\frac{\pi}{h_1}\right)^{-3/2}\frac{\left\{\left(\frac{\pi}{h_1}\right)^{3/2}C_4\Gamma\left(\frac{\alpha}{2}\right)\right\}^{m+1}}{\Gamma\left((m+1)\frac{\alpha}{2}\right)}\right](t-\tau)^{-5/2+\alpha(m+1)/2}$$

(2. 38)′

$$\cdot\exp\left\{-h_1\frac{|x-\xi|^2}{t-\tau}\right\},\qquad (m=0,1,2,\cdots).$$

$$|K_m(x,t;\xi,\tau)|\leqq\left[\left(\frac{\pi}{h_1}\right)^{-3/2}\frac{T^{-\alpha/2}\left\{\left(\frac{\pi}{h_1}\right)^{3/2}C_4\Gamma\left(\frac{\alpha}{2}\right)T^{\alpha/2}\right\}^{m+1}}{\left((m+1)\frac{\alpha}{2}\right)}\right](t-\tau)^{-5/2+\alpha/2}$$

(2. 38)″

$$\cdot\exp\left\{-h_1\frac{|x-\xi|^2}{t-\tau}\right\},\qquad (m=0,1,2,\cdots).$$

By (2. 38)′, the righthand side of (2. 37) converges uniformly for $t>\tau\geqq0$, and by (2. 38)″ we have:

LEMMA 2. 16.

$$|\Phi(x,t;\xi,\tau)|\leqq\left\{\sum_{m=0}^{\infty}\left[\left(\frac{\pi}{h_1}\right)^{-3/2}T^{-\alpha/2}\frac{\left(\left(\frac{\pi}{h_1}\right)^{3/2}C_4\Gamma\left(\frac{\alpha}{2}\right)T^{\alpha/2}\right)^{m+1}}{\Gamma\left(\frac{\alpha}{2}(m+1)\right)}\right]\right\}(t-\tau)^{-5/2+\alpha/2}$$

(2. 39)

$$\cdot\exp\left\{-h_1\frac{|x-\xi|^2}{t-\tau}\right\}\equiv C_5(t-\tau)^{-5/2+\alpha/2}\exp\left\{-h_1\frac{|x-\xi|^2}{t-\tau}\right\}.$$

REMARK. The linear equation connected with (1. 9)[3]

(2. 40)
$$\begin{cases}\dfrac{\partial\theta}{\partial t}=\tilde{\sigma}(x,t)\Delta\theta+g, & (\tilde{\sigma},\ g\in H^{\alpha}_{T},\ 0<\tilde{\sigma}_0\leqq\sigma(x,t)\leqq\tilde{\sigma}_1<+\infty),\\ \theta(x,0)=\theta_0(x)(\in H^{2+\alpha}),\end{cases}$$

is, of course, uniformly parabolic in Petrowsky's sense, so that we can treat (2. 40) in parallel with (1. 11). The corresponding constants and variables will be denoted, e.g., by $'C_1$, $'\Phi$, etc.

§ 3. Estimates for the fundamental solution.

We want to know concrete ways of estimate the solution $v(x,t)$ of (1. 11), for a later use in demonstrating the existence of a solution of the original problem (1. 9)-(1. 10). It will be shown in § 3 and § 4 that, by utilizing lemmas of § 2, we can estimate Γ and $u(x,t)$ almost along the line of [24], Ch. IV, which treats the case of a scalar dependent variable. Practically, we need too lengthy calculations. For this reason, we make it our general principle to give rough outlines.

3.1. **Hölder-continuity in x of $\Phi(x,t;\xi,\tau)$ (I).**

$$
\begin{aligned}
&|K(x,t;\xi,\tau)-K(x',t;\xi,\tau)|\\
&\leqq|\{\sigma(x,t)-\sigma(x',t)\}P_0(D_x)Z(x-\xi,t;\xi,\tau)|\\
&\quad+|\sigma(x',t)-\sigma(\xi,\sigma)|\cdot|P_0(D_x)Z(x-\xi,t;\xi,\tau)-P_0(D_{x'})Z(x'-\xi,t;y,\tau)|.
\end{aligned}\tag{3.1}
$$

$$
\begin{aligned}
&|\{\sigma(x,t)-\sigma(x',t)\}|\cdot|P_0(D_x)Z(x-\xi,t;\xi,\tau)|\\
&\leqq|\sigma|_T^{(\alpha)}|x-x'|^\alpha\cdot 36C_1^{(2)}(t-\tau)^{-5/2}\exp\left\{-2h_1\frac{|x-\xi|^2}{t-\tau}\right\}.
\end{aligned}\tag{3.2}
$$

We denote by x'' the nearer of x and x' to ξ if $|x-\xi|\neq|x'-\xi|$, and x' if $|x-\xi|=|x'-\xi|$. By Lemma 2.9, we have

$$
\begin{aligned}
&|P_0(D_x)Z(x-\xi,t;\xi,\tau)-P_0(D_{x'})Z(x'-\xi,t;\xi,\tau)|\\
&\leqq 36C_1^{(3)}(t-\tau)^{-3}\exp\left\{-2h_1\frac{|\hat{x}-\xi|^2}{t-\tau}\right\}\cdot|x-x'|,\qquad(\hat{x}\in\overline{xx'}).
\end{aligned}\tag{3.3}
$$

i) If $4h_1(t-\tau)^{-1}|x-x'|^2<1$, then

$$
\exp\left\{-2h_1\frac{|\hat{x}-\xi|^2}{t-\tau}\right\}\leqq e^{-2h_1(t-\tau)^{-1}\{|\hat{x}-x'|-|x'-\xi|\}^2}\leqq e\cdot e^{-h_1(t-\tau)^{-1}|x'-\xi|^2}.\tag{3.4}
$$

Thus,

$$
|P_0(D_x)Z-P_0(D_{x'})Z|\leqq 36\cdot eC_1^{(3)}(t-\tau)^{-3}\cdot|x-x'|e^{-h_1(t-\tau)|x'-\xi|^2}.\tag{3.5}
$$

ii) If $4h_1(t-\tau)^{-1}|x-x'|^2>1$, then

$$
\begin{aligned}
&|P_0(D_x)Z(x-\xi,t;\xi,\tau)|\\
&\leqq 36C_1^{(2)}\{4h_1(t-\tau)^{-1}|x-x'|^2\}^{1/2}(t-\tau)^{-5/2}e^{-2h_1(t-\tau)^{-1}|x-\xi|^2}\\
&\leqq 72C_1^{(2)}(h_1)^{1/2}(t-\tau)^{-3}e^{-2h_1(t-\tau)^{-1}|x-\xi|^2}\cdot|x-x'|.
\end{aligned}\tag{3.6}
$$

Hence,

$$
|P_0(D_x)Z-P_0(D_{x'})Z|\leqq 44C_1^{(2)}(h_1)^{1/2}(t-\tau)^{-3}|x-x'|e^{-h_1(t-\tau)^{-1}|x''-\xi|^2}.\tag{3.6$'$}
$$

From i) and ii), we have

$$
\begin{aligned}
&|P_0(D_x)Z(x-\xi,t;\xi,\tau)-P_0(D_{x'})Z(x'-\xi,t;\xi,\tau)|\\
&\leqq 36\{eC_1^{(3)}+4C_1^{(2)}(h_1)^{1/2}\}|x-x'|(t-\tau)^{-3}\exp\left\{-h_1\frac{|x''-\xi|^2}{t-\tau}\right\}.
\end{aligned}\tag{3.7}
$$

Thus, for $\gamma\in[0,1]$,

$$|P_0(D_x)Z-P_0(D_{x'})Z|\leqq C_6(t-\tau)^{-(5+\gamma)/2}|x-x'|^{\gamma}\exp\left\{-h_1\frac{|x''-\xi|^2}{t-\tau}\right\}. \tag{3. 8}$$

Putting $\gamma=\alpha$ in (3. 8), we obtain the inequality,

$$\begin{aligned}&|\sigma(x',t)-\sigma(\xi,\tau)|\cdot|P_0(D_x)Z-P_0(D_{x'})Z|\\&\leqq C_6|\sigma|_T^{(\alpha)}\{|x-\xi|^{\alpha}+|t-\tau|^{\alpha/2}\}|x-x'|^{\alpha}(t-\tau)^{-(5+\alpha)/2}\exp\left\{-h_1\frac{|x''-\xi|^2}{t-\tau}\right\}.\end{aligned} \tag{3. 9}$$

When $x''=x'$,

$$\begin{aligned}&|\sigma(x',t)-\sigma(\xi,\tau)|\cdot|P_0(D_x)Z-P_0(D_{x'})Z|\\&\leqq\bar{C}_6(t-\tau)^{-5/2}||x-x'|^{\alpha}\exp\left\{-h_1\frac{|x''-\xi|^2}{t-\tau}\right\}.\end{aligned} \tag{3. 10}$$

For the case that $x''=x$, we can estimate in an analogous way. Therefore, we have:

LEMMA 3. 1.

$$|K(x,t;\xi,\tau)-K(x',t;\xi,\tau)|\leqq C_{6,1}|x-x'|^{\alpha}(t-\tau)^{-5/2}\exp\left\{-h_1\frac{|x''-\xi|^2}{t-\tau}\right\}. \tag{3. 11}$$

LEMMA 3. 2. *For* $\gamma\in[0,1]$,

$$|K(x,t;\xi,\tau)-K(x',t;\xi,\tau)|\leqq C_{6,2}|x-x'|^{\alpha}(t-\tau)^{-5/2+\alpha(1-\gamma)/2}\exp\left\{-h_1\frac{|x''-\xi|^2}{t-\tau}\right\}. \tag{3. 12}$$

LEMMA 3. 3. *For* $\beta\in[0,\alpha)$,

$$|\Phi(x,t;\xi,\tau)-\Phi(x',t;\xi,\tau)|\leqq C_{6,3}|x-x'|^{\beta}(t-\tau)^{-5/2+(\alpha-\beta)/2}\exp\left\{-h_1\frac{|x''-\xi|^2}{t-\tau}\right\}. \tag{3. 13}$$

Proof. This lemma follows directly from (2. 30) and the lemmas 2. 14, 2. 16, and 3. 2.

3. 2. **Hölder-continuity in x of $\Phi(x,t;\xi,\tau)$ (II).** We show that we can improve (3. 13), i.e., that Φ is Hölder-continuos in x with the exponent α.

$$\begin{aligned}&\int_{\tau}^{t}d\tau_0\int_{R^3}\{K(x,t;y,\tau_0)-K(x',t;y,\tau_0)\}\Phi(y,\tau_0;\xi,\tau)dy\\&=\int_{\tau}^{t}d\tau_0\int_{\Sigma_0}K(x,t;y,\tau_0)\Phi(y,\tau_0;\xi,\tau)dy-\int_{\tau}^{t}d\tau_0\int_{\Sigma_0}K(x',t;y,\tau_0)\Phi(y,\tau_0;\xi,\tau)dy\\&+\int_{\tau}^{t}d\tau_0\int_{R^3-\Sigma_0}\{K(x,t;y,\tau_0)-K(x',t;y,\tau_0)\}\Phi(y,\tau_0;\xi,\tau)dy=I_1+I_2+I_3,\end{aligned} \tag{3. 14}$$

where $\Sigma_0=\Sigma_1-\Sigma_2$, Σ_1 is a closed ball with $x''(\xi)$ the center and with the radius $2|x-x'|$, and Σ_2 is a closed ball with ξ the center and with the radius $(1/2)|x''-\xi|$.

1°) Estimate for I_1.

$$|y-\xi|\geqq\frac{|x''-\xi|}{2}, \qquad (y\in\Sigma_0\subset R^3-\Sigma_2).$$

$$\begin{aligned}|I_1|&\leqq C_4C_5\int_\tau^t(t-\tau_0)^{-(5-\alpha)/2}(\tau_0-\tau)^{-(5-\alpha)/2}d\tau_0\int_{\Sigma_0}\exp\left\{-h_1\left\{\frac{|x-y|^2}{t-\tau_0}+\frac{|y-\xi|^2}{\tau_0-\tau}\right\}\right\}dy\\ &\leqq C_4C_5\left[\int_{\Sigma_0}dy\int_\tau^{(t+\tau)/2}\left(\frac{t-\tau}{2}\right)^{-(5-\alpha)/2}(\tau_0-\tau)^{-(5-\alpha)/2}\exp\left\{-h_1\left\{\frac{|y-\xi|^2}{2(\tau_0-\tau)}+\frac{|x''-\xi|^2}{4(t-\tau)}\right\}\right\}d\tau_0\right.\\ &\qquad\left.+\int_{\Sigma_0}dy\int_{(t+\tau)/2}^t(t-\tau_0)^{-(5-\alpha)/2}\left(\frac{t-\tau}{2}\right)^{-(5-\alpha)/2}\exp\left\{-h_1\frac{|x-y|^2}{t-\tau_0}-\frac{h_1}{4}\frac{|x''-\xi|^2}{t-\tau}\right\}d\tau_0\right]\\ &\leqq C_{7,1}|x-x'|^\alpha(t-\tau)^{-5/2+\alpha/2}\exp\left\{-\frac{h_1}{4}\frac{|x''-\xi|^2}{t-\tau}\right\}.\end{aligned}\tag{3.15}$$

2°) In the same way, we have

$$|I_2|\leqq C_{7,2}|x-x'|^\alpha(t-\tau)^{-5/2+\alpha/2}\exp\left\{-\frac{h_1}{4}\frac{|x''-\xi|^2}{t-\tau}\right\}, \qquad (C_{7,2}\equiv C_{7,1}).\tag{3.16}$$

3°) For clearness' sake, we assume that $x''=x'$.

$$\begin{aligned}I_3&=\int_\tau^t d\tau_0\int_{R^3-\Sigma_0}\{\sigma(x,t)-\sigma(x',t)\}P_0(D_x)Z(x-y,t;y,\tau_0)\Phi(y,\tau_0;\xi,\tau)dy\\ &\quad+\int_\tau^t d\tau_0\int_{R^3-\Sigma_0}\{\sigma(x',t)-\sigma(y,\tau_0)\}\{P_0(D_x)Z(x-y,t;y,\tau_0)\\ &\qquad\qquad -P_0(D_{x'})Z(x'-y,t;y,\tau_0)\}\Phi(y,\tau_0;\xi,\tau)dy\\ &\equiv I_{3,1}+I_{3,2}.\end{aligned}\tag{3.17}$$

$$I_{3,1}=\int_\tau^{(t+\tau)/2}d\tau_0\int_{R^3-\Sigma_0}\cdots dy+\int_{(t+\tau)/2}^t d\tau_0\int_{R^3-\Sigma_0}\cdots dy\equiv I_{3,1,1}+I_{3,1,2}.\tag{3.18}$$

$$|I_{3,1,1}|\leqq C_{7,3,1,1}|x-x'|^\alpha(t-\tau)^{-5/2+\alpha/2}\exp\left\{-h_1\frac{|x-\xi|^2}{t-\tau}\right\}.\tag{3.19}$$

$$
\begin{aligned}
I_{3,1,2}=&\{\sigma(x,t)-\sigma(x',t)\}\int_{(t+\tau)/2}^{t} d\tau_0 \int_{R^3-\Sigma_0} P_0(D_x)Z(x-y,t;y,\tau_0)\{\Phi(y,\tau_0;\xi,\tau) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -\Phi(x,\tau_0;\xi,\tau)\}dy \\
&+\{\sigma(x,t)-\sigma(x',t)\}\int_{(t+\tau)/2}^{t} d\tau_0 \int_{R^3-\Sigma_0} \{P_0(D_x)Z(x-y,t;y,\tau_0) \\
&\qquad\qquad\qquad -P_0(D_x)Z(x-y,t;x_0,\tau_0)|_{x_0=x}\}\Phi(x,\tau_0;\xi,\tau)dy \\
&+\{\sigma(x,t)-\sigma(x',t)\}\int_{(t+\tau)/2}^{t} d\tau_0 \int_{R^3-\Sigma_0} P_0(D_x)Z(x-y,t;x_0,\tau_0)|_{x_0=x}\cdot\Phi(x,\tau_0;\xi,\tau)dy \\
=&I_{3,1,2,1}+I_{3,1,2,2}+I_{3,1,2,3}.
\end{aligned}
\tag{3. 20}
$$

$$(P_0(D_x)Z(x-y,t;x_0,\tau)|_{x_0=x}=P_0(D_y)Z(x-y,t;x,\tau)).$$

By (3. 13),

$$|I_{3,1,2,1}|\leqq C_{7,3,1,2,1}|x-x'|^{\alpha}(t-\tau)^{-5/2+\alpha/2}\exp\left\{-\frac{h_1}{2}\frac{|x-\xi|^2}{t-\tau}\right\}. \tag{3. 21}$$

$$|I_{3,1,2,2}|\leqq C_{7,3,1,2,2}|x-x'|^{\alpha}(t-\tau)^{-5/2+\alpha/2}\exp\left\{-\frac{h_1}{2}\frac{|x-\xi|^2}{t-\tau}\right\}, \quad \text{(cf. Lemma 2. 8).} \tag{3. 22}$$

For $P_0(D_x)=(a_{kl}^{ij}D_{x_k}D_{x_l})$, we define

$$\tilde{P}_0(D_x,\nu)\equiv(\nu_k a_{kl}^{ij}D_{x_l}), \qquad (\nu,\ \text{outer normal to the furface } S_{\Sigma_0} \text{ of } \Sigma_0). \tag{3. 23}$$

$$I_{3,1,2,3}=\{\sigma(x,t)-\sigma(x',t)\}\int_{(t+\tau)/2}^{t} d\tau_0 \int_{S_{\Sigma_0}} \tilde{P}_0(D_y,\nu)Z(y-x,t;x,\tau_0)\Phi(x,\tau_0;\xi,\tau)dS_y.$$

Hence, by Lemma 2. 5 and Lemma 2. 15,

$$
\begin{aligned}
|I_{3,1,2,3}|\leqq& |\sigma|_T^{(\alpha)}\cdot 36C_5C_1^{(1)}|x-x'|^{\alpha}\exp\left\{-h_1\frac{|x-\xi|^2}{t-\tau}\right\}\int_{(t+\tau)/2}^{t}(t-\tau_0)^{-2}(\tau_0-\tau)^{-5/2+\alpha/2}d\tau_0 \\
&\times\left\{\int_{S_1}\exp\left\{-h_1\frac{|x-y|^2}{t-\tau_0}\right\}dS_y+\int_{S_2}\exp\left\{-h_1\frac{|x-y|^2}{t-\tau_0}\right\}dS_y\right\} \\
\leqq& C_{7,3,1,2,3}|x-x'|^{\alpha}(t-\tau)^{-5/2+\alpha/2}\exp\left\{-h_1\frac{|x-\xi|^2}{t-\tau}\right\}, \quad (S_1=S_{\Sigma_1}-\Sigma_2,\ S_2=S_{\Sigma_2}-\Sigma_1).
\end{aligned}
\tag{3. 23$'$}
$$

Thus,

$$|I_{3,1}|\leqq C_{7,3,1}|x-x'|^{\alpha}(t-\tau)^{-5/2+\alpha/2}\exp\left\{-\frac{h_1}{2}\frac{|x-\xi|^2}{t-\tau}\right\}. \tag{3. 24}$$

4°) Estimate for $I_{3,2}$.

$$I_{3,2}=\int_{\tau}^{t}d\tau_0\int_{R^3-\Sigma_1}\cdots dy+\int_{\tau}^{t}d\tau_0\int_{\Sigma_1\cap\Sigma_2}\cdots dy\equiv I_{3,2,1}+I_{3,2,2}. \tag{3.25}$$

$$\begin{aligned}|I_{3,2,1}|\leqq|\sigma|_T^{(\alpha)}C_5C_6\Big[\int_{\tau}^{(t+\tau)/2}&(t-\tau_0)^{-(5+\alpha)/2}(\tau_0-\tau)^{-5/2+\alpha/2}d\tau_0\int_{R^3-\Sigma_1}2\exp\left\{-\frac{h_1}{2}\frac{|y-x'|^2}{t-\tau_0}\right\}\\ &\cdot\exp\left\{-h_1\frac{|y-\xi|^2}{\tau_0-\tau}\right\}\{|x'-y|^\alpha+|t-\tau_0|^{\alpha/2}|x-x'|^\alpha dy\}\\ &+|x-x'|\int_{(t+\tau)/2}^{t}(t-\tau_0)^{-3}(\tau_0-\tau)^{-5/2+\alpha/2}d\tau_0\\ &\cdot\int_{R^3-\Sigma_1}2\{|x'-y|^\alpha+|t-\tau_0|^{\alpha/2}\}\exp\left\{-\frac{h_1}{2}\frac{|y-x'|^2}{t-\tau_0}\right\}\exp\left\{-h_1\frac{|y-\xi|^2}{\tau_0-\tau}\right\}dy\Big].\end{aligned} \tag{3.26}$$

From (3. 26), we have

$$|I_{3,2,1}|\leqq C_{7,3,2,1}|x-x'|^\alpha(t-\tau)^{-5/2+\alpha/2}\exp\left\{-\frac{h_1}{16}\frac{|x'-\xi|^2}{t-\tau}\right\}, \tag{3.27}$$

$$\left(\text{N.B.: }\frac{1}{2}|x'-y|\leqq|x-y|\leqq\frac{3}{2}|x'-y|,\quad\text{for } y\in S_{\Sigma_0}\right).$$

If $y\in\Sigma_2$, then

$$\frac{1}{3}|x'-y|\leqq\frac{1}{2}|x'-\xi|\leqq|x'-y|,\qquad |x-y|\geqq\frac{1}{3}|x'-y|. \tag{3.28}$$

$$\begin{aligned}|I_{3,2,2}|&\leqq|\sigma|_T^{(\alpha)}C_5C_6\cdot2\int_{\tau}^{t}(t-\tau_0)^{-5/2}(\tau_0-\tau)^{-5/2+\alpha/2}d\tau_0\\ &\cdot\int_{\Sigma_1\cap\Sigma_2}\{|x'-y|^\alpha+|t-\tau_0|^{\alpha/2}\}\exp\left\{-\left\{\frac{h_1}{9}\frac{|x'-y|^2}{t-\tau_0}+\frac{h_1}{9}\frac{|y-\xi|^2}{\tau_0-\tau}\right\}\right\}dy\\ &\leqq C_{7,3,2,2}|x-x'|^\alpha(t-\tau)^{-5/2+\alpha/2}\exp\left\{-\frac{h_1}{72}\frac{|x'-\xi|^2}{t-\tau}\right\}.\end{aligned} \tag{3.29}$$

From (3. 26) and (3. 29), we have

$$|I_{3,2}|\leqq C_{7,3,2}|x-x'|^\alpha(t-\tau)^{-5/2+\alpha/2}\cdot\exp\left\{-\frac{h_1}{72}\frac{|x'-\xi|^2}{t-\tau}\right\}, \tag{3.30}$$

$$(C_{7,3,2}\equiv C_{7,3,2,1}+C_{7,3,2,2}).$$

We obtain analogous results in the case when $x''=x$. Thus, we have:

LEMMA 3. 4.

$$
(3.31)\quad \begin{aligned}
&\left|\int_\tau^t d\tau_0 \int_{R^3} \{K(x, t; y, \tau)-K(x', t; y, \tau_0)\}\Phi(y, \tau_0; \xi, \tau)dy\right| \\
&\leqq C_7|x-x'|^\alpha(t-\tau)^{-5/2+\alpha/2}\exp\left\{-\frac{h_1}{72}\frac{|x''-\xi|^2}{t-\tau}\right\}, \qquad (C_7\equiv C_{7,1}+C_{7,2}+C_{7,3,1}+C_{7,3,2}).
\end{aligned}
$$

From this lemma follows:

LEMMA 3. 5.

$$
(3.33)\quad \begin{aligned}
&|\Phi(x, t; \xi, \tau)-\Phi(x', t; \xi, \tau_0)| \\
&\leqq |K(x, t; \xi, \tau_0)-K(x', t; \xi, \tau_0)| \\
&\quad +\left|\int_\tau^t d\tau_0 \int_{R^3} \{K(x, t; y, \tau_0)-K(x', t; y, \tau_0)\}\Phi(y, \tau_0; \xi, \tau)dy\right| \\
&\leqq \bar{C}_7|x-x'|^\alpha(t-\tau)^{-5/2}\exp\left\{-\frac{h_1}{72}\frac{|x''-\xi|^2}{t-\tau}\right\}, \qquad (\bar{C}_7\equiv C_{6,1}+C_7T^{\alpha/2}).
\end{aligned}
$$

3. 3. **Hölder-continuity in t of $\Phi(x, t; \xi, \tau)$.**

1°) We assume that $t\geqq t'>\tau$ and $t-t'<(t-\tau)/3$, $(\tau\geqq 0)$.

LEMMA 3. 6. *For* $t\geqq t'>\tau\geqq 0$, $t-t'<(t-\tau)/3$,

$$
(3.34)\quad |K(x, t; \xi, \tau)-K(x, t'; \xi, \tau)|\leqq C_8(t-t')^{\alpha/2}(t'-\tau)^{-5/2}\exp\left\{-h_1\frac{|x-\xi|^2}{t-\tau}\right\}.
$$

Proof.

$$
\begin{aligned}
&|K(x, t; \xi, \tau)-K(x, t'; \xi, \tau)| \\
&\leqq |\sigma(x, t)-\sigma(x, t')|\cdot|P_0(D_x)Z(x-\xi, t; \xi, \tau)| \\
&\quad +|\sigma(x, t')-\sigma(\xi, \tau)|\cdot|P_0(D_x)Z(x-\xi, t; \xi, \tau)-P_0(D_x)Z(x-\xi, t'; \xi, \tau)|.
\end{aligned}
$$

From this follows (3. 34). Q.E.D.

LEMMA 3. 7. *For* $t\geqq t'>\tau$, $t-t'<(t-\tau)/3$,

$$
(3.35)\quad |\Phi(x, t; \xi, \tau)-\Phi(x, t'; \xi, \tau)|\leqq C_9(t-t')^{\alpha/2}(t'-\tau)^{-5/2}\exp\left\{-\frac{h_1}{144}\frac{|x-\xi|^2}{t-\tau}\right\}.
$$

Proof. We estimate the difference

$$\int_\tau^t d\tau_0 \int_{R^3} K(x, t; y, \tau_0)\Phi(y, \tau_0; \xi, \tau)dy - \int_\tau^{t'} d\tau_0 \int_{R^3} K(x, t'; y, \tau_0)\Phi(y, \tau_0; \xi, \tau)dy$$

$$= \int_{2t'-t}^t d\tau_0 \int_{R^3} K(x, t; y, \tau_0)\Phi(y, \tau_0; \xi, \tau)dy - \int_{2t'-t}^{t'} d\tau_0 \int_{R^3} K(x, t'; y, \tau_0)\Phi(y, \tau_0; \xi, \tau)dy$$

$$+\{\sigma(x, t)-\sigma(x, t')\}\int_t^{2t'-t} d\tau_0 \int_{R^3} P_0(D_x)Z(x-y, t; y, \tau_0)\Phi(y, \tau_0; \xi, \tau)dy \tag{3.36}$$

$$+\int_\tau^{2t'-t} d\tau_0 \int_{R^3} \{\sigma(x, t)-\sigma(y, \tau_0)\}\{P_0(D_x)Z(x-y, t; y, \tau_0)$$

$$-P_0(D_x)Z(x-y, t'; y, \tau)\}\Phi(y, \tau_0; \xi, \tau)dy$$

$$\equiv \sum_{i=1}^4 I_i.$$

$$|I_1| \leqq C_{9,1}(t-t')^{\alpha/2}(t-\tau)^{-5/2+\alpha/2} \exp\left\{-h_1\frac{|x-\xi|^2}{t-\tau}\right\}, \tag{3.37}$$

$$|I_2| \leqq C_{9,2}(t-t')^{\alpha/2}(t'-\tau)^{-5/2+\alpha/2} \exp\left\{-h_1\frac{|x-\xi|^2}{t'-\tau}\right\}. \tag{3.37$'$}$$

$$I_3 = \{\sigma(x, t)-\sigma(x, t')\}\int_{(t'+\tau)/2}^{2t'-t} d\tau_0 \int_{R^3} P_0(D_x)Z(x-y, t; y, \tau_0)\{\Phi(y, \tau_0; \xi, \tau)-(x, \tau_0; \xi, \tau)\}dy$$

$$+\{\sigma(x, t)-\sigma(x, t')\}\int_{(t'+\tau)/2}^{2t'-t} d\tau_0 \int_{R^3} P_0(D_x)Z(x-y, t; y, \tau_0)\Phi(x, \tau_0; \xi, t)dy \tag{3.38}$$

$$+\{\sigma(x, t)-\sigma(x, t')\}\int_\tau^{(t'+\tau)/2} d\tau_0 \int_{R^3} P_0(D_x)Z(x-y, t; y, \tau_0)\Phi(y, \tau_0; \xi, \tau)dy$$

$$\equiv I_{3,1}+I_{3,2}+I_{3,3}.$$

$$|I_{3,1}| \leqq C_{9,3,1}(t-t')^{\alpha/2}(t'-\tau)^{-5/2+\alpha/2} \exp\left\{-\frac{h_1}{2\cdot 72}\frac{|x-\xi|^2}{t-\tau}\right\}. \tag{3.39}$$

$$|I_{3,2}| \leqq |\sigma|_T^{(\alpha)}(t-t')^{\alpha/2}$$

$$\times\left|\int_{(t'+\tau)/2}^{2t'-t} d\tau_0 \int_{R^3} \{P_0(D_x)Z(x-y, t; y, \tau_0)-P_0(D_x)Z(x-y, t; x_0, \tau_0)|_{x_0=x}\}\right. \tag{3.40}$$

$$\left.\cdot\Phi(x, \tau_0; \xi, \tau)dy\right|$$

$$\leqq C_{9,3,2}(t-t')^{\alpha/2}(t'-\tau)^{-5/2+\alpha/2} \exp\left\{-h_1\frac{|x-\xi|^2}{t-\tau}\right\}, \qquad (2k_1 \geqq h_1).$$

$$|I_{3,3}| \leqq C_{9,3,3}(t-t')^{\alpha/2}(t'-\tau)^{-5/2+\alpha/2} \exp\left\{-h_1\frac{|x-\xi|^2}{t-\tau}\right\}. \tag{3.41}$$

Thus,

$$(3.42)\qquad |I_3|\leqq C_{9,3}(t-t')^{\alpha/2}(t'-\tau)^{-5/2+\alpha/2}\exp\left\{-\frac{h_1}{2\cdot 72}\frac{|x-\xi|^2}{t-\tau}\right\},\quad \left((C_{9,3}\equiv T^{\alpha/2}\sum_i C_{9,3,1}\right).$$

$$(3.43)\qquad I_4=\int_{(t'+\tau)/2}^{2t'-t}d\tau_0\int_{R^3}\cdots dy+\int_{\tau}^{(t'+\tau)/2}d\tau_0\int_{R^3}\cdots dy\equiv I_{4,1}+I_{4,2}.$$

$$(3.44)\qquad |I_{4,1}|\leqq C_{9,4,1}(t-t')^{\alpha/2}(t'-\tau)^{-5/2+\alpha/2}\exp\left\{-\frac{h_1}{2}\frac{|x-\xi|^2}{t-\tau}\right\}.$$

$$|I_{4,2}|\leqq|\sigma|_T^{(\alpha)}C_5\int_{\tau}^{(t'+\tau)/2}d\tau_0\int_{R^3}\{|x-y|^\alpha+|t'-\tau_0|^{\alpha/2}\}|P_0(D_x)Z(x-y,t;y,\tau_0)$$
$$-P_0(D_x)Z(x-y,t';y,\tau_0)|^{\alpha/2}$$
$$(3.45)\qquad \times\{|P_0(D_x)Z(x-y,t;y,\tau_0)|$$
$$+|P_0(D_x)Z(x-y,t';y,\tau_0)|\}^{1-(\alpha/2)}(\tau_0-\tau)^{-5/2+\alpha/2}\exp\left\{-h_1\frac{|y-\xi|^2}{\tau_0-\tau}\right\}dy$$
$$\leqq C_{9,4,2}(t-t')^{\alpha/2}(t'-\tau)^{-5/8+\alpha/2}\exp\left\{-h_1\frac{|x-\xi|^2}{t-\tau}\right\},\qquad \left(t'-\tau\geqq\frac{2}{3}(t-\tau)\right).$$

$$(3.46)\qquad |I_4|\leqq C_{9,4}(t-t')^{\alpha/2}(t'-\tau)^{-5/2+\alpha/2}\exp\left\{-\frac{h_1}{2}\frac{|x-\xi|^2}{t-\tau}\right\},\quad (C_{9,4}\equiv C_{9,4,1}+C_{9,4,2}).$$

Hence, we have (3. 35), where

$$C_9\equiv C_8+(C_{9,1}+C_{9,2}+C_{9,3}+C_{9,4})T^{\alpha/2}.\qquad \text{Q.E.D.}$$

2°) For $t>t'>\tau$, $t-t'\geqq(t-\tau)/3$,

$$(3.47)\qquad \begin{cases}(t-\tau)^{-5/2+\alpha/2}\leqq 3^{\alpha/2}(t-t')^{\alpha/2}(t'-\tau)^{-5/2},\\ (t'-\tau)^{-5/2+\alpha/2}\leqq 2^{\alpha/2}(t-t')^{\alpha/2}(t'-\tau)^{-5/2}.\end{cases}$$

Thus, directly from Lemma 2. 16 it follows that

$$(3.48)\qquad |\Phi(x,t;\xi,\tau)-\Phi(x,t;\xi,\tau)|\leqq C_5(3^{\alpha/2}+2^{\alpha/2})(t-t')^{\alpha/2}(t'-\tau)^{-5/2}\exp\left\{-h_1\frac{|x-\xi|^2}{t-\tau}\right\}.$$

3°) After all, from Lemma 3. 7 and (3. 48) we have:

LEMMA 3. 8. *For* $t\geqq t'>\tau\geqq 0$,

$$(3.49)\qquad |\Phi(x,t;\xi,\tau)-\Phi(x,t';\xi,\tau)|\leqq\bar{C}_9(t-t')^{\alpha/2}(t'-\tau)^{-5/2}\exp\left\{-\frac{h}{2\cdot 72}\frac{|x-\xi|^2}{t-\tau}\right\},$$
$$(\bar{C}_9\equiv C_9+C_5(3^{\alpha/2}+2^{\alpha/2})).$$

3.4. **Estimates for** $q(x, t)$. We define $q(x, t)$ as follows:

$$q(x, t) \equiv \int_0^t d\tau \int_{R^3} \Phi(x, t; \xi, \tau) d\xi. \tag{3.50}$$

LEMMA 3.9.

$$|q(x, t) - q(x', t)| \leqq C_{10} |x - x'|^{\alpha}. \tag{3.51}$$

Proof.

$$\begin{aligned} q(x, t) - q(x', t) &= \int_{\Sigma_1(x')} d\xi \int_0^t \Phi(x, t; \xi, \tau) d\tau - \int_{\Sigma_1(x')} d\xi \int_0^t \Phi(x', t; \xi, \tau) d\xi \\ &\quad + \int_{R^3 - \Sigma_1} d\xi \int_0^t \{\Phi(x, t; \xi, \tau) - \Phi(x', t; \xi, \tau)\} d\tau = I_1 + I_2 + I_3, \end{aligned} \tag{3.52}$$

$$(\Sigma_1(x') \equiv \{\xi : |\xi - x'| \leqq 2|x - x'|\}).$$

$$\begin{aligned} |I_1| &\leqq C_5 \int_{\Sigma_1(x')} d\xi \int_0^t (t-\tau)^{-5/2+\alpha/2} \exp\left\{-h_1 \frac{|x-\xi|^2}{t-\tau}\right\} d\tau \\ &\leqq C_5 \cdot 2h_1^{-(3-\alpha)/2} I(2-\alpha) \int_{\Sigma_1(x')} |x-\xi|^{-3+\alpha} d\xi \\ &\leqq C_{10,1} |x - x'|^{\alpha}. \end{aligned} \tag{3.53}$$

$$|I_2| \leqq C_{10,2} |x - x'|^{\alpha}. \tag{3.53$'$}$$

$$\begin{aligned} I_3 &= \int_{R^3 - \Sigma_1(x')} d\xi \int_0^t \{K(x, t; \xi, \tau) - K(x', t; \xi, \tau)\} d\tau \\ &\quad + \int_{R^3 - \Sigma_1(x')} d\xi \int_0^t d\tau \int_\tau^t d\tau_0 \int_{R^3} \{K(x, t; y, \tau_0) - K(x', t; y, \tau_0)\} \Phi(y, \tau_0; \xi, \tau) dy \equiv I_{3,1} + I_{3,2}. \end{aligned} \tag{3.54}$$

$$\begin{aligned} I_{3,1} &= \{\sigma(x, t) - \sigma(x', t)\} \int_0^t d\tau \int_{R^3 - \Sigma_1(x')} P_0(D_x) Z(x - \xi, t; \tau) d\xi \\ &\quad + \int_0^t d\tau \int_{R^3 - \Sigma_1(x')} \{\sigma(x', t) - \sigma(\xi, \tau)\} \{P_0(D_x) Z(x - \xi, t; \xi, \tau) \\ &\qquad\qquad - P_0(D_{x'}) Z(x' - \xi, t; \xi, \tau)\} d\xi \\ &\equiv I_{3,1,1} + I_{3,1,2}. \end{aligned} \tag{3.55}$$

$$|I_{3,1,1}|\leqq|\sigma|_T^{(\alpha)}|x-x'|^\alpha\left|\int_0^{t/2}d\tau\int_{R^3-\Sigma_1(x')}P(D_x)Z(x-\xi,t;\xi,\tau)d\xi\right.$$

$$\left.+\int_{t/2}^t d\tau\int_{R^3-\Sigma_1}P_0(D_x)Z(x-\xi,t;\xi,\tau)dy\right|$$

$$\leqq|\sigma|_T^{(\alpha)}|x-x'|^\alpha\Big[36C_1^{(2)}\cdot\log 2\cdot(2h_1)^{-3/2}\pi^{3/2} \tag{3. 56}$$

$$+\left|\int_{t/2}^t d\tau\int_{R^3-\Sigma_12x')}\{P_0(D_x)Z(x-\xi,t;x,\tau)-P_0(D_x)Z(x-\xi,t;x_0,\tau)|_{x_0=x}\}d\xi\right|$$

$$+\left|\int_{t/2}^t d\tau\int_{R^3-\Sigma_1}P_0(D_x)Z(x-\xi,t;x,\tau)d\xi\right|\Big]\leqq C_{10,3,1,1}|x-x'|^\alpha.$$

$$|I_{3,1,2}|\leqq\bar{C}_6\int_0^{t/2}d\tau\int_{R^3-\Sigma_1}|x-x'|^\alpha(t-\tau)^{-5/2}\exp\left\{-h_1\frac{|x''-\xi|^2}{t-\tau}\right\}d\xi$$

$$+|\sigma|_T^{(\alpha)}C_6\int_{t/2}^t d\tau\int_{R^3-\Sigma_1}|x-x'|(t-\tau)^{-3}\exp\left\{-h_1\frac{|x''-\xi|^2}{t-\tau}\right\}(|x'-\xi|^\alpha+|t-\tau|^{\alpha/2})d\xi \tag{3. 57}$$

$$\leqq C_{10,3,1,2}|x-x'|^\alpha.$$

Hence,

$$|I_{3,1}|\leqq C_{10,3,1}|x-x'|^\alpha,\qquad (C_{10,3,1}=C_{10,3,1,1}+C_{10,3,1,2}). \tag{3. 58}$$

$$|I_{3,2}|\leqq C_7|x-x'|^\alpha\int_0^t(t-\tau)^{-5/2+\alpha/2}d\tau\int_{R^3-\Sigma_1}\exp\left\{-\frac{h_1}{72}\frac{|x''-\xi|^2}{t-\tau}\right\}d\xi$$

$$\leqq C_{10,3,2}|x-x'|^\alpha,\qquad \text{(cf. Lemma 3. 4)}. \tag{3. 59}$$

From (3. 53), (3. 53)′, (3. 58), and (3. 59) follows the lemma. Q.E.D.

LEMMA 3. 10. *For* $t\geqq t'>0$,

$$|q(x,t)-q(x,t')|\leqq C_{11}(t-t')^{\alpha/2}. \tag{3. 60}$$

Proof. For $t\geqq t'>0$,

$$q(x,t)-q(x,t')$$

$$=\int_{0\vee(2t'-t)}^t d\tau\int_{R^3}\Phi(x,t;\xi,\tau)d\xi-\int_{0\vee(2t'-t)}^{t'}d\tau\int_{R^3}\Phi(x,t;\xi,\tau)d\xi$$

$$+\int_0^{0\vee(2t'-t)}d\tau\int_{R^3}d\xi\Big[\int_\tau^t d\tau_0\int_{R^3}K(x,t;y,\tau_0)\Phi(y,\tau_0;\xi,\tau)dy$$

$$-\int_\tau^{t'}d\tau_0\int_{R^3}K(x,t';y,\tau_0)\Phi(y,\tau_0;\xi,\tau)dy\Big] \tag{3. 61}$$

$$+\{\sigma(x,t)-\sigma(x,t')\}\int_0^{oV(2t'-t)}d\tau\int_{R^3}P_0(D_x)Z(x-\xi,t;\xi,\tau)d\xi$$

$$+\int_0^{oV(2t'-t)}d\tau\int_{R^3}\{\sigma(x,t')-\sigma(\xi,\tau)\}\{P_0(D_x)Z(x-\xi,t;\xi,\tau)-P_0(D_x)Z(x-\xi,t';\xi,\tau)\}d\xi$$

$$\equiv\sum_{i=1}^{5}J_i,\qquad (oV(2t'-t')\equiv\max\,[0,2t'-t]).$$

$$|J_1|\leqq C_{11,1}(t-t')^{\alpha/2}. \tag{3.62}$$

$$|J_2|\leqq C_5(2^{1+(\alpha/2)}-1)\alpha^{-1}\left(\frac{\pi}{h_1}\right)^{3/2}(t-t')^{\alpha/2}\equiv C_{11,2}(t-t')^{\alpha/2}. \tag{3.63}$$

Next, we note that, for $t\geqq t'>\tau\geqq 0$ and $t-t'<(t-\tau)/3$,

$$\left|\int_\tau^t d\tau_0\int_{R^3}K(x,t;y,\tau_0)\Phi(y,\tau_0;\xi,\tau)dy-\int_\tau^{t'}d\tau_0\int_{R^3}K(x,t';y,\tau_0)\Phi(y,\tau_0;\xi,\tau)dy\right|$$

$$\leqq\left(\sum_{i=1}^{4}C_{9,i}\right)(t-t')^{\alpha/2}(t'-\tau)^{-5/2}\exp\left\{-\frac{h_1}{2\cdot 72}\frac{|x-\xi|^2}{t-\tau}\right\}, \tag{3.64}$$

(cf. (3. 35), (3. 36), (3. 40), and (3. 45)),

and that, for $t-t'\geqq(t-\tau)/3$,

(3. 65)

$$\begin{cases}\left|\int_\tau^t d\tau_0\int_{R^3}K(x,t;y,\tau_0)\Phi(y,\tau_0;\xi,\tau)dy\right|\leqq 3^{\alpha/2}C_4C_5(t-t')^{\alpha/2}(t'-\tau)^{-5/2+\alpha/2}\exp\left\{-h_1\frac{|x-\xi|^2}{t-\tau}\right\},\\ \left|\int_\tau^{t'} d\tau_0\int_{R^3}K(x,t';y,\tau_0)\Phi(y,\tau_0;\xi,\tau)dy\right|\leqq 2^{\alpha/2}C_4C_5(t-t')^{\alpha/2}(t'-\tau)^{-5/2+\alpha/2}\exp\left\{-h_1\frac{|x-\xi|^2}{t-\tau}\right\}.\end{cases}$$

Thus,

$$|J_3|\leqq\left[\sum_{i=1}^{4}C_{9,i}+C_4C_5(3^{\alpha/2}+2^{\alpha/2})\right]T^{\alpha/2}(t-t')^{\alpha/2}\equiv C_{11,3}(t-t')^{\alpha/2}. \tag{3.66}$$

$$\begin{aligned}|J_4|&\leqq|\sigma|_T^{(\alpha)}(t-t')^{\alpha/2}\left|\int_0^{oV(2t'-t)}d\tau\int_{R^3}\{P_0(D_x)Z(x-\xi,t;\xi,\tau)-P_0(D_\xi)Z(x-\xi,t;x,\tau)\}d\xi\right|\\ &\leqq C_{11,4}(t-t')^{\alpha/2},\end{aligned} \tag{3.67}$$

$$|J_5| \leqq |\sigma|_T^{(\alpha)} \int_0^{0 \vee (2t'-t)} d\tau \int_{R^3} \{|x-\xi|^\alpha + (t'-\tau)^{\alpha/2}\} |P_0(D_x) Z(x-\xi, t; \xi, \tau) - P_0(D_x) Z(x-\xi, t'; \xi, \tau)| d\xi$$

$$\text{(3.68)} \quad \leqq |\sigma|_T^{(\alpha)} \left\{ 2^{\alpha/2} K\left(\frac{\alpha}{2}, \frac{1}{2}\right) + 1 \right\} 11^2 \sigma_1 C_1^{(4)} \int_0^{0 \vee (2t'-t)} d\tau \int_{R^3} (t-t')(t'-\tau)^{-7/2+\alpha/2} \cdot \exp\left\{ -h_1 \frac{|x-\xi|^2}{t-\tau} \right\} d\xi$$

$$= \begin{cases} 0, & (0 \geqq 2t'-t), \\ C_{11,5}(t-t')^{\alpha/2}, & (0 < 2t'-t), \end{cases}$$

$$\left(C_{11,5} \equiv |\sigma|_T^{(\alpha)} \left\{ 2^{\alpha/2} K\left(\frac{\alpha}{2}, \frac{1}{2}\right) + 1 \right\} 11^2 \sigma_1 C_1^{(4)} 2^{5/2} \left(\frac{\pi}{h_1}\right)^{3/2} (2-\alpha)^{-1} \right).$$

Thus,

$$\text{(3.69)} \qquad |J_5| \leqq C_{11,5}(t-t')^{\alpha/2}.$$

From the above follows the lemma, where $C_{11} \equiv \sum_{i=1}^5 C_{11,i}$. Q.E.D.

3.5. **Estimates for $\Gamma(x, t; \xi, \tau)$.**

$$\text{(3.70)} \quad \Gamma(x, t; \xi, \tau) \equiv Z(x-\xi, t; \xi, \tau) + \int_\tau^t d\tau_0 \int_{R^3} Z(x-y, t; y, \tau) \Phi(y, \tau_0; \xi, \tau) dy \equiv Z + \Gamma_0.$$

LEMMA 3.11. *For* $0 \leqq 2r + |m| \leqq 2$,

$$\text{(3.71)} \quad |D_t^r D_x^m \Gamma(x, t; \xi, \tau)| \leqq C_{12}^{(r,|m|)} (t-\tau)^{-(3+2r+|m|)/2} \exp\left\{ -d(r, |m|) \frac{|x-\xi|^2}{t-\tau} \right\}.$$

Proof. i) $2r + |m| = 0$.

$$\text{(3.72)} \quad \begin{aligned} |\Gamma| &\leqq |Z| + |\Gamma_0| \leqq 9C_1^{(0)} \left\{ 1 + C_5 \left(\frac{\pi}{h_1}\right)^{3/2} \cdot \frac{2}{\alpha} T^{\alpha/2} \right\} (t-\tau)^{-3/2} \exp\left\{ -h_1 \frac{|x-\xi|^2}{t-\tau} \right\} \\ &\equiv C_{12}^{(0,0)} (t-\tau)^{-3/2} \exp\left\{ -d(0,0) \frac{|x-\xi|^2}{t-\tau} \right\}. \end{aligned}$$

ii) $r=0$, $|m|=1$.

$$\text{(3.73)} \quad \begin{aligned} |D_x^m \Gamma| &\leqq |D_x^m Z| + \int_\tau^t d\tau_0 \int_{R^3} |D_x^m Z(x-y, t; y, \tau_0)| \cdot |\Phi(y, \tau_0; \xi, \tau)| dy \\ &\leqq C_{12}^{(0,1)} (t-\tau)^{-2} \exp\left\{ -d(0,1) \frac{|x-\xi|^2}{t-\tau} \right\}, \qquad (d(0,1) = h_1). \end{aligned}$$

iii) $|m|=2$.

$$
\begin{aligned}
|D_x^m\Gamma| \leqq |D_x^m Z| &+ \left|\int_\tau^{(t+\tau)/2} d\tau_0 \int_{R^3} D_x^m Z(x-y, t; y, \tau_0)\Phi(y, \tau_0; \xi, \tau)dy\right| \\
&+ \left|\int_{(t+\tau)/2}^t d\tau_0 \int_{R^3} D_x^m Z(x-y, t; y, \tau_0)\{\Phi(y, \tau_0; \xi, \tau)-\Phi(x, \tau_0; \xi, \tau)\}dy\right| \\
&+ \left|\int_{(t+\tau)/2}^t d\tau_0 \int_{R^3} D_x^m Z(x-y, t; y, \sigma_0)\Phi(x, \tau_0; \xi, \tau)dy\right| \\
&\leqq C_{12}^{(0,2)}(t-\tau)^{-5/2}\exp\left\{-d(0,2)\frac{|x-\xi|^2}{t-\tau}\right\}, \qquad \left(d(0,2)=\frac{h_1}{72}\right).
\end{aligned} \tag{3. 74}
$$

iv) $r=1$, $|m|=0$.

$$
\begin{aligned}
D_t^r\Gamma(x, t; \xi, \tau) &= D_tZ + D_t\Gamma_0 = D_tZ + \Phi + \int_\tau^{(t+\tau)/2} d\tau_0 \int_{R^3} D_tZ(x-y, t; y, \tau_0)\Phi(y, \tau_0; \xi, \tau)dy \\
&+ \int_{(t+\tau)/2}^t d\tau_0 \int_{R^3} D_tZ(x-y, t; y, \tau_0)\{\Phi(y, \tau_0; \xi, \tau)-(x, \tau_0; \xi, \tau)\}dy \\
&+ \int_{(t+\tau)/2}^t d\tau_0 \int_{R^3} \{D_tZ(x-y, t; y, \tau_0)-D_tZ(x-y, t; x, \tau_0)\}\Phi(x, \tau_0; \xi, \tau)dy.
\end{aligned} \tag{3. 75}
$$

From (3. 75), we have

$$
|D_t\Gamma| \leqq C_{12}^{(1,0)}(t-\tau)^{-5/2}\exp\left\{-d(1,0)\frac{|x-\xi|^2}{t-\tau}\right\}, \qquad \left(d(1,0)\equiv\frac{h_1}{72}\right). \tag{3. 75}'
$$

Q.E.D.

LEMMA 3. 12. *For $|m|=2$, and for ε and $\gamma\in[0, 1]$,*

$$
\begin{aligned}
&|D_x^m\Gamma(x, t; \xi, \tau)-D_{x'}^m\Gamma(x, t; \xi, \tau)| \\
&\leqq C_{13}\{|x-x'|^\gamma(t-\tau)^{-(5+\gamma)/2}+|x-x'|^{\alpha\varepsilon}(t-\tau)^{-5/2+\alpha/2(1-\varepsilon)}\}\exp\left\{-\bar{h}_1\frac{|x''-\xi|^2}{t-\tau}\right\}, \\
&\qquad\qquad \left(\bar{h}_1\equiv\frac{h_1}{72}\right).
\end{aligned} \tag{3. 76}
$$

Proof.

$$
\begin{aligned}
D_x^m\Gamma - D_{x'}^m\Gamma &= \{D_x^mZ - D_{x'}^mZ\} + \int_\tau^{(t+\tau)/2} d\tau_0 \int_{R^3}\{D_x^mZ-D_{x'}^mZ\}\Phi dy \\
&+ \int_{(t+\tau)/2}^t d\tau_0 \int_{R^3}\{D_x^mZ-D_{x'}^mZ\}\Phi dy \equiv \{D_x^mZ-D_{x'}^mZ\}+I_1+I_2.
\end{aligned} \tag{3. 77}
$$

It is easy to estimate $\{D_x^mZ-D_{x'}^mZ\}$ and I_1. I_2 is transformed into:

$$I_2=\int_{(t+\tau)/2}^{t} d\tau_0 \int_{\Sigma_1=\{y;\, y-x''<2|x-x'|\}} D_x^m Z(x-y, t; y, \tau_0)\{\Phi(y, \tau_0; \xi, \tau)-(x', \tau_0; \xi, \tau)\} dy$$

$$-\int_{(t+\tau)/2}^{t} d\tau_0 \int_{\Sigma_1} D_{x'}^m Z(x'-y, t; y, \tau_0)\{\Phi(y, \tau_0; \xi, \tau)-\Phi(x', \tau_0; \xi, \tau)\} dy$$

$$(3.77)' \quad +\int_{(t+\tau)/2}^{t} d\tau_0 \int_{R^3-\Sigma_1} \{D_x^m Z(x-y, t; y, \tau_0)-D_{x'}^m (x'-y, t; y, \tau_0)\}\{\Phi(y, \tau_0; \xi, \tau) -\Phi(x', \tau_0; \xi, \tau)\} dy$$

$$+\int_{(t+\tau)/2}^{t} d\tau_0 \int_{\Sigma_1} D_x^m Z(x-y, t; y, \tau_0)\{\Phi(x, \tau_0; \xi, \tau)-\Phi(x', \tau_0; \xi, \tau)\} dy$$

$$+\int_{(t+\tau)/2}^{t} d\tau_0 \int_{\Sigma_1} \{D_x^m Z(x-y, t; y, \tau_0)-D_{x'}^m Z(x'-y, t; y, \tau_0)\}\Phi(x', t; \xi, \tau) dy.$$

We have only to estimate I_2 on the basis of (3. 77)$'$, utilizing lemmas of § 2 and the Hölder-continuity of Φ in x. As a result, we have (3. 76). Q.E.D.

LEMMA 3. 13. *For* $\beta\in[0, \alpha]$ *and* $\gamma\in[0, 1]$,

$$(3.78) \quad \begin{aligned} &|D_t\Gamma(x, t; \xi, \tau)-D_t\Gamma(x', t; \xi, \tau)| \\ &\leqq \bar{C}_{13}\{|x-x'|^{\beta}(t-\tau)^{-5/2+(\alpha-\beta)/2}\}+|x-x'|^{\gamma}(t-\tau)^{-(5+\gamma)/2} \exp\left\{-\bar{h}_1\frac{|x''-\xi|^2}{t-\tau}\right\}. \end{aligned}$$

Proof. We have only to note that $D_t\Gamma$ satisfies

$$D_t\Gamma(x, t; \xi, \tau)=\sigma(x, t)P_0(D_x)\Gamma(x, t; \xi, \tau). \qquad \text{Q.E.D.}$$

LEMMA 3. 14. *For* $t\geqq t'>\tau$ *and* $2\geqq|m|\geqq 0$,

$$(3.79) \quad \begin{aligned} &|D_x^m\Gamma(x, t; \xi, \tau)-D_x^m\Gamma(x, t'; \xi, \tau)| \\ &\leqq C_{14}^{(|m|)}\{(t-t')(t'-\tau)^{-(5+|m|)/2}\}+(t-t')^{(2-|m|+\alpha)/2}(t'-\tau)^{-5/2} \exp\left\{-\bar{h}_1\frac{|x-\xi|^2}{t-\tau}\right\}, \\ &\hspace{20em} (|m|=1, 2), \end{aligned}$$

$$|D_x^0\Gamma(x, t; \xi, \tau)-D_x^0(x, t'; \xi, \tau)|\leqq C_{14}^{(0)}(t-t')(t'-\tau)^{-5/2} \exp\left\{-\frac{\bar{h}_1}{2}\frac{|x-\xi|^2}{t-\tau}\right\}.$$

Proof. i) For $t\geqq t_{,}>\tau$ and $t-t'<(t'-\tau)/4$, we utilize the expression,

$$D_x^m\Gamma_0(x, t; \xi, \tau)-D_x^m\Gamma_0(x, t'; \xi, \tau)$$

$$=\int_t^{(t'+\tau)/2} d\tau_0\int_{R^3}\{D_x^mZ(x-y, t; y, \tau_0\}-D_x^mZ(x-y, t'; y, \tau_0)\}\Phi(y, \tau_0; \xi, \tau)dy$$

$$+\int_{(t'+\tau)/2}^{2t'-t} d\tau_0\int_{R^3}\{D_x^mZ(x-y, t; y, \tau_0)-D_x^mZ(x-y, t'; y, \tau_0)\}$$
$$\{\Phi(y, \tau_0; \xi, \tau)-\Phi(x, \tau_0; \xi, \tau)\}dy$$

$$(3.80)\qquad +\int_{(t'+\tau)/2}^{2t'-t} d\tau_0\int_{R^3}\{D_x^mZ(x-y, t; y, \tau_0)-D_x^mZ(x-y, t'; y, \tau_0)\}\Phi(x, \tau_0; \xi, \tau)dy$$

$$+\int_{2t'-t}^{t} d\tau_0\int_{R^3} D_x^mZ(x-y, t; y, \tau_0)\{\Phi(y, \tau_0; \xi, \tau)-\Phi(x, \tau_0; \xi, \tau)\}dy$$

$$-\int_{2t'-t}^{t'} d\tau_0\int_{R^3} D_x^mZ(x-y, t; y, \tau_0)\{\Phi(y, \tau_0; \xi, \tau)-\Phi(x, \tau_0; \xi, \tau)\}dy$$

$$+\int_{2t'-t}^{t} d\tau_0\int_{R^3} D_x^mZ(x-y, t; y, \tau_0)\Phi(x, \tau_0; \xi, \tau)dy$$

$$-\int_{2t'-t}^{t'} d\tau_0\int_{R^3} D_x^mZ(x-y, t'; y, \tau_0)\Phi(x, \tau_0; \xi, \tau)dy.$$

After lengthy calculations, we have, for $t\geqq t'>\tau$ and $t-t'<(t'-\tau)/4$,

$$(3.81)\qquad \begin{aligned}&|D_x^m\Gamma(x, t; \xi, \tau)-D_x^x\Gamma(x, t'; \xi, \tau)|\\ &\leqq C_{14,1}^{(|m|)}(t-t')(t'-\tau)^{-(5+|m|)/2}+(t-t')^{(2-|m|+\alpha)/2}(t'-\tau)^{-5/2}\exp\left\{-\frac{\bar{h}_1}{2}\frac{|x-\xi|^2}{t-\tau}\right\},\\ &\qquad (|m|=1, 2),\end{aligned}$$

$$|\Gamma(x, t; \xi, \tau)-\Gamma(x, t'; \xi, \tau)|\leqq C_{14,1}^{(0)}(t-t')(t'-\tau)^{-5/2}\exp\left\{-\frac{\bar{h}_1}{2}\frac{|x-\xi|^2}{t-\tau}\right\}.$$

ii) For $t>t'>\tau$, $t-t'\geqq(t'-\tau)/4$, and $|m|\leqq 2$, the following inequalities hold:

$$(3.82)\qquad \begin{aligned}|D_x^m\Gamma_0|&\leqq\bar{C}_{12}^{(0,|m|)}(t-\tau)^{-(3+|m|-\alpha)/2}\exp\left\{-d(0, |m|)\frac{|x-\xi|^2}{t-\tau}\right\}\\ &\leqq\bar{C}_{12}^{(0,|m|)}2^{2-|m|+\alpha}(t-t')^{(2-|m|+\alpha)/2}(t'-\tau)^{-5/2}\exp\left\{-d(0, |m|)\frac{|x-\xi|^2}{t-\tau}\right\},\end{aligned}$$

$$|D_x^mZ|\leqq 36C_1^{(|m|)}(t-t')(t'-\tau)^{-(5+|m|)/2}\exp\left\{-2h_1\frac{|x-\xi|^2}{t-\tau}\right\},\qquad \text{(cf. Lemma 3.11).}$$

From i) and ii), we have the lemma, where

(3. 83)
$$\begin{cases} C_{14}^{(|m|)} \equiv C_{14,1}^{(|m|)} + \bar{C}_{12}^{(0,|m|)}(2^{2-|m|+\alpha}+1)9C_1^{(|m|)}(2^2+1), & (|m|=1,2), \\ C_{14}^{(0)} \equiv C_{14,1}^{(0)} + T^{\alpha/2}\bar{C}_{12}^{(0,0)}(2^\alpha+1)+9C_1^{(0)}(2^2+1). \end{cases}$$
Q.E.D.

In an anlogous way, we have:

LEMMA 3. 15. *For* $t \geqq t' > \tau$,

(3. 84)
$$\begin{aligned} &|D_t\Gamma(x,t;\xi,\tau)-D_{t'}\Gamma(x,t';\xi,\tau)| \\ &\leqq C_{15}(t-t')(t'-\tau)^{-7/2}+(t-t')^{\alpha/2}(t'-\tau)^{-5/2}\exp\left(-\frac{h_1}{2}\frac{|x-\xi|^2}{t-\tau}\right). \end{aligned}$$

Proof. This is obvious from the expression that Γ satisfies. Q.E.D.

§4. Estimates for the solution in $H_T^{2+\alpha}$ of a linear problem.

The unique solution in $H_T^{2+\alpha}$ of the Cauchy problem

(4. 1)
$$\begin{cases} D_t v=\sigma(x,t)P_0(D_x)v+f, & (f,\sigma\in H_T^\alpha,\ 0<\sigma_0\leqq\sigma\leqq\sigma_1<+\infty), \\ v(x,0)=0, \end{cases}$$

is given by

(4. 2)
$$v(x,t)=\int_0^t d\tau\int_{R^3}\Gamma(x,t;\xi,\tau)f(\xi,\tau)d\xi\equiv u(x,t),$$

for the righthand side of (4. 2) certainly satisfies (4. 1) and belongs to $H_T^{2+\alpha}$ as we shall see below.

LEMMA 4. 1.

(4. 3)
$$\begin{cases} |D_x^m u(x,t)|\leqq C_{16}^{(|m|)}t^{(2-|m|+\alpha)/2}\|f\|_T^{(\alpha)}, & (|m|=1,2), \\ |D_x^0 u(x,t)|\leqq C_{16}^{(0)}\cdot t\cdot\|f\|_T^{(\alpha)}. \end{cases}$$

Proof. The second inequality is obvious. For the first one, we utilize the expression,

(4. 4)
$$\begin{aligned} D_x^m u(x,t)=&\int_0^t d\tau\int_{R^3}D_x^m\Gamma(x,t;\xi,\tau)\{f(\xi,\tau)-f(x,\tau)\} \\ &+\int_0^t d\tau\int_{R^3}\{D_x^m Z(x-\xi,t;\xi,\tau)+D_x^m\Gamma_0(x,t;\xi,\tau)\}d\xi. \end{aligned}$$
Q.E.D.

COROLLARY of Lemma 4. 1.

(4. 5)
$$|D_t u(x,t)|\leqq|\sigma(x,t)P_0(D_x)u(x,t)|+|f|\leqq\bar{C}_{16}\|f\|_T^{(\alpha)}.$$

LEMMA 4. 2. *For* $t\geqq t'>0$,

$$(4.6)\quad \begin{cases} |D_x^m u(x,t)-D_x^m u(x,t')| \leqq C_{17}^{(|m|)}(t-t')^{(2-|m|+\alpha)/2}\|f\|_T^{(\alpha)}, & (|m|=1,2), \\ |D_x^0 u(x,t)-D_x^0 u(x,t')| \leqq C_{17}^{(0)}(t-t')\|f\|_T^{(\alpha)}. \end{cases}$$

Proof. i) For $t>t'>0$ anp $t>2t'$ (i.e., $t'<t-t'$, $t<2(t-t')$), by Lemma 4. 1,

$$(4.7)\quad \begin{cases} |D_x^m u(x,t)-D_x^m u(x,t')| \leqq C_{16}^{(|m|)}(2^{(2-|m|+\alpha)/2}+1)(t-t')^{(2-|m|+\alpha)/2}\|f\|_T^{(\alpha)}, & (|m|=1,2), \\ |D_x^0 u(x,t)-D_x^0 u(x,t')| \leqq 3C_{16}^{(0)}(t-t')\|f\|_T^{(\alpha)}. \end{cases}$$

ii) For $0\leqq t-t'\leqq t'$, we utilize the following expression:

$$(4.8)\quad \begin{aligned} & D_x^m u(x,t)-D_x^m u(x,t') \\ &= \int_{2t'-t}^{t}\int_{R^3} D_x^m\Gamma(x,t;\xi,\tau)\{f(\xi,\tau)-f(x,t)\} \\ &\quad -\int_{2t'-t}^{t} d\tau\int_{R^3} D_x^m\Gamma(x,t';\xi,\tau)\{f(\xi,\tau)-f(x,\tau)\}d\xi \\ &\quad +\int_0^{2t'-t} d\tau\int_{R^3}\{D_x^m\Gamma(x,t;\xi,\tau)-D_x^m\Gamma(x,t';\xi,\tau)\}\{f(\xi,\tau)-f(x,\tau)\}d\xi \\ &\quad +\int_{2t'-t}^{t} d\tau\int_{R^3} D_x^m\Gamma(x,t;\xi,\tau)f(\xi,\tau)d\xi-\int_{2t'-t}^{t'} d\tau\int_{R^3} D_x^m\Gamma(x,t';\xi,\tau)f(x,\tau)d\xi \\ &\quad +\int_0^{2t'-t} d\tau\int_{R^3}\{D_x^m\Gamma(x,t;\xi,\tau)-D_x^m\Gamma(x,t';\xi,\tau)\}\{f(x,\tau)-f(x,t)\}d\xi \\ &\quad +\int_0^{2t'-t} d\tau\int_{R^3}\{D_x^m\Gamma(x,t;\xi,\tau)-D_x^m\Gamma(x,t';\xi,\tau)\}f(x,t)d\xi \equiv \sum_{i=1}^{7} I_i^{(|m|)}. \end{aligned}$$

$I_i^{(|m)|}$ $(i=1,2,\cdots,6)$ can easily be estimated, i.e.,

$$(4.9)\quad \begin{cases} |I_i^{(|m|)}| \leqq C_{17,i}^{(|m|)}(t-t')^{(2-|m|+\alpha)/2}\|f\|_T^{(\alpha)}, & (|m|=1,2;\ i=1,2,\cdots,6), \\ |I_i^{(0)}| \leqq C_{17,i}^{(0)}(t-t')\|f\|_T^{(\alpha)}, & (i=1,2,\cdots,6). \end{cases}$$

In order to estimate $I_7^{(|m|)}$, we transform this into the form,

$$(4.10)\quad \begin{aligned} I_7^{(|m|)} &= \int_0^{2t'-t} d\tau\int_{R^3}\{D_x^m Z(x-\xi,t;\xi,\tau)-D_x^m Z(x-\xi,t';\xi,\tau)\}f(x,t)d\xi \\ &\quad -\int_{2t'-t}^{t} d\tau\int_{R^3} D_x^m\Gamma(x,t;\xi,\tau)f(x,t)d\xi+\int_{2t'-t}^{t} d\tau\int_{R^3} D_x^m\Gamma(x,t';\xi,\tau)f(x,t)d\xi \\ &\quad +\left[\int_0^t d\tau\int_{R^3} D_x^m\Gamma_0(x,t;\xi,\tau)f(x,t)d\xi-\int_0^{t'} d\tau\int_{R^3} D_x^m\Gamma_0(x,t';\xi,\tau)f(x,t)d\xi\right] \\ &\equiv \sum_{i=1}^{4} I_{7,i}^{(|m|)}. \end{aligned}$$

For $i=1, 2, 3$, we have easily,

(4. 11) $$\begin{cases} |I_{7,i}^{(|m|)}| \leqq C_{17,7,i}^{(|m|)}(t-t')^{(2-|m|+\alpha)/2}\|f\|_T^{(\alpha)}, & (|m|=1, 2), \\ |I_{7,i}^{(0)}| \leqq C_{17,7,i}^{(0)}(t-t')\|f\|_T^{(\alpha)}. \end{cases}$$

(4. 12) $$\begin{aligned} I_{7,4}^{(|m|)} &= \left[\int_0^t d\tau \int_{R^3} D_x^m \Gamma_0(x, t; \xi, \tau)\, d\xi - \int_0^{t'} d\tau \int_{R^3} D_x^m \Gamma_0(x, t'; \xi, \tau) d\xi\right] f(x, t) \\ &= \left[D_x^m \int_0^t d\tau_0 \int_{R^3} Z(x-y, t; y, \tau_0) q(y, \tau_0) dy \right. \\ &\qquad \left. - D_x^m \int_0^{t'} d\tau_0 \int_{R^3} Z(x-y, t'; y, \tau_0) q(y, \tau_0) dy\right] f(x, t) \\ &\equiv \hat{I}^{(|m|)} \cdot f(x, t), \end{aligned}$$

$$\left(q(y, \tau_0) = \int_0^t d\tau_0 \int_{R^3} \Phi(y, \tau_0; \xi, \tau) d\xi, \quad \text{cf. (3. 50)}\right).$$

We note that

(4. 13) $\hat{I}^{(|m|)}=$[the righthand side of (4. 8) as obtained by replacing Γ, f, ξ, and τ by Z, q, y, and τ_0, respectively]$\equiv \sum_{i=1}^{7} \hat{I}_i^{(|m|)}$.

(4. 14) $$\begin{cases} |\hat{I}_i^{(|m|)}| \leqq C_{17,7,4,i}^{(|m|)}(t-t')^{(2-|m|+\alpha)/2}, & (|m|=1, 2;\ i=1, 2, \cdots, 6), \\ |I_i^{(0)}| \leqq C_{17,7,4,i}^{(0)}(t-t'), & (i=1, 2, \cdots, 6), \end{cases}$$

(4. 14)′ $$|\hat{I}_7^{(m)}| = [I_{7,1}^{(|m|)} \text{ with } f \text{ replaced by } q] \leqq \begin{cases} C_{17,7,1}^{(|m|)} \cdot \|q\|_T^{(\alpha)}(t-t')^{(2-|m|+\alpha)/2}, & (|m|=1, 2), \\ C_{17,7,1}^{(0)} \cdot \|q\|_T^{(\alpha)}(t-t'), & (|m|=0), \end{cases}$$

$$(\|q\|_T^{(\alpha)} \leqq C_{10}+C_{11};\ C_{17,7,4,7}^{(|m|)} \equiv C_{17,7,1}^{(|m|)}(C_{10}+C_{11})).$$

Thus,

(4. 14)″ $$|I_7^{(m)}| \leqq \sum_{i=1}^{3} C_{17,7,i}^{(|m|)} + \sum_{i=1}^{7} C_{17,7,4,i}^{(|m|)} \|f\|_T^{(\alpha)} \times \begin{cases} (t-t')^{(2-|m|+\alpha)/2}, & (|m|=1, 2), \\ (t-t'), & (|m|=0). \end{cases}$$

Hence, for $0 \leqq t-t' \leqq t'$, we have

(4. 15) $$\begin{cases} |D_x^m u(x, t) - D_x^m u(x, t')| \leqq \left(\sum_{i=1}^{7} C_{17,i}^{(|m|)}\right)(t-t')^{(2-|m|+\alpha)/2}\|f\|_T^{(\alpha)}, & (|m|=1, 2), \\ |D_x^0 u(x, t) - D_x^0 u(x, t')| \leqq \left(\sum_{i=1}^{7} C_{17,i}^{(0)}\right)(t-t')\|f\|_T^{(\alpha)}, & (m=0), \end{cases}$$

$$\left(C_{17,7} \equiv \sum_{i=1}^{7} C_{17,7,4,i} + \sum_{i=1}^{3} C_{17,7,i}\right).$$

From (4. 7) and (4. 15) follows (4. 6), where

$$(4.16)\qquad \begin{cases} C_{17}^{(|m|)} \equiv C_{16}^{(|m|)}(2^{(2-|m|+\alpha)/2}+1)+\sum_{i=1}^{7} C_{17,i}^{(|m|)}, & (|m|=1,2), \\ C_{17}^{(0)} \equiv 3C_{16}^{(0)}+\sum_{i=1}^{7} C_{17,i}^{(0)}. \end{cases}$$

Q.E.D.

LEMMA 4. 3. *For* $|m|=2$,

$$(4.17)\qquad |D_x^m u(x,t)-D_{x'}^m u(x',t)| \leqq C_{18}|x-x'|^\alpha \|f\|_T^\alpha.$$

Proof.

$$\begin{aligned} & D_x^m u(x,t)-D_{x'}^m u(x',t) \\ &= \int_0^t d\tau \int_{\Sigma_1=\{\xi;|\xi-x|\leqq 2|x-x'|\}} D_x^m \Gamma(x,t;\xi,\tau)[\{f(\xi,\tau)-f(x,\tau)\}+\{f(x,\tau)-f(x',\tau)\}]d\xi \\ (4.18)\qquad & -\int_0^t d\tau \int_{\Sigma_1} D_x^m \Gamma(x',t;\xi,\tau)[\{f(\xi,\tau)-f(x',\tau)\}+\{f(x',\tau)-f(x',t)\}]d\xi \\ & +\int_0^t d\tau \int_{R^3-\Sigma_1} \{D_x^m \Gamma(x,t;\xi,\tau)-D_{x'}^m \Gamma(x',t;\xi,\tau)\}\{f(\xi,\tau)-f(x',t)\}d\xi \\ & +\left[\int_0^t d\tau \int_{R^3} \{D_x^m \Gamma(x,t;\xi,\tau)-D_{x'}^m \Gamma(x',t;\xi,\tau)\}d\xi\right] f(x',t) \equiv \sum_{i=1}^{4} J_i. \end{aligned}$$

For $i=1,2,3$, we have easily

$$(4.19)\qquad |J_i| \leqq C_{18,i}|x-x'|^\alpha \|f\|_T^{(\alpha)}, \qquad (i=1,2,3).$$

$$\begin{aligned} J_4 &= \left[\int_0^t d\tau \int_{R^3} \{D_x^m Z(x-\xi,t;\xi,\tau)-D_{x'}^m Z(x'-\xi,t;\xi,\tau)\}d\xi\right] f(x',t) \\ (4.20)\qquad & +\left[\int_0^t d\tau \int_{R^3} \{D_x^m \Gamma_0(x-\xi,t;\xi,\tau)-D_{x'}^m \Gamma_0(x'-\xi,t;\xi,\tau)\}d\xi\right] f(x',t) \\ & \equiv J_{4,1}+J_{4,2}. \end{aligned}$$

$$(4.21)\qquad |J_{4,1}| \leqq C_{18,4,1}|x-x'|^\alpha \|f\|_T^{(\alpha)}.$$

$$J_{4,2}=\left[D_x^m \int_0^t d\tau \int_{R^3} Z(x-y,t;y,\tau)q(y,\tau)dy - D_{x'}^m \int_0^t d\tau \int_{R^3} Z(x'-y,t;y,\tau)q(y,\tau)\right] f(x',t)$$

(4. 22) $=$[the righthand side of (4. 18) with Γ and f replaced by Z and q, respectively] $f(x',t)$

$$\equiv \sum_{i=1}^{4} J_{4,2,i}.$$

For $i=1,2,3$,

$$|J_{4,2,i}|\leqq C_{18,4,2,i}|x-x'|^{\alpha}\|f\|_T^{(\alpha)}, \qquad (i=1,2,3). \tag{4.23}$$

$$J_{4,2,4}=[J_{4,1} \text{ with } f \text{ replaced by } q]\ f(x',t).$$

Thus,

$$|J_{4,2,4}|\leqq C_{18,4,2,4}|x-x'|^{\alpha}\|f\|_T^{(\alpha)}. \tag{4.24}$$

Therefore,

$$|J_4|\leqq C_{18,4,1}+\sum_{i=1}^{4} C_{18,4,2,i}|x-x'|^{\alpha}\|f\|_T^{(\alpha)}\equiv C_{18,4}|x-x'|^{\alpha}\|f\|_T^{(\alpha)}. \tag{4.25}$$

From (4.19) and (4.25) follows the lemma, where

$$C_{18}\equiv \sum_{i=1}^{4} C_{18,i}. \tag{4.26}$$

Q.E.D.

The discussion above shows that $u(x,t)\in H_T^{2+\alpha}$. Therefore, from the uniqueness of the solution in $H_T^{2+\alpha}$ of (4.1) follows that $u(x,t)=v(x,t)$. Next, we consider the linear problem,

$$\begin{cases} D_t v=\sigma P_0(D_x)v+f, & (\sigma, f\in H_T^{\alpha}),\\ v(x,0)=v_0(x), & (v_0\in H^{2+\alpha}). \end{cases} \tag{1.11}$$

The solution $v(x,t)$ in $H_T^{2+\alpha}$ is written,

$$v(x,t)=v_0(x)+\{v(x,t)-v_0(x)-u(x,t)\}+u(x,t)\equiv v_0(x)+\bar{u}(x,t)+u(x,t), \tag{4.27}$$

where $\bar{u}(x,t)$ satisfies

$$\begin{cases} D_t\bar{u}=D_tv-D_tu=\sigma P_0v+f-(\sigma P_0u+f)=\sigma P_0(v-v_0-u)+\sigma P_0v_0=\sigma P_0\bar{u}+\sigma P_0v_0\\ \bar{u}(x,0)=0. \end{cases} \tag{4.28}$$

Hence, by (4.2),

$$\bar{u}(x,t)=\int_0^t d\tau\int_{R^3}\Gamma(x,t;\xi,\tau)\sigma(\xi,\tau)P_0(D_\xi)v_0(\xi)d\xi, \qquad (\sigma P_0v_0\in H_T^{\alpha}). \tag{4.28$'$}$$

Thus,

$$v(x,t)=v_0(x)+\int_0^t d\tau\int_{R^3}\Gamma(x,t;\xi,\tau)\{f(\xi,\tau)+\sigma(\xi,\tau)P_0(D_\xi)v_0(\xi)\}d\xi. \tag{4.29}$$

From (4.29), we have the following four lemmas.

LEMMA 4.4. *For the solution in $H_T^{2+\alpha}$ of the problem* (1.11),

$$|D_x^m v(x,t)|\leqq|D_x^m v_0(x)|+C_{16}^{(|m|)}t^{(2-|m|+\alpha)/2}\|f+\sigma P_0v_0\|_T^{(\alpha)}, \qquad (|m|=1,2). \tag{4.30}$$

$$|D_tv(x,t)|\leqq|\sigma P_0v+f|\leqq\frac{4}{3}\sigma_1\sum_{|m|=2}\{|D_x^m v_0(x)|+C_{16}^{(|m|)}t^{\alpha/2}\|f+\sigma P_0v_0\|_T^{(\alpha)}\}+\|f\|_T^{(\alpha)}, \tag{4.30$'$}$$

$$(4.30)'' \qquad |D_x^0 v(x,t)| \leqq |v_0(x)| + t\cdot C_{16}^{(0)} \|f+\sigma P_0 v_0\|_T^{(\alpha)}.$$

LEMMA 4. 5. *For $T \geqq t,\ t'>0$,*

$$(4.31) \qquad \begin{cases} |D_x^m v(x,t) - D_x^m v(x,t')| \leqq C_{17}^{(|m|)} |t-t'|^{(2-|m|+\alpha)/2} \cdot \|f+\sigma P_0 v_0\|_T^{(\alpha)}, & (|m|=1,2), \\ |D_x^0 v(x,t) - D_x^0 v(x,t')| \leqq C_{17}^{(0)} |t-t'| \cdot \|f+\sigma P_0 v_0\|_T^{(\alpha)}, \end{cases}$$

$$(|D_t v(x,t) - D_{t'} v(x,t')| \leqq |\sigma(x,t) P_0 v(x,t) - \sigma(x,t') P_0 v(x,t')| + |f(x,t) - f(x,t')|).$$

LEMMA 4. 6. *For $|m|=2$,*

$$(4.32) \qquad |D_x^m v(x,t) - D_x^m v(x',t)| \leqq |x-x'|^\alpha \{|D_x^m v_0|^\alpha + C_{18} \|f+\sigma P_0 v_0\|_T^{(\alpha)}\},$$

$$(|D_t v(x,t) - D_t v(x',t)| \leqq |\sigma(x,t) P_0(D_x) v(x,t) - \sigma(x',t) P_0(D_{x'}) v(x',t)| + |f(x,t) - f(x',t)|).$$

LEMMA 4. 7. *$C_{16}^{(|m|)}$, $C_{17}^{(|m|)}$, and C_{18}[1] are positive functions continuous in σ_0^{-1}, σ_1, $|\sigma|_T^{(\alpha)}$, and T, and monotonically increasing in each argument.*

Proof. We obtain this by tracing the lengthy calculations made in § 2~§ 4.

§ 5. The existence of a bounded solution of (1. 9)-(1. 10).

5. 1. Estimates for the characteristic curves. We define:

$$(5.1) \qquad \begin{cases} \langle v\rangle_T \equiv \sum_{|m|=0}^{2} |D_x^m v|_T^{(0)} + \sum_{|m|=0}^{1} |D_x^m v|_{t,T}^{(\alpha/2)}, \\ \langle v\rangle_T' \equiv \sum_{|m|=2} \{|D_x^m v|_{x,T}^{(\alpha)} + |D_x^m v|_{t,T}^{(\alpha/2)}\}, \qquad \langle\langle v\rangle\rangle_T \equiv \langle v\rangle_T + \langle v\rangle_T', \\ (\theta)_T \equiv \sum_{|m|=0}^{1} \{|D_x^m \theta|_T^{(0)} + |D_x^m \theta|_{t,T}^{(\alpha/2)}\} + \sum_{|m|=1} |D_x^m \theta|_{x,T}^{(\alpha)}, \end{cases}$$

$$(5.1)' \qquad \begin{cases} \hat{H}_T^{2+\alpha}: & \text{Banach space of functions } v(x,t) \text{ defined on } \bar{R}_T^3 \text{ such that } \langle\langle v\rangle\rangle_T < +\infty, \quad (\text{the norm is } \langle\langle\cdot\rangle\rangle_T); \\ \hat{H}_T^{1+\alpha}: & \text{Banach space of functions } \theta(x,t) \text{ defined on } \bar{R}_T^3 \text{ such that } (\theta)_T < +\infty, \quad (\text{the norm is } (\cdot)_T). \end{cases}$$

If $\bar{v} \in \hat{H}_T^{2+\alpha}$, (1. 9)^1 is written in the following way: (v is replaced by $\bar{v}$)

$$(5.2) \qquad D_t \rho + \bar{v}\cdot\nabla\rho = -\rho \operatorname{div} \bar{v}.$$

---

1) Particularly speaking, $C_{16}^{(|m|)}$, $C_{17}^{(|m|)}$, and C_{18} are bounded above by polynominals with non-negative coefficients in T^{α_i}, $(|\sigma|_T^{(\alpha)})^{\delta_l}$, $\sigma_0^{-\beta_j}$, $\sigma_1^{\gamma_k}$, and $\exp\{S(T^{\alpha/2}, |\sigma|_T^{(\alpha)}, \sigma_0^{-(5/2)}, \sigma_0^{-(5-\alpha)/2}, \sigma_1^{3/2})\}$ (α_i, β_j, γ_k, and $\delta_l>0$, and the index sets $\{i\}$, etc., are finite), where S is a polynominal with positive coefficients in the five arguments.

(5. 2) is a partial differential equation of the first order in ρ, so that $\rho(x, t)$ can be expressed by

$$\rho(x, t)=\rho_0(x_0(x, t)) \exp\left\{-\int_0^t \operatorname{div} \bar{v}(\bar{x}(\tau; x, t), \tau) d\tau\right\}, \tag{5. 3}$$

where $\bar{x}(\tau; x, t)$ is the solution of the characteristic equation of (5. 2),

$$\begin{cases} \dfrac{d\bar{x}_i(\tau)}{d\tau}=\bar{v}_i(\bar{x}(\tau), \tau), \qquad (i=1, 2, 3), \quad (\tau\in[0, T]), \\ \bar{x}(t)=x, \end{cases} \tag{5. 4}$$

and

$$x_0(x, t)\equiv\bar{x}(0; x, t). \tag{5. 5}$$

Since $\bar{v}\in\hat{H}_T^{2+\alpha}$, by the theory of ordinary differential equations [differentiation in parameters, etc.],

$$\frac{\partial\bar{x}_i}{\partial x_j}(\tau; x, t), \quad \frac{\partial\bar{x}_i}{\partial t}(\tau; x, t), \quad \frac{\partial^2\bar{x}_i}{\partial x_k\partial x_j}(\tau; x, t), \quad \text{and} \quad \frac{\partial^2\bar{x}_i}{\partial t\partial x_j}(\tau; x, t) \qquad (i, j, k=1, 2, 3)$$

exist, being continuous on $[0, T]$ and satisfying, respectively,

$$\begin{cases} \dfrac{d}{d\tau}\dfrac{\partial\bar{x}_i}{\partial x_j}(\tau; x, t)=\dfrac{\partial}{\partial\bar{x}_k}\bar{v}_i(\bar{x}(\tau; x, t), \tau)\dfrac{\partial\bar{x}_k}{\partial x_j}(\tau; x, t), \qquad (i, j=1, 2, 3), \quad (0\leqq\tau\leqq T), \\ \dfrac{\partial x_i}{\partial x_j}(t; x, t)=\delta_{ij}, \end{cases} \tag{5. 6}$$

$$\begin{cases} \dfrac{d}{d\tau}\dfrac{\partial\bar{x}_i}{\partial t}(\tau; x, t)=\dfrac{\partial}{\partial\bar{x}_k}\bar{v}_i(\bar{x}(\tau; x, t), \tau)\dfrac{\partial\bar{x}_k}{\partial t}(\tau; x, t), \\ \dfrac{\partial\bar{x}_i}{\partial t}(\tau; x, t)|_{\tau=t}=\bar{v}_i(x, t), \end{cases} \tag{5. 7}$$

$$\begin{cases} \dfrac{d}{d\tau}\dfrac{\partial^2\bar{x}_i}{\partial x_k\partial x_j}(\tau; x, t)=\dfrac{\partial^2}{\partial\bar{x}_m\partial\bar{x}_l}\bar{v}_i(\bar{x}(\tau; x, t), \tau)\dfrac{\partial\bar{x}_m}{\partial x_k}\cdot\dfrac{\partial\bar{x}_l}{\partial x_j} \\ \qquad\qquad +\dfrac{\partial}{\partial\bar{x}_l}\bar{v}_i(\bar{x}(\tau; x, t), \tau)\dfrac{\partial^2\bar{x}}{\partial x_k\partial x_j}(\tau; x, t), \\ \dfrac{\partial^2\bar{x}_i}{\partial x_k\partial x_j}(t; x, t)=0, \end{cases} \tag{5. 8}$$

$$(5.9)\quad \begin{cases} \dfrac{d}{d\tau}\dfrac{\partial^2 \bar{x}_i}{\partial t \partial x_j}(\tau; x, t) = \dfrac{\partial^2}{\partial \bar{x}_m \partial \bar{x}_l}\bar{v}_i(\bar{x}(\tau; x, t), \tau)\dfrac{\partial \bar{x}_m}{\partial t}\cdot\dfrac{\partial \bar{x}_l}{\partial x_j} \\ \qquad\qquad + \dfrac{\partial}{\partial \bar{x}_l}\bar{v}_i(\bar{x}(\tau; x, t), \tau)\dfrac{\partial^2 \bar{x}_l}{\partial t \partial x_j}(\tau; x, t), \\ \dfrac{\partial^2 \bar{x}_i}{\partial t \partial x_j}(\tau; x, t)|_{\tau=t} = 0. \end{cases}$$

From (5.6) we have

$$(5.10)\quad X_1(\tau; x, t) \equiv \max_{i,j}\left|\frac{\partial \bar{x}_i}{\partial x_j}(\tau; x, t)\right| \leqq 1 + \int_\tau^t \langle \bar{v} \rangle_T X_1(\tau_1; x, t)d\tau_1, \qquad (0 \leqq \tau \leqq t \leqq T).$$

Hence,

$$(5.10)'\quad X_1(\tau; x, t) \leqq e^{\langle \bar{v} \rangle_T (t-\tau)}, \qquad X_1(0; x, t) \leqq e^{\langle \bar{v} \rangle_T \cdot t}.$$

From (5.7) we have

$$(5.11)\quad X_2(\tau; x, t) \equiv \max_{i,j}\left|\frac{\partial \bar{x}_i}{\partial t}(\tau; x, t)\right| \leqq \langle \bar{v} \rangle_T + \int_\tau^t \langle \bar{v} \rangle_T \cdot X_2(\tau_1; x, t)d\tau_1.$$

Hence,

$$(5.11)'\quad X_2(\tau; x, t) \leqq \langle \bar{v} \rangle_T e^{\langle \bar{v} \rangle_T (t-\tau)}, \qquad X_2(0; x, t) \leqq \langle \bar{v} \rangle_T e^{\langle \bar{v} \rangle_T \cdot t}.$$

From (5.8) and (5.9) we have, respectively,

$$(5.12)\quad \begin{aligned} X_3(\tau; x, t) &\equiv \max_{i,j,k}\left|\frac{\partial^2 \bar{x}_i}{\partial x_k \partial x_j}(\tau; x, t)\right| \\ &\leqq \langle \bar{v} \rangle_T \int_\tau^t X_3(\tau_1; x, t)d\tau_1 + \langle \bar{v} \rangle_T \int_\tau^t \{X_1(\tau_1; x, t)\}^2 d\tau_1, \end{aligned}$$

$$(5.12)'\quad \begin{aligned} X_4(\tau; x, t) &\equiv \max_{i,j}\left|\frac{\partial^2 \bar{x}_i}{\partial t \partial x_j}(\tau; x, t)\right| \\ &\leqq \langle \bar{v} \rangle_T \int_\tau^t X_4(\tau_1; x, t)d\tau_1 + \langle \bar{v} \rangle_T \int_\tau^t X_1(\tau_1; x, t)X_2(\tau_1; x, t)d\tau_1. \end{aligned}$$

Hence,

$$(5.13)\quad X_3(\tau; x, t) \leqq e^{\langle \bar{v} \rangle_T (t-\tau)}\{e^{\langle \bar{v} \rangle_T (t-\tau)} - 1\}, \qquad X_3(0; x, t) \leqq e^{\langle \bar{v} \rangle_T t}\{e^{\langle \bar{v} \rangle_T t} - 1\};$$

$$(5.14)\quad X_4(\tau; x, t) \leqq \langle \bar{v} \rangle_T e^{\langle \bar{v} \rangle_T (t-\tau)}\{e^{\langle \bar{v} \rangle_T (t-\tau)} - 1\}, \qquad X_4(0; x, t) \leqq \langle \bar{v} \rangle_T e^{\langle \bar{v} \rangle_T t}\{e^{\langle \bar{v} \rangle_T t} - 1\}.$$

Thus, we have:

LEMMA 5. 1.

$$|D_x^m \bar{x}_i(\tau; x, t)| \leqq B_1^{(|m|)}(t-\tau; \langle\bar{v}\rangle_T), \qquad (|m|=1, 2;\ 0\leqq\tau\leqq t\leqq T), \tag{5.15}$$

$$[B_1^{(1)} \equiv e^{\langle\bar{v}\rangle_T(t-\tau)},\ B_1^{(2)} \equiv e^{\langle\bar{v}\rangle_T(t-\tau)}\{e^{\langle\bar{v}\rangle_T(t-\tau)}-1\}],$$

$$|D_x^m D_t \bar{x}_i(\tau; x, t)| \leqq B_2^{(|m|)}(t-\tau; \langle\bar{v}\rangle_T), \qquad (|m|=0, 1;\ 0\leqq\tau\leqq t\leqq T), \tag{5.15$'$}$$

$$[B_2^{(0)} \equiv \langle\bar{v}\rangle_T e^{\langle\bar{v}\rangle_T(t-\tau)},\ B_2^{(1)} \equiv \langle\bar{v}\rangle_T e^{\langle\bar{v}\rangle_T(t-\tau)}\{e^{\langle\bar{v}\rangle_T(t-\tau)}-1\}],$$

$$\begin{cases} |D_x^m x_{0,i}(x, t)| \leqq B_1^{(|m|)}(t; \langle\bar{v}\rangle_T), & (|m|=1, 2), \\ |D_x^m D_t x_{0,i}(x, t)| \leqq B_2^{(|m|)}(t; \langle\bar{v}\rangle_T), & (|m|=0, 1). \end{cases} \tag{5.16}$$

5. 2. **Operator** G_{T_1}. For the system (1. 9)-(1. 10),

$$\begin{cases} \dfrac{\partial\rho}{\partial t} + \operatorname{div} \rho v = 0, \\ \rho\dfrac{\partial v}{\partial t} - \dfrac{\mu}{3}(3\Delta + \nabla \operatorname{div})v = -\rho(v\cdot\nabla)v - \nabla\rho\dfrac{\partial p}{\partial\rho} - \nabla\theta\dfrac{\partial p}{\partial\rho} + \rho f \equiv N_1(x, t), \\ C_v\rho\dfrac{\partial\theta}{\partial t} = \kappa\Delta\theta + \Psi - \theta\dfrac{\partial p}{\partial\theta}\operatorname{div} v - C_v\rho v\cdot\nabla\theta \equiv C_v\rho N_2(x, t), \end{cases} \tag{1.9}$$

$$v(x, 0) = v_0(x), \qquad \theta(x, 0) = \theta_0(x), \qquad \rho(x, 0) = \rho_0(x), \tag{1.10}$$

we assume that

$$\theta_0(x),\ \ v_0(x) \in H^{2+\alpha}, \quad \rho_0(x) \in H^{1+\alpha}, \quad \text{and} \quad f(x, t) \in H_T^\alpha, \tag{5.17}$$

$$(0 < \bar{\rho}_0 \leqq \rho_0(x) \leqq \bar{\bar{\rho}}_0 \equiv |\rho_0|^{(0)} < \rho^* < +\infty).$$

For $\theta < 0$, we define p by the righthand side of (1. 8).

We take an arbitrary number $M_1 > ||v_0||^{(2)}$ and choose T_1 such that

$$0 < T_1 < M_1^{-1} \log\frac{\rho^*}{\bar{\bar{\rho}}_0}, \qquad \left(||v_0||^{(2)} = \sum_{|m|=0}^{2} |D_x^m v_0(x)|^{(0)}\right). \tag{5.18}$$

Next, we define a mapping G_{T_1} from

$$S_{T_1}^0 \equiv \{(v, \theta) : (v, \theta) \in \hat{H}_{T_1}^{2+\alpha} \times \hat{H}_{T_1}^{1+\alpha}\}, \quad \langle v\rangle_{T_1} \leqq M_1, \quad v(x, 0) = v_0(x), \quad \theta(x, 0) = \theta_0(x) \tag{5.19}$$

into $\hat{H}_{T_1}^{2+\alpha} \times \hat{H}_{T_1}^{1+\alpha}$ in the following way:

$$\begin{cases} \hat{v}(x, t) = v_0(x) + \displaystyle\int_0^t d\tau \int_{R^3} \Gamma(x, t; \xi, \tau; \rho_{(v)}) \left\{N_1 + \frac{\mu}{\rho_{(v)}} P_0 v_0\right\}(\xi, \tau) d\xi, \\ \hat{\theta}(x, t) = \theta_0(x) + \displaystyle\int_0^t d\tau \int_{R^3} {}'\Gamma(x, t; \xi, \tau; \rho_{(v)}) \left\{N_2 + \frac{\kappa}{C_v\rho_{(v)}} \Delta\theta_0\right\}(\xi, \tau) d\xi, \end{cases} \tag{5.20}$$

where $(\hat{v}, \hat{\theta}) = G_{T_1}(v, \theta)$. As for $'\Gamma$, cf. § 2, Remark. The norm in $\hat{H}^{2+\alpha}_{T_1} \times \hat{H}^{1+\alpha}_{T_1}$ is defined by

$$\|(v, \theta)\|_{T_1} \equiv \langle\!\langle v \rangle\!\rangle_{T_1} + (\theta)_{T_1}. \tag{5.21}$$

LEMMA 5.2. *G_{T_1} is well-defined.*

Proof. It suffices to prove that

$$N_1 + \frac{\mu}{\rho_{(v)}} P_0 v_0 \quad \text{and} \quad N_2 + \frac{\kappa}{C_v \rho_{(v)}} \Delta\theta_0 \in H^{\alpha}_{T_1}, \tag{5.22}$$

$$(\text{especially, } \rho_{(v)} \text{ and } \nabla\rho_{(v)} \in H^{\alpha}_{T_1}).$$

i)

$$\begin{aligned} &|\rho_{(v)}(x, t) - \rho_{(v)}(x', t')| \\ &\leqq |\rho_{(v)}(x, t) - \rho_{(v)}(x', t)| + |\rho_{(v)}(x', t) - \rho_{(v)}(x', t')| \\ &\leqq \{|\nabla\rho_{(v)}(\tilde{x}, t)|\, |x - x'|\}^{\alpha} \{|\rho_{(v)}(x, t)| + |\rho_{(v)}(x', t)|\}^{1-\alpha} \\ &\quad + \left\{ \left| \frac{\partial\rho_{(v)}}{\partial t}(x', \tilde{t}) \right| |t - t'| \right\}^{\alpha/2} \{|\rho_{(v)}(x', t)| + |\rho_{(v)}(x', t')|\}^{1-(\alpha/2)}. \end{aligned} \tag{5.23}$$

$$\left| \frac{\partial\rho_{(v)}}{\partial t} \right| \leqq |v|\, |\nabla\rho| + |\rho|\, |\mathrm{div}\, v| \leqq \{|\rho| + |\nabla\rho|\} \langle v \rangle_{T_1}, \tag{5.24}$$

$$\begin{aligned} |\nabla\rho_{(v)}(x, t)| &= \left| \rho(x, t) \left\{ \frac{D_{x_{0,l}}\rho_0(x_0(x, t))}{\rho_0(x_0(x, t))} \nabla_x x_{0,l}(x, t) \right. \right. \\ &\quad \left. \left. - \int_0^t \mathrm{div}\, D_{\bar{x}_l} v(\bar{x}(\tau; x, t), \tau) \nabla_x \bar{x}_l(\tau; x, t) d\tau \right\} \right| \\ &\leqq \bar{\bar{\rho}}_0 e^{\langle v \rangle_{T_1} T_1} \{(\bar{\rho}_0)^{-1} \|\rho_0\|^{(1+\alpha)} B_1^{(1)}(T_1; \langle v \rangle_{T_1}) + \langle v \rangle_{T_1} B_1^{(1)}(T_1; \langle v \rangle_{T_1}) T_1\}, \end{aligned} \tag{5.24'}$$

$$(|\rho| \leqq \bar{\bar{\rho}}_0 e^{\langle v \rangle_{T_1} T_1}).$$

Thus,

$$\begin{aligned} |\rho_{(v)}|^{(\alpha)}_{T_1} &\leqq 2^{1-\alpha} \|\rho_0\|^{(1+\alpha)} e^{\langle v \rangle_{T_1} T_1} \left\{ \left(\frac{\|\rho_0\|^{(1+\alpha)}}{\bar{\rho}_0} + \langle v \rangle_{T_1} \cdot T_1 \right) B_1^{(1)}(T_1; \langle v \rangle_{T_1}) \right\}^{\alpha} \\ &\quad + \{2\|\rho_0\|^{(1+\alpha)} e^{\langle v \rangle_{T_1} \cdot T_1}\}^{1+(\alpha/2)} [\{\|\rho_0\|^{(1+\alpha)} e^{\langle v \rangle_{T_1} \cdot T_1} + (\bar{\rho}_0)^{-1} \|\rho_0\|^{(1+\alpha)} \\ &\quad + \langle v \rangle_{T_1} T_1\} B_1^{(1)}(T_1; \langle v \rangle_{T_1})]^{\alpha/2} \end{aligned} \tag{5.25}$$

$$\left| \frac{1}{\rho_{(v)}} \right|^{(\alpha)}_{T_1} \leqq (\bar{\rho}_0)^{-2} |\rho_{(v)}|^{(\alpha)}_{T_1}. \tag{5.25'}$$

ii)

$$\frac{\nabla\rho_{(v)}}{\rho_{(v)}}(x,t)=\nabla_x\log\rho_{(v)}=\nabla_x\left\{\log\rho_0(x_0(x,t))-\int_0^t\operatorname{div}v(\bar{x}(\tau;x,t),\tau)d\tau\right\}. \tag{5.26}$$

$$\begin{aligned}&\nabla_x\log\rho_{(v)}(x,t)-\nabla_{x'}\log\rho_{(v)}(x',t)\\&=\left\{\frac{D_{x_{0,l}}\rho_0(x_0(x,t))}{\rho_0(x_0(x,t))}\nabla_x x_{0,l}(x,t)-\frac{D_{x_{0,l}}\rho_0(x_0(x',t))}{\rho_0(x_0(x',t))}\nabla_{x'}x_{0,l}(x',t)\right\}\\&\quad-\int_0^t\{\operatorname{div}D_{\bar{x}_l}v(\bar{x}(\tau;x,t),\tau)\nabla_x\bar{x}_l(\tau;x,t)-\operatorname{div}D_{\bar{x}_l}v(\bar{x}(\tau;x',t),\tau)\nabla_{x'}\bar{x}_l(\tau;x',t)\}d\tau.\end{aligned} \tag{5.27}$$

Hence,

$$\begin{aligned}&|\nabla\log\rho_{(v)}(x,t)-\nabla\log\rho_{(v)}(x',t)|\\&\leqq|D_{x_{0,l}}\rho_0(x_0(x,t))-D_{x_{0,l}}\rho_0(x_0(x',t))|\cdot\left|\frac{\nabla_x x_{0,l}(x,t)}{\rho_0(x_0(x,t))}\right|\\&\quad+\left|D_{x_{0,l}}\rho_0(x_0(x,t))\left\{\frac{1}{\rho_0(x_0(x,t))}-\frac{1}{\rho_0(x_0(x',t))}\right\}\nabla_x x_0(x,t)\right|\\&\quad+\left|\frac{D_{x_{0,l}}\rho_0(x_0(x',t))}{\rho_0(x_0(x,t))}\{\nabla_x x_{0,l}(x,t)-\nabla_{x'}x_{0,l}(x',t)\}\right|\\&\quad+\int_0^t|\operatorname{div}D_{\bar{x}_l}v(\bar{x}(\tau;x,t),\tau)-\operatorname{div}D_{\bar{x}_l}v(\bar{x}(\tau;x',t),\tau)|\cdot|\nabla_x\bar{x}_l(\tau;x,t)|d\tau\\&\quad+\int_0^t|\operatorname{div}D_{x_l}v(\bar{x}(\tau;x',t))|\cdot|\nabla_x\bar{x}_l(\tau;x,t)-\nabla_{x'}\bar{x}_l(\tau;x',t)|d\tau.\end{aligned} \tag{5.28}$$

$$|D_{x_{0,l}}\rho_0(x_0(x,t))-D_{x_{0,l}}\rho_0(x_0(x',t))|\leqq||\rho_0||^{(1+\alpha)}9B_1^{(1)}(T_1;\langle v\rangle_{T_1})^\alpha|x-x'|^\alpha, \tag{5.29}$$

$$\left|\frac{1}{\rho_0(x_0(x,t))}-\frac{1}{\rho_0(x_0(x',t))}\right|\leqq(\rho_0)^{-2}||\rho_0||^{(1+\alpha)}\{3B_1^{(1)}(T_1;\langle v\rangle_{T_1})\}^\alpha 2^{1+\alpha}|x-x'|^\alpha. \tag{5.29$'$}$$

$$\begin{aligned}&|\nabla_x x_{0,l}(x,t)-\nabla_{x'}x_{0,l}(x',t)|\\&\leqq 3^{1+\alpha}\{B_1^{(2)}(T_1;\langle v\rangle_{T_1})\}^\alpha\{2B_1^{(2)}(T_1;\langle v\rangle_{T_1})\}^{1-\alpha}|x-x'|^\alpha\end{aligned} \tag{5.29$''$}$$

$$\cdot\int_0^t|\operatorname{div}D_{\bar{x}_l}v(\bar{x}(\tau;x,t),\tau)-\operatorname{div}D_{\bar{x}_l}v(\bar{x}(\tau;x',t),\tau)|\cdot|\nabla_x\bar{x}_l(\tau;x,t)|d\tau.$$

$$\begin{aligned}&\leqq 3^{1+2\alpha}\langle v\rangle'_{T_1}T_1\{B_1^{(1)}(T_1;\langle v\rangle_{T_1})\}^{1+\alpha}|x-x'|^\alpha\\&\cdot\int_0^t|\operatorname{div}D_{\bar{x}_l}v(\bar{x}(\tau;x',t),\tau)\{\nabla_x\bar{x}_l(\tau;x,t)-\nabla_{x'}\bar{x}_l(\tau;x',t)\}|d\tau\end{aligned} \tag{5.30}$$

$$\leqq T_1\langle v\rangle_{T_1}3^{1+\alpha}\{B_1^{(2)}(T_1;\langle v\rangle_{T_1})\}^\alpha\{2B_1^{(1)}(T_1;\langle v\rangle_{T_1})^{1-\alpha}|x-x'|^\alpha. \tag{5.30$'$}$$

Thus, there exist non-negative constants D_1 and D_2 ($<+\infty$) depending on $\langle v\rangle_{T_1}$, T_1, $(\bar{\rho}_0)^{-1}$, and $||\rho_0||^{(1+\alpha)}$ such that

$$(5.31)\quad \begin{aligned}&|\nabla \log \rho|(x,t)-\nabla \log \rho(x',t)|\\ &\leqq\{D_1(\langle v\rangle_{T_1}, T_1; (\bar{\rho}_0)^{-1}, ||\rho_0||^{(1+\alpha)})+\langle v\rangle'_{T_1}T_1D_2(\langle v\rangle_{T_1}, T_1; (\bar{\rho}_0)^{-1}, ||\rho_0||^{(1+\alpha)})\}|x-x'|^{\alpha}.\end{aligned}$$

In an analogous way, we have an inequality, $(t\geqq t')$

$$(5.32)\quad \begin{aligned}&|\nabla \log \rho_{(v)}(x,t)-\nabla \log \rho_{(v)}(x,t')|\\ &\leqq\left|\{D_{x_{0,l}}\rho_0(x_0(x,t))-D_{x_{0,l}}\rho_0(x_0(x,t'))\}\frac{\nabla_x x_{0,l}(x,t)}{\rho_0(x_0(x,t))}\right|\\ &\quad+\left|D_{x_{0,l}}\rho_0(x_0(x,t'))\left\{\frac{1}{\rho_0(x_0(x,t))}-\frac{1}{\rho_0(x_0(x,t'))}\right\}D_x x_{0,l}(x,t)\right|\\ &\quad+\left|\frac{D_{x_{0,l}}\rho_0(x_0(x,t))}{\rho_0(x_0(x,t))}\{\nabla_x x_{0,l}(x,t)-\nabla_x x_{0,l}(x,t')\}\right|\\ &\quad+\int_{t'}^{t}|\operatorname{div} D_{\bar{x}_l}v(\bar{x}(\tau;x,t),\tau)\nabla_x\bar{x}_l(\tau;x,t)|d\tau\\ &\quad+\int_0^{t'}|\{\operatorname{div} D_{\bar{x}_l}v(\bar{x}(\tau;x,t),\tau)-\operatorname{div} D_{\bar{x}_l}v(\bar{x}(\tau;x,t'),\tau)\}\nabla_x\bar{x}_l(\tau;x,t)|d\tau\\ &\quad+\int_0^{t'}|\operatorname{div} D_{\bar{x}_l}v(\bar{x}(\tau;x,t'),\tau)\{\nabla_x\bar{x}_l(\tau;x,t)-\nabla_x\bar{x}_l(\tau;x,t')\}|d\tau.\end{aligned}$$

From this follows that there exist non-negative constants D_3 and D_4 depending on $\langle v\rangle_{T_1}$, T_1, $(\bar{\rho}_0)^{-1}$, and $||\rho_0||^{(1+\alpha)}$ such that

$$(5.32)\quad \begin{aligned}&|\nabla \log \rho_{(v)}(x,t)-\nabla \log \rho_{(v)}(x,t')|\\ &\leqq\{D_3(\langle v\rangle_{T_1}, T_1)+D_4(\langle v\rangle_{T_1}, T_1)T_1\langle v\rangle'_{T_1}\}|t-t'|^{\alpha/2}.\end{aligned}$$

From (5.31) and (5.32)′ we have

$$(5.33)\quad |\nabla \log \rho_{(v)}|_{T_1}^{(\alpha)}\leqq D_1(\langle v\rangle_{T_1}, T_1)+D_3(\langle v\rangle_{T_1}, T_1)+\langle v\rangle'_{T_1}T_1\{D_2(\langle v\rangle_{T_1}, T_1)+D_4(\langle v\rangle_{T_1}, T_1)\}.$$

E_i ($i=1, 2\,3, 4$) are continuous functions of $\langle v\rangle_{T_1}$ and T_1, monotonically increasing in each argument.

iii) It is easily shown that $\partial P/\partial\rho$, $\partial p/\partial\rho$, and $\Psi\in H_{T_1}^{\alpha}$, because p is assumed to be virially expanded and $\rho_{(v)}\in H_{T_1}^{\alpha}$.

From i), ii), and iii) follows the result that

$$(5.34)\quad \begin{cases} \left\| N_1 + \dfrac{\mu}{\rho_{(v)}} P_0 v_0 \right\|_{T_1}^{(\alpha)} \leqq A_1(\langle v\rangle_{T_1}, (\theta)_{T_1}, T_1; (\bar{\rho}_0)^{-1}, \|v_0\|^{(2+\alpha)}, \|f\|_T^{(\alpha)}) \\ \qquad\qquad + T_1 A_2(\langle v\rangle_{T_1}, (\theta)_{T_1}, T_1; (\bar{\rho}_0)^{-1}, \|v_0\|^{(2+\alpha)}, \|f\|_T^{(\alpha)}) \langle v\rangle'_{T_1} < +\infty, \\ \left\| N_2 + \dfrac{\kappa}{C_v \rho_{(v)}} \Delta\theta_0 \right\|_{T_1}^{(\alpha)} \leqq A_3(\langle v\rangle_{T_1}, (\theta)_{T_1}, T_1; (\bar{\rho}_0)^{-1}, \|v_0\|^{(2+\alpha)}, \|f\|_T^{(\alpha)} < +\infty, \end{cases}$$

where A_i $(i=1, 2, 3)$ are positive constants continuous in the six parameters and increasing monotonically in each parameter. Q.E.D.

5.3. **The subset** $S_{T'}$.

LEMMA 5.3. *There exist* $T' \in (0, T_1]$, $M_2(>0)$, *and* $M_3(>0)$ *such that, for a subset* $S_{T'}$ *of* $S_{T'}^0$ *defined by*

$$(5.35)\qquad S_{T'} \equiv \{(v, \theta) : (v, \theta) \in S_{T'}^0, (\theta)_{T'} \leqq M_2, \langle v\rangle'_{T'} \leqq M_3\},$$

the following holds:

$$(5.35)'\qquad G_{T'} S_{T'} \subset S_{T'} (\subset S_{T'}^0 \subset \hat{H}_{T'}^{2+\alpha} \times \hat{H}_{T'}^{1+\alpha}).$$

Proof. The lemmas 4.4~4.7 and the procedure of the proof of Lemma 5.2 guarantee that, for an arbitrary $T_0 \in (0, T_1]$ and for an arbitrary $(v, \theta) \in S_{T_0}$, we have

$$(5.36)\quad \begin{cases} \langle \hat{v}\rangle_{T_0} \leqq \|v_0\|^{(2)} + C(\langle v\rangle_{T_0}, T_0)\{A_1(\langle v\rangle_{T_0}, (\theta)_{T_0}, T_0) + T_0 A_2(\langle v\rangle_{T_0}, (\theta)_{T_0}, T_0)\langle v\rangle'_{T_0}\}, \\ (\hat{\theta})_{T_0} \leqq \|\theta_0\|^{(1+\alpha)} + {}'C(\langle v\rangle_{T_0}, T_0) A_3(\langle v\rangle_{T_0}, (\theta)_{T_0}, T_0), \\ \langle \hat{v}\rangle'_{T_0} \leqq |\nabla\nabla v_0|^{(\alpha)} + \hat{C}(\langle v\rangle_{T_0}, T_0) A_1(\langle v\rangle_{T_0}, (\theta)_{T_0}, T_0) + T_0 A_2(\langle v\rangle_{T_0}, (\theta)_{T_0}, T_0)\langle v\rangle'_{T_0}, \end{cases}$$

where $C(\cdot, T_0)$ and ${}'C(\cdot, T_0) \searrow 0$, and $C(\cdot, T_0) \searrow C_0 > 0$, as $T_0 \searrow 0$. Taking and arbitrary $M_2 > \|\theta_0\|^{(1+\alpha)}$, we can choose $T_2 \in (0, T_1]$ such that

$$(5.37)\quad \begin{cases} \|v_0\|^{(2)} + C(M_1, T_2) A_1(M_1, M_2, T_2) < M_1, \\ \|\theta_0\|^{(1+\alpha)} + {}'C(M_1, T_2) A_3(M_1, M_2, T_2) \leqq M_2, \\ \hat{\theta}(M_1, T_2) T_2 A_2(M_1, M_2, T_2) < 1. \end{cases}$$

Moreover, there exists $T' \in (0, T_2]$ such that

$$(5.37)'\quad \begin{aligned} &|\nabla\nabla v_0|^{(\alpha)} + C(M_1, T') A_1(M_1, M_2, T') \\ &\leqq \frac{1 - C(M_1, T') T' A_2(M_1, M_2, T')}{C(M_1, T') T' A_2(M_1, M_2, T')} \{M_1 - \|v_0\|^{(2)} - C(M_1, T') A_1(M_1, M_2, T')\}. \end{aligned}$$

Hence, there exists $M_3 > 0$ such that

(5. 37)''

$$\frac{|\nabla\nabla v_0|^{(\alpha)}+\hat{C}(M_1, T')A_1(M_1, M_2, T')}{1-\hat{C}(M_1, T')T'A_2(M_1, M_2, T')}\leqq M_3\leqq\frac{M_1-\{\|v_0\|^{(2)}+C(M_1, T')A_1(M_1, M_2, T')\}}{C(M_1, T')T'A_2(M_1, M_2, T')},$$

$$(C(M_1, T'), A_1(M_1, M_2, T'), C(M_1, T'), A_2(M_1, M_2, T')>0).$$

From (5. 36), (5. 37), and (5. 37)'' follows (5. 35)'. Q.E.D.

We note that

(5. 38) $$G_{T'}S_{T'}\subset S_{T'}\cap\{H^{2+\alpha}_{T'}\times H^{2+\alpha}_{T'}\}\subset S_{T'}\cap\{\hat{H}^{2+\alpha}_{T'}\times\hat{H}^{2+\alpha}_{T'}\}\subset S_{T'},\qquad (\text{cf. } \S\, 4),$$

and that

(5. 38)' $$G_{T'}S_{T'}\subset S_{T'}\subset\hat{H}^{2+\alpha}_{T'}\times\hat{H}^{1+\alpha}_{T'}\subset\dot{H}^{2+\alpha}_{T'}\times\dot{H}^{1+\alpha}_{T'}\subset\dot{H}^{2+\alpha\beta}_{T'}\times\dot{H}^{1+\alpha\beta}_{T'},$$

$$(\beta\in(0,1) \text{ is arbitrary}),$$

where $\dot{H}^{2+\gamma}_{T'}$ $\dot{H}^{1+\gamma}_{T'}$ ($\gamma\in(0,1)$) is a Fréchet space defined by a countable system of seminorms

(5. 39) $$[(v, \theta)]^{(\gamma)}_{N,T'}\equiv\langle\!\langle v\rangle\!\rangle^{(\gamma)}_{N,T'}+(\theta)^{(\gamma)}_{N,T'},\qquad (N=1, 2, 3, \cdots),$$

[the suffix "N, T'" indicates that the supremum is taken on $R^3_{N,T'}\equiv\{(x, t); |x|\leqq N+M_1(T'-t),\ 0\leqq t\leqq T',\ x\in R^3\}$ instead of $R^3_{T'}$].

LEMMA 5. 4. *$S_{T'}$ is a convex compact subset in $\dot{H}^{2+\alpha\beta}_{T'}\times\dot{H}^{1+\alpha\beta}_{T'}$.*

Proof. i) Convexity. For (v_1, θ_1) and $(v_2, \theta_2)\in S_{T'}$ and for $\lambda\in[0, 1]$,

(5. 40) $$(v_1, \theta_1)+(1-\lambda)(v_2, \theta_2)=(\lambda v_1+(1-\lambda)v_2, \lambda\theta_1+(1-\lambda)\theta_2)\in S_{T'},$$

because

(5. 40)' $$\begin{cases}\langle\lambda v_1+(1-\lambda)v_2\rangle^{(\alpha)}_{T'}\leqq M_1, & [\lambda\theta_1+(1-\lambda)\theta_2]^{(\alpha)}_{T'}\leqq M_2, \qquad \langle\lambda v_1+(1-\lambda)v_2\rangle'^{(\alpha)}_{T'}\leqq M_3,\\ (\lambda v_1+(1-\lambda)v_2, \lambda\theta_1+(1-\lambda)\theta_2)|_{t=0}=(v_0, \theta_0).\end{cases}$$

ii) Compactness. By a simple calculation it is shown that

(5. 41) $$[(v, \theta)]^{(\alpha\beta)}_{N,T'}\leqq 3[(v, \theta)]^{(\alpha)}_{N,T'}\leqq 3\langle\!\langle(v, \theta)\rangle\!\rangle^{(\alpha)}_{T'}.$$

Next, for an arbitrary sequence $\{(v_i, \theta_i)\}^{\infty}_{i=1}\subset S_{T'}$, by Ascolic-Arzelà's theorem, there exist a subsequence $\{(v_{i_k}, \theta_{i_k})\in S_{T'}\}$ and $(\bar{v}, \bar{\theta})\in S_{T'}$ such that, in a compact uniform way,

(5. 42) $$\begin{cases}D^m_x v_{i_k}\to D^m_x\bar{v}, & (k\to+\infty;\ |m|\leqq 2),\\ D^m_x\theta_{i_k}\to D^m_x\bar{\theta}, & (k\to+\infty;\ |m|\leqq 1).\end{cases}$$

It remains to show that (v_{i_k}, θ_{i_k}) converges to $(\bar{v}, \bar{\theta})$ in the topology of $\dot{H}^{2+\alpha\beta}_{T'}\times\dot{H}^{1+\alpha\beta}_{T'}$.

Firstly, in $R^3_{N,T'}$,

$$
\begin{aligned}
&|\{D^m_x v_{i_k}(x,t)-D^m_x\bar{v}(x,t)\}-\{D^m_x v_{i_k}(x,t')-D^m_x\bar{v}(x,t')\}| \\
&\leqq |\{D^m_x v_{i_k}(x,t)-D^m_x v_{i_k}(x,t')\}-\{D^m_x\bar{v}(x,t)-D^m_x\bar{v}(x,t')\}|^{\beta} \\
&\quad\times\{|D^m_x v_{i_k}(x,t)-D^m_x\bar{v}(x,t)|+|D^m_x v_{i_k}(x,t')-D^m_x\bar{v}(x,t')|\}^{1-\beta} \\
&\leqq \{|D^m_x v_{i_k}|^{(\alpha/2)}_{t,N,T'}+|D^m_x\bar{v}|^{(\alpha/2)}_{t,N,T'}\}^{\beta}|t-t'|^{\alpha\beta/2}\{2|D^m_x v_{i_k}-D^m_x\bar{v}|^{(0)}_{N,T'}\}^{1-\beta}, \qquad (|m|\leqq 2).
\end{aligned}
\tag{5.43}
$$

Hence,

$$
|D^m_x v_{i_k}-D^m_x\bar{v}|^{(\alpha\beta/2)}_{t,N,T'}\leqq\{|D^m_x v_{i_k}|^{(\alpha/2)}_{t,N,T'}+|D^m_x\bar{v}|^{(\alpha/2)}_{t,N,T'}\}^{\beta}\{2|D^m_x v_{i_k}-D^m_x\bar{v}|^{(0)}_{N,T'}\}^{1+\beta}, \qquad (|m|\leqq 2).
\tag{5.43$'$}
$$

Therefore,

$$
|D^m_x v_{i_k}-D^m_x\bar{v}|^{(\alpha\beta/2)}_{t,N,T'}\to 0 \qquad (k\to+\infty;\ N=1,2,\cdots;\ |m|\leqq 2).
\tag{5.43$''$}
$$

In the same way, as $k\to+\infty$,

$$
\begin{aligned}
&|D^m_x v_{i_k}-D^m_x\bar{v}|^{(\alpha\beta)}_{x,N,T'} \quad (|m|=2), \qquad |D^{m'}_x\theta_{i_k}-D^{m'}_x\bar{\theta}|^{(\alpha\beta/2)}_{t,N,T'} \quad (|m'|=0,1), \\
&\text{and} \qquad |D^{m''}_x\theta_{i_k}-D^{m''}_x\bar{\theta}|^{(\alpha\beta)}_{x,N,T'} \quad (|m''|=1)\to 0.
\end{aligned}
\tag{5.44}
$$

Thus,

$$
d^{(\alpha\beta)}_{T'}((v_{i_k},\theta_{i_k}),(\bar{v},\bar{\theta}))\to 0 \qquad (k\to+\infty),
\tag{5.44$'$}
$$

where $d^{(\gamma)}_{T'}(\cdot,\cdot)$ is a distance defined by

$$
d^{(\gamma)}_{T'}((v_1,\theta_1),(v_2,\theta_2))\equiv\sum_{N=1}^{\infty}\left(\frac{1}{2}\right)^{N}\frac{[(v_1,\theta_1)-(v_2,\theta_2)]^{(\gamma)}_{N,T'}}{1+[(v_1,\theta_1)-(v_2,\theta_2)]^{(\gamma)}_{N,T'}}. \qquad \text{Q.E.D.}
\tag{5.44$''$}
$$

5.4. The continuity of $G_{T'}$ on $S_{T'}$ as a subset of $\dot{H}^{2+\alpha\beta}_{T'}\times\dot{H}^{1+\alpha\beta}_{T'}$.

LEMMA 5.5. *$G_{T'}$ is continuous as an operator from $S_{T'}$ as a subset of $\dot{H}^{2+\alpha\beta}_{T'}\times\dot{H}^{1+\alpha\beta}_{T'}$ into itself.*

Proof. We put

$$
(\hat{v},\hat{\theta})\equiv G_{T'}(v,\theta), \qquad ((v,\theta)\in S_{T'}).
$$

We show that, for an arbitrary sequence $\{(v_n,\theta_n)\in S_{T'}\}$ that converges to an arbitrary fixed element $(v,\theta)\in S_{T'}$ in the topology of $\dot{H}^{2+\alpha\beta}_{T'}\times\dot{H}^{1+\alpha\beta}_{T'}$,

$$
\begin{cases}
G_{T'}(v_n,\theta_n)=(\hat{v}_n,\hat{\theta}_n)\to(\hat{v},\hat{\theta})=G_{T'}(v,\theta) \quad (\text{in } \dot{H}^{2+\alpha\beta}_{T'}\times\dot{H}^{1+\alpha\beta}_{T'}), \qquad (n\to+\infty), \\
\text{i.e., for } \forall\,\varepsilon>0,
\end{cases}
\tag{5.45}
$$

$$
[(\hat{v}_n,\hat{\theta}_n)-(\hat{v},\hat{\theta})]^{(\alpha\beta)}_{N,T'}<\varepsilon, \qquad (n\geqq n_0(\varepsilon,N)).
$$

Obviously, $(\hat{v}, \hat{\theta})$ and $(\hat{v}_n, \hat{\theta}_n)$ satisfy, respectively,

$$(5.46)\qquad \begin{cases} \dfrac{\partial \hat{v}}{\partial t} = \dfrac{\mu}{\rho_{(v)}} P_0(D_x)\hat{v} + N_1(x, t; v, \theta), & \hat{v}(x, 0) = v_0(x), \\ \dfrac{\partial \hat{\theta}}{\partial t} = \dfrac{\kappa}{C_v \rho_{(v)}} \Delta\theta + N_2(x, t; v, \theta), & \hat{\theta}(x, 0) = \theta_0(x), \end{cases}$$

and

$$(5.46)'\qquad \begin{cases} \dfrac{\partial \hat{v}_n}{\partial t} = \dfrac{\mu}{\rho_{(v_n)}} P_0(D_x)\hat{v}_n + N_1(x, t; v_n, \theta_n), & v_n(x, 0) = v_0(x), \\ \dfrac{\partial \hat{\theta}_n}{\partial t} = \dfrac{\kappa}{C_v \rho_{(v_n)}} \Delta\theta_n + N_2(x, t; v_n, \theta_n), & \hat{\theta}_n(x, 0) = \theta_0(x). \end{cases}$$

Therefore,

$$(5.47)\qquad (\bar{w}_n, \zeta_n) \equiv (\hat{v} - \hat{v}_n, \hat{\theta} - \hat{\theta}_n)$$

satisfies

$$(5.47)'\qquad \begin{cases} \dfrac{\partial \bar{w}_n}{\partial t} - \dfrac{\mu}{\rho_{(v)}} P_0(D_x)\bar{w}_n = \mu\left\{\dfrac{1}{\rho_{(v)}} - \dfrac{1}{\rho_{(v_n)}}\right\} P_0(D_x)\hat{v}_n + \{N_1(x, t; v, \theta) - N_1(x, t; v_n, \theta_n)\}, \\ \dfrac{\partial \zeta_n}{\partial t} - \dfrac{\kappa}{C_v \rho_{(v)}} \Delta\zeta_n = \dfrac{\kappa}{C_v}\left\{\dfrac{1}{\rho_{(v)}} - \dfrac{1}{\rho_{(v_n)}}\right\} \Delta\hat{\theta}_n + \{N_2(x, t; v, \theta) - N_2(x, t; v_n, \theta_n)\}, \\ \bar{w}_n(x, 0) = 0, \qquad \zeta_n(x, 0) = 0. \end{cases}$$

Obviously, $\bar{w}_n$ and $\zeta_n \in H_{T'}^{2+\alpha}$, and each of the two right-hand sides of (5.47)$'$ belongs to $H_{T'}^{\alpha}$, so that $(\bar{w}_n, \zeta_n)$ can be expressed as

$$(5.47)''\qquad \begin{cases} \bar{w}_n(x, t) = \displaystyle\int_0^t d\tau \int_{R^3} \Gamma(x, t; \xi, \tau; \rho_{(v)}) \mu\left(\frac{1}{\rho_{(v)}} - \frac{1}{\rho_{(v_n)}}\right) P_0(D_\xi)\hat{v}_n(\xi, \tau) \\ \qquad\qquad + [N_1(\xi, \tau; v, \theta) - N_1(\xi, \tau, v_n, \theta_n)], \\ \zeta_n(x, t) = \displaystyle\int_0^t d\tau \int_{R^3} {}'\Gamma(x, t; \xi, \tau; \rho_{(v)}) \frac{\kappa}{C_v}\left(\frac{1}{\rho_{(v)}} - \frac{1}{\rho_{(v_n)}}\right) \Delta_\xi\hat{\theta}_n(\xi, \tau) \\ \qquad\qquad + [N_2(\xi, \tau, v, \theta) - N_2(\xi, \tau; v_n, \theta_n)]. \end{cases}$$

We put

$$(5.48)\qquad \begin{cases} \tilde{N}_1(x, t; v, \theta; v_n, \theta_n) \equiv \mu\left(\dfrac{1}{\rho_{(v)}} - \dfrac{1}{\rho_{(v_n)}}\right) P_0(D_x)\hat{v}_n + \{N_1(x, t; v, \theta) - N_1(x, t; v_n, \theta_n)\}, \\ \tilde{N}_2(x, t; v, \theta; v_n, \theta_n) \equiv \dfrac{\kappa}{C_v}\left\{\dfrac{1}{\rho_{(v)}} - \dfrac{1}{\rho_{(v_n)}}\right\} \Delta\theta_n + \{N_2(x, t; v, \theta) - N_2(x, t; v_n, \theta_n)\}. \end{cases}$$

In the first place, we estimate $\|\tilde{N}_1\|^{(\alpha\beta)}_{N,T'}$ and $\|\tilde{N}_2\|^{(\alpha\beta)}_{N,T'}$ $(N=1,2,\cdots)$. We note that

$$\|\tilde{N}_i\|^{(\alpha\beta)}_{1,T'}\leqq\|\tilde{N}_i\|^{(\alpha\beta)}_{2,T'}\leqq\cdots\leqq\|\tilde{N}_i\|^{(\alpha\beta)}_{T'}\leqq\text{const}\,\|\tilde{N}_i\|^{(\alpha)}_{T'},\qquad (i=1,2). \tag{5.49}$$

i)

$$\|\rho_{(v)}-\rho_{(v_n)}\|^{(\alpha\beta)}_{N,T'}=|\rho_{(v)}-\rho_{(v_n)}|^{(0)}_{N,T'}+|\rho_{(v)}-\rho_{(v_n)}|^{(\alpha\beta)}_{x,N,T'}+|\rho_{(v)}-\rho_{(v_n)}|^{(\alpha\beta)}_{t,N,T'}.$$

For $(\xi,\tau)\in R^3_{N,T'}$,

$$\begin{aligned}
&|\rho_{(v)}(\xi,\tau)-\rho_{(v_n)}(\xi,\tau)|\\
&=\Big|\rho_0(x_0(\xi,\tau))\exp\Big\{-\int_0^\tau \operatorname{div} v(\bar{x}(\tau_0;\xi,\tau),\tau_0)d\tau_0\Big\}\\
&\qquad -\rho_0(x_{n,0}(\xi,\tau))\exp\Big\{-\int_0^\tau \operatorname{div} v_n(\bar{x}_n(\tau_0;\xi,\tau),\tau_0)d\tau_0\Big\}\Big|\\
&\leqq|\nabla\rho_0|^{(0)}|(x_0-x_{n,0})(\xi,\tau)|e^{M_1T'}\\
&\quad+|\rho_0|^{(0)}e^{M_1T'}\Big\{\int_0^\tau|\operatorname{div} v(\bar{x}(\tau_0;\xi,\tau),\tau_0)-\operatorname{div} v(\bar{x}_n(\tau_0;\xi,\tau),\tau_0)|\,d\tau_0\Big\}\\
&\quad+\int_0^\tau|\operatorname{div} v(\bar{x}_n(\tau_0;\xi,\tau),\tau_0)-\operatorname{div} v_n(\bar{x}_n(\tau_0;\xi,\tau),\tau_0)|d\tau_0\\
&\leqq\Big[|\nabla\rho_0|^{(0)}|(x_0-x_{n,0})(\xi,\tau)|+|\rho_0|^{(0)}\Big\{|\nabla\nabla v|^{(0)}_{T'}\int_0^\tau|(\bar{x}-\bar{x}_n)(\tau_0;\xi,\tau)|d\tau_0\\
&\qquad+\int_0^\tau|\nabla v-\nabla v_n|^{(0)}_{N,T'}d\tau_0\Big\}\Big]e^{M_1T'}.
\end{aligned}\tag{5.50}$$

Now, we note that

$$\begin{aligned}
(\bar{x}-\bar{x}_n)(\tau_0;\xi,\tau)=-\int_0^\tau&\{(v(x(\tau_1;\xi,\tau),\tau_1)-v_n(\bar{x}(\tau_1;\xi,\tau),\tau_1))\\
&+(v_n(\bar{x}(\tau_1;\xi,\tau),\tau_1)-v_n(\bar{x}_n(\tau_1;\xi,\tau)\tau_1))\}d\tau_1,
\end{aligned}\tag{5.51}$$

and, therefore, that

$$\begin{cases}
|(\bar{x}-\bar{x}_n)(\tau_0;\xi,\tau)|\leqq|v-v_n|^{(0)}_{N,T'}(\tau-\tau_0)+M_1\displaystyle\int_{\tau_0}^\tau|(\bar{x}-\bar{x}_n)(\tau_1;\xi,\tau)|d\tau_1,\\[2ex]
|(\bar{x}-\bar{x}_n)(\tau_0;\xi,\tau)|\leqq\dfrac{1}{M_1}|v-v_n|^{(0)}_{N,T'}(e^{M_1(\tau-\tau_0)}-1).
\end{cases}\tag{5.51$'$}$$

Thus,

$$|\rho_{(v)}-\rho_{(v_n)}|^{(0)}_{N,T'}\leqq E_1(M_1,T';\|\rho_0\|^{(1)})\langle\!\langle v-v_n\rangle\!\rangle^{(\alpha\beta)}_{N,T'},\tag{5.52}$$

where E_1 is a constant depending on M_1, T', and $\|\rho_0\|^{(1)}$. Next, we have

$$
\begin{aligned}
&|(\rho_{(v)}-\rho_{(v_n)})(\xi,\tau)-(\rho_{(v)}-\rho_{(v_n)})(\xi',\tau)|\\
&\leqq\{|(\rho_{(v)}-\rho_{(v_n)})(\xi,\tau)|+|(\rho_{(v)}-\rho_{(v_n)})(\xi',\tau)|\}^{1-\beta}\cdot\{|\rho_{(v)}(\xi,\tau)-\rho_{(v)}(\xi',\tau)|\\
&\qquad\qquad+|\rho_{(v_n)}(\xi,\tau)\rho_{(v_n)}(\xi',\tau)|\}^{\beta}\\
&\leqq\{2|\rho_{(v)}-\rho_{(v_n)}|^{(0)}_{N,T'}\}^{1-\beta}\{|\rho_{(v)}|^{(\alpha)}_{x,T'}+|\rho_{(v_n)}|^{(\alpha)}_{x,T'}\}^{\beta}|\xi-\xi'|^{\alpha\beta}.
\end{aligned}
\tag{5.53}
$$

Hence,

$$
|\rho_{(v)}-\rho_{(v_n)}|^{(\alpha\beta)}_{x,N,T'}\leqq(2E_1)^{1-\beta}\{|\rho_{(v)}|^{(\alpha)}_{x,T'}+|\rho_{(v_n)}|^{(\alpha)}_{x,T'}\}^{\beta}\{\langle\!\langle v-v_n\rangle\!\rangle^{(\alpha\beta)}_{N,T'}\}^{1-\beta}. \tag{5.53$'$}
$$

In the same way, we have

$$
|\rho_{(v)}-\rho_{(v_n)}|^{(\alpha\beta)}_{t,N,T'}\leqq(2E_1)^{1-\beta}\{|\rho_{(v)}|^{(\alpha/2)}_{t,T'}+|\rho_{(v_n)}|^{(\alpha/2)}_{t,T'}\}^{\beta}\{\langle\!\langle v-v_n\rangle\!\rangle^{(\alpha\beta)}_{N,T'}\}^{1-\beta}. \tag{5.53$''$}
$$

ii)

$$
\begin{aligned}
&\|\nabla\log\rho_{(v)}-\nabla\log\rho_{(v_n)}\|^{(\alpha\beta)}_{N,T'}\\
&=|\nabla\log\rho_{(v)}-\nabla\log\rho_{(v_n)}|^{(0)}_{N,T'}+|\nabla\log\rho_{(v)}-\nabla\log\rho_{(v_n)}|^{(\alpha\beta)}_{x,N,T'}\\
&\quad+|\nabla\log\rho_{(v)}-\nabla\log\rho_{(v_n)}|^{(\alpha\beta/2)}_{t,N,T'}.
\end{aligned}
$$

$$
\begin{aligned}
&(\nabla\log\rho_{(v)}-\nabla\log\rho_{(v_n)})(\xi,\tau)\\
&=\frac{(\nabla\rho_{(v)}-\nabla\rho_{(v_n)})(\xi,\tau)}{\rho_{(v)}(\xi,\tau)}+\frac{\rho_{(v_n)}(\xi,\tau)(\rho_{(v_n)}-\rho_{(v)})(\xi,\tau)}{\rho_{(v)}(\xi,\tau)\rho_{(v_n)}(\xi,\tau)}.
\end{aligned}
\tag{5.54}
$$

$$
\begin{aligned}
&(\nabla\rho_{(v)}-\nabla\rho_{(v_n)})(\xi,\tau)\\
&=\nabla_\xi\rho_0(x_0(\xi,\tau))\exp\left\{-\int_0^\tau \operatorname{div} v(\bar{x}(\tau_0;\xi,\tau),\tau_0)d\tau_0\right\}\\
&\quad+\rho_0(x_0(\xi,\tau))\exp\left\{-\int_0^\tau \operatorname{div} v(\bar{x}(\tau_0;\xi,\tau),\tau_0)d\tau_0\right\}\cdot\nabla_\xi\int_0^\tau \operatorname{div} v(\bar{x}(\tau_0;\xi,\tau),\tau_0)d\tau_0\\
&\quad-\left\{\nabla_\xi\rho_0(x_{n,0}(\xi,\tau))\exp\left\{-\int_0^\tau \operatorname{div} v_n(\bar{x}_n(\tau_0;\xi,\tau),\tau_0)\,d\tau_0\right\}\right.\\
&\quad\left.+\rho_0(x_{n,0}(\xi,\tau))\exp\left\{-\int_0^\tau \operatorname{div} v_n(\bar{x}_n(\tau_0;\xi,\tau),\tau_0)\,d\tau_0\right\}\cdot\int_0^\tau \operatorname{div} v_n(\bar{x}_n(\tau_0;\xi,\tau),\tau_0)\,d\tau_0\right\}.
\end{aligned}
\tag{5.55}
$$

Therefore,

$$
\begin{aligned}
&|(\nabla\rho_{(v)}-\nabla\rho_{(v_n)})(\xi,\tau)| \\
&\leqq\{|\nabla_{x_0}\rho_0(x_0(\xi,\tau))|\cdot|(\nabla_\xi x_0-\nabla_\xi x_{n,0})(\xi,\tau)| \\
&\quad+|\nabla_{x_0}\rho_0(x_0(\xi,\tau))-\nabla_{x_{n,0}}\rho_0(x_{n,0}(\xi,\tau))|\cdot|\nabla_\xi x_{n,0}(\xi,\tau)|\}e^{M_1T'} \\
&\quad+|\nabla\rho_0|^{(0)}|\nabla_\xi x_0|^{(0)}\left\{|\nabla\nabla v|^{(0)}_{T'}\int_0^\tau|(\bar{x}-\bar{x}_n)(\tau_0;\xi,\tau)|d\tau_0+|\nabla v-\nabla v_n|^{(0)}_{N,T'}\cdot T'\right\} \\
&\quad+|\rho_{(v)}-\rho_{(v_n)}|^{(0)}_{N,T'}|\nabla\nabla v|^{(0)}_{T'}\int_0^\tau|\nabla_\xi\bar{x}(\tau_0;\xi,\tau)|d\tau_0 \\
&\quad+|\rho_{(v_n)}|^{(0)}_{T'}\int_0^\tau\{|\nabla\nabla v|^{(\alpha)}_{x,T'}|(\bar{x}-\bar{x}_n)(\tau_0;\xi,\tau)|^\alpha+|\nabla\nabla v-\nabla\nabla v_n|^{(0)}_{N,T'}\}|\nabla_\xi\bar{x}(\tau_0;\xi,\tau)|d\tau_0 \\
&\quad+|\rho_{(v_n)}|^{(0)}_{T'}\int_0^\tau|\nabla\nabla v_n|^{(0)}_{T'}(\nabla_\xi\bar{x}-\nabla_\xi\bar{x}_n)(\tau_0;\xi,\tau)d\tau_0.
\end{aligned}
\tag{5.55$'$}
$$

There is a necessity to estimate $|(\nabla_\xi\bar{x}-\nabla_\xi\bar{x}_n)(\tau_0;\xi,\tau)|$. The matrices

$$
\bar{X}(\tau_0;\xi,\tau)=\left(\frac{\bar{x}_i(\tau_0;\xi,\tau)}{\partial\xi_j}\right),\qquad \bar{X}_n(\tau_0;\xi,\tau)\equiv\left(\frac{\bar{x}_{n,i}(\tau_0;\xi,\tau)}{\partial\xi_j}\right),
\tag{5.56}
$$

satisfy, respectively,

$$
\begin{cases}
\dfrac{d}{d\tau_0}\bar{X}(\tau_0;\xi,\tau)=\left(\dfrac{\partial v_i}{\partial\bar{x}_j}(\bar{x}(\tau_0;\xi,\tau),\tau_0)\bar{X}(\tau_0;\xi,\tau)\right) \\
\qquad\qquad\equiv\hat{\nabla}(\tau_0;\xi,\tau)\bar{X}(\tau_0;\xi,\tau),\ \bar{X}(\tau;\xi,\tau)=I, \\
\dfrac{d}{d\tau_0}\bar{X}_n(\tau_0;\xi,\tau)=\left(\dfrac{\partial v_{n,i}}{\partial\bar{x}_j}(\bar{x}_n(\tau_0;\xi,\tau),\tau_0)\bar{X}_n(\tau_0;\xi,\tau)\right) \\
\qquad\qquad\equiv\hat{\nabla}_n(\tau_0;\xi,\tau)\bar{X}_n(\tau_0;\xi,\tau),\ \bar{X}_n(\tau;\xi,\tau)=I.
\end{cases}
\tag{5.56$'$}
$$

From this follows that, for $\tau_0\in[0,\tau]$,

$$
\begin{aligned}
&|(\bar{X}-\bar{X}_n)(\tau_0;\xi,\tau)| \\
&\leqq\int_{\tau_0}^\tau|(\bar{X}-\bar{X}_n)(\tau_1;\xi,\tau)|\,|\hat{\nabla}_n(\tau_1;\xi,\tau)|d\tau_1+\int_{\tau_0}^\tau|(\hat{\nabla}-\hat{\nabla}_n)(\tau_1;\xi,\tau)|\,|\bar{X}(\tau_1)|d\tau_1.
\end{aligned}
\tag{5.57}
$$

Thus,

$$
\begin{aligned}
&|(\bar{X}-X_n)(\tau_0;\xi,\tau)| \\
&\leqq\{|v-v_n|^{(0)}_{N,T'}(e^{M_1T'}-1)+M_1|\nabla v-\nabla v_n|^{(0)}_{N,T'}\}(e^{M_1T'}-1)9B_1^{(1)}(T';M_1).
\end{aligned}
\tag{5.58}
$$

From the above, we have

$$(5.59)\qquad |\nabla\rho_{(v)}-\nabla\rho_{(v_n)}|^{(0)}_{N,T'}\leqq E_{2,1}(M_1, T')\langle\!\langle v-v_n\rangle\!\rangle^{(\alpha\beta)}_{N,T'}+E_{2,2}(M_1, T')\{\langle\!\langle v-v_n\rangle\!\rangle^{(\alpha\beta)}_{N,T'}\}^{\alpha},$$

where $E_{2,1}$ and $E_{2,2}$ are non-negative constants depending on M_1 and T'. Therefore, from (5.54) we obtain

$$(5.60)\qquad \begin{aligned}&|\nabla\log\rho_{(v)}-\nabla\log\rho_{(v_n)}|^{(0)}_{N,T'}\\ &\leqq E_{3,1}(M_1, T')\langle\!\langle v-v_n\rangle\!\rangle^{(\alpha\beta)}_{N,T'}+E_{3,2}(M_1, T')\{\langle\!\langle v-v_n\rangle\!\rangle^{(\alpha\beta)}_{N,T'}\}^{\alpha},\end{aligned}$$

where $E_{3,1}$ and $E_{3,2}$ are non-negative constants depending on M_1 and T'. Next,

$$(5.61)\qquad \begin{aligned}&|\{\nabla\log\rho_{(v)}(\xi,\tau)-\nabla\log\rho_{(v_n)}(\xi,\tau)\}-\{\nabla\log\rho_{(v)}(\xi',\tau)-\nabla\log\rho_{(v_n)}(\xi',\tau)\}|\\ &\leqq\{|\nabla\log\rho_{(v)}(\xi,\tau)-\nabla\log\rho_{(v)}(\xi',\tau)|+|\nabla\log\rho_{(v_n)}(\xi,\tau)-\nabla\log\rho_{(v_n)}(\xi',\tau)|\}^{\beta}\\ &\quad\times\{|\nabla\log\rho_{(v)}(\xi,\tau)-\nabla\log\rho_{(v_n)}(\xi,\tau)|+|\nabla\log\rho_{(v)}(\xi',\tau)-\nabla\log\rho_{(v_n)}(\xi',\tau)|\}^{1-\beta}\\ &\leqq|\xi-\xi'|^{\alpha\beta}\{|\nabla\log\rho_{(v)}|^{(\alpha)}_{x,T'}+|\nabla\log\rho_{(v_n)}|^{(\alpha)}_{x,T'}\}^{\beta}\cdot\{2|\nabla\log\rho_{(v)}-\nabla\log\rho_{(v_n)}|^{(0)}_{N,T'}\}^{1-\beta}.\end{aligned}$$

Hence, we have

$$(5.61)'\qquad \begin{aligned}&|\nabla\log\rho_{(v)}-\nabla\log\rho_{(v_n)}|^{(\alpha\beta)}_{x,N,T'}\\ &\leqq 2^{1-\beta}\{|\nabla\log\rho_{(v)}|^{(\alpha)}_{x,T'}+|\nabla\log\rho_{(v_n)}|^{(\alpha)}_{x,T'}\}^{\beta}\times\{E_{3,1}\langle\!\langle v-v_n\rangle\!\rangle^{(\alpha\beta)}_{N,T'}+E_{3,2}(\langle\!\langle v-v_n\rangle\!\rangle^{(\alpha\beta)}_{N,T'})^{\alpha}\}^{1-\beta}.\end{aligned}$$

In an analogous way, we have

$$(5.61)''\qquad \begin{aligned}&|\nabla\log\rho_{(v)}-\nabla\log\rho_{(v_n)}|^{(\alpha\beta)}_{t,N,T'}\\ &\leqq 2^{1-\beta}\{|\nabla\log\rho_{(v)}|^{(\alpha/2)}_{t,T'}+|\nabla\log\rho_{(v_n)}|^{(\alpha/2)}_{t,T'}\}^{\beta}\times\{E_{3,1}\langle\!\langle v-v_n\rangle\!\rangle^{(\alpha\beta)}_{N,T'}+E_{3,2}(\langle\!\langle v-v_n\rangle\!\rangle^{(\alpha\beta)}_{N,T'})^{\alpha}\}^{1-\beta}.\end{aligned}$$

iii) From i) and ii) it follows that

$$(5.62)\qquad \begin{cases}||\tilde{N}_1||^{(\alpha\beta)}_{N,T'}\leqq E_{4,1}[(v,\theta)-(v_n,\theta_n)]^{(\alpha\beta)}_{N,T'}+E_{4,2}\{[(v,\theta)-(v_n,\theta_n)]^{(\alpha\beta)}_{N,T'}\}^{1-\beta},\\ ||\tilde{N}_2||^{(\alpha\beta)}_{N,T'}\leqq E_{5,1}[(v,\theta)-(v_n,\theta_n)]^{(\alpha\beta)}_{N,T'}+E_{5,2}\{[(v,\theta)-(v_n,\theta_n)]^{(\alpha\beta)}_{N,T'}\}^{1-\beta},\end{cases}$$

where $E_{4,1}$ and $E_{4,2}$ depend on M_1, M_2, M_3, and T', and $E_{5,1}$ and $E_{5,2}$ depend on M_1, M_2, and T'.

iv) Finally we estimate $(\bar{w}_n, \bar{\zeta}_n)$.

If $(x,t)\in R^3_{N',T'}$, then, for an arbitrary $\varepsilon>0$, there exists a number $n_0(N',\varepsilon,N(\varepsilon))$ such that, for $n\geqq n_0$,

$$|\bar{w}_n(x,t)| \leqq \int_0^t d\tau \int_{R^3} |\Gamma_{(v)}(x,t;\xi,\tau)\tilde{N}_1(\xi,\tau;v,\theta;v_n,\theta_n)|d\xi$$

$$\leqq \int_0^t d\tau \int_{R_N^3} |\Gamma_{(v)}(x,t;\xi,\tau)\tilde{N}_1(\xi,\tau)|d\xi + \int_0^t d\tau \int_{R^3-R_N^3} |\Gamma_{(v)}\cdot\tilde{N}_1|d\xi$$

$$(R_N^3 \equiv \{x: x \in R^3, |x| \leqq N\})$$

(5.63)

$$\leqq \{E_{4,1}[(v,\theta)-(v_n,\theta_n)]_{N,T'}^{(\alpha\beta)} + (E_{4,2}[(v,\theta)-(v_n,\theta_n)]_{N,T'}^{(\alpha\beta)})^{1-\beta}\}$$

$$\times C_{12}^{(0,0)}(M_1,T')\left(\frac{\pi}{h_1(v)}\right)^{3/2}$$

$$+\int_0^t d\tau \int_{R^3-R_N^3} |\Gamma_{(v)}\tilde{N}_1|d\xi < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence,

(5.63)′ $\qquad |\bar{w}_n|_{N',T'}^{(0)} < \varepsilon, \qquad$ (for $n \geqq n_0(N',\varepsilon,N(\varepsilon))$; $N(\varepsilon) \geqq N' + M_1T'$).

For $|m| = 1, 2$,

$$|D_x^m \bar{w}_n(x,t)|$$

$$\leqq \int_0^t d\tau \int_{R_N^3} |D_x^m \Gamma_{(v)}(x,t;\xi,\tau)|\cdot|\tilde{N}_1(\xi,\tau) - \tilde{N}_1(x,t)|d\xi$$

$$+\int_0^t d\tau \int_{R^3-R_N^3} |D_x^m \Gamma_{(v)}\cdot\{\tilde{N}_1(\xi,\tau)-\tilde{N}_1(x,t)\}|d\xi$$

(5.63)″

$$+\int_0^t d\tau \int_{R_N^3} D_x^m Z_{(v)}(x-\xi,t;\xi,\tau) + D_x^m \Gamma_{(v),0}(x,t;\xi,\tau)\}\tilde{N}_1(x,t)d\xi$$

$$+\int_{R_N^3} D^m Z_{(v)}(x-\xi,t;x,\tau)\tilde{N}_1(x,t)d\xi + \int_{R^3-R_N^3} D^m Z_{(v)}(x-\xi,t;x,\tau)\tilde{N}_1(x,t)d\xi$$

$$+\int_{R^3-R_N^3} D_x^m Z_{(v)}(x-\xi,t;\xi,\tau) + D_x^m \Gamma_{(v),0}(x,t;\xi,\tau)\tilde{N}_1(x,t)d\xi|,$$

$$\left(\int_{R^3} D_\xi^m Z_{(v)}(x-\xi,t;x,\tau)\tilde{N}_1(x,t)d\xi = 0\right).$$

From this, we have, for an arbitrary number $\varepsilon > 0$,

$$|D_x^m \bar{w}_n(x,t)|$$

$$\leqq |\tilde{N}_1(x,t;v,\theta;v_n,\theta_n)|_{N,T'}^{(\alpha\beta)} C t^{(2-|m|+\alpha)/2} + |\tilde{N}_1|_{T'}^{(\alpha\beta)} C t^{(2-|m|+\alpha)/2} \int_{h_1(N-N')/\sqrt{T'}}^{+\infty} 4\pi r^2 e^{-r^2} dr$$

(5.64) $$+|\tilde{N}_1(n)|_{N,T'}^{(0)} C' t^{(2-|m|+\alpha)/2} + |\tilde{N}_1|_{T'}^{(0)} C' t^{(2-|m|+\alpha)/2} \int_{h_1(N-N')/\sqrt{T'}}^{+\infty} 4\pi r^2 e^{-r^2} dr$$

$$\leqq t^{(2-|m|+\alpha)/2}(C+C')\|\tilde{N}_1\|_{N,T'}^{(\alpha\beta)} + \|\tilde{N}_1\|_{T'}^{(\alpha\beta)} \cdot \hat{\varepsilon}(N(\varepsilon), N') < \varepsilon,$$

$(n \geqq n_0(N', \varepsilon, N(\varepsilon)))$; $\hat{\varepsilon} \searrow 0$ as $N \nearrow +\infty$; C and C' depend on M_1, T', and $\alpha\beta$).

Thus,

(5.64)' $$|D_x^m \bar{w}_n|_{N',T'}^{(0)} < \varepsilon, \qquad (n \geqq n_0(N', \varepsilon, N(\varepsilon))).$$

By (5.64), for $x \in R_{N'}^3$ and $t-t' \geqq t' > 0$ and for an arbitrary number $\varepsilon > 0$, there exists a number $n_0(N', \varepsilon, N(\varepsilon))$ such that, for $n \geqq n_0$,

(5.65) $$\begin{aligned} &|D_x^m \bar{w}_n(x,t) - D_x^m \bar{w}_n(x,t')| \\ &\leqq (C+C')(2^{(2-|m|+\alpha\beta)/2}+1)(t-t')^{(2-|m|+\alpha\beta)/2}\{\|\tilde{N}_1(n)\|_{N,T'}^{(\alpha\beta)} + \|\tilde{N}_1(n)\|_{T'}^{(\alpha\beta)} \hat{\varepsilon}(N(\varepsilon), N')\}. \end{aligned}$$

For $t-t' \leqq t'$, we estimate along the line of (4.8). After tedious calculations, we have

(5.65)' $$\begin{aligned} &|D_x^m \bar{w}_n(x,t) - D_x^m \bar{w}_n(x,t')| \\ &\leqq C_{17}^{(|m|)}(\alpha\beta(t-t')^{(2-|m|+\alpha\beta)/2}\{\|\tilde{N}_1(n)\|_{N,T'}^{(\alpha\beta)} + \|\tilde{N}_1(n)\|_{T'}^{(\alpha\beta)} \hat{\varepsilon}(N(\varepsilon), N')\}. \end{aligned}$$

Therefore, by (5.65) and (5.65)', for an arbitrary number $\varepsilon > 0$

(5.65)'' $$|D_x^m \bar{w}_n|_{t,N',T'}^{(\alpha\beta/2)} < \varepsilon, \qquad (n \geqq n_1(N', \varepsilon, N(\varepsilon))).$$

For $|m|=2$ and $x, x' \in R_{N'+M_1(T'-t)}^3$ $(0 \leqq t \leqq T')$,

(5.66) $$|D_x^m \bar{w}_n(x,t) - D_x^m \bar{w}_n(x',t)|$$

is estimated along the line of (4.18). We have only to note that, for an arbitrary function $\varphi(x,t;\xi,\tau)$,

(5.66)' $$\begin{aligned} &\int_0^t d\tau \int_{R^3-\Sigma_1} \varphi(\cdot;\xi,\tau) d\xi \\ &= \int_0^t d\tau \int_{R^3-\Sigma_1^k} \varphi(\cdot;\xi,\tau) d\xi + \int_0^t d\tau \int_{\Sigma_1^k-\Sigma_1} \varphi(\cdot;\xi,\tau) d\xi, \end{aligned}$$

where

(5.66)'' $$\Sigma_1^k \equiv \{\xi : |x-\xi| \leqq 2k|x-x'|,\ k \text{ is an integer } \geqq 1\}.$$

Obviously,

$$\Sigma_1 \subset \Sigma_1^k \subset R^3_{(2k+1)(N'+M_1T')}.$$

For example, the integral (cf. I_3 of (4. 18)),

$$\int_{R^3-\Sigma_1^k} d\xi \int_0^t |x-x'|\,|x'-\xi|^{\alpha\beta}(t-\tau)^{-3} \exp\left\{-h_1 \frac{|x'-\xi|^2}{t-\tau}\right\} d\tau \equiv \hat{I}_k \tag{5. 67}$$

is estimated as follows:

(5. 67)′
$$\begin{aligned}
\hat{I}_k &\leqq |x-x'|2h_1^{-2}I(3)\int_{R^3-\Sigma_1^k} |x'-\xi|^{-4+\alpha\beta}d\xi \\
&\leqq 2^{5-\alpha\beta}|x-x'|\cdot h_1^{-2}I(3)\int_{R^3-\Sigma_1^k} |x-\xi|^{-4+\alpha\beta}d\xi \\
&\qquad \left(\frac{1}{2} \leqq \frac{x'-\xi}{x-\xi} \leqq \frac{3}{2}, \quad \text{for} \quad \xi\in(\Sigma_1)^c\right) \\
&\leqq 2^{5-\alpha\beta}|x-x'|h_1^{-2}I(3)\int_{2k|x-x'|}^{+\infty} 4\pi r^{-2+\alpha\beta}dr = 2^6\pi I(3)(1-\alpha\beta)^{-1}k^{-1+\alpha\beta}|x-x'|^{\alpha\beta}.
\end{aligned}$$

Therefore,

$$I_k/|x-x'|^{\alpha\beta} \to 0 \qquad (k\to+\infty). \tag{5. 67}''$$

In this way follows the result that, for $|m|=2$ and an arbitrary number $\varepsilon>0$, there exists a number $n_0(N', \varepsilon, k(\varepsilon))$ such

$$|D_x^m \bar{w}_n|_{x,N',T'}^{(\alpha\beta)} \leqq C_{18}{}^{(\alpha\beta)}\|N_1(n)\|_{N(k),T'}^{(\alpha\beta)} + \|\tilde{N}_1(n)\|_{T'}^{(\alpha\beta)}\hat{\varepsilon}_1(k(\varepsilon), N') < \varepsilon, \tag{5. 68}$$

$$(\hat{\varepsilon}_1 \searrow 0 \text{ as } \varepsilon \searrow 0).$$

$|D_x^m \bar{\zeta}_n|_{N,T'}^{(0)}$, etc., are estimated in an analogous way. From iii) and iv) follows the assertion of this lemma. Q.E.D.

5. 5. **The existence of a bounded solution of** (1. 9)-(1. 10). Now, to complete the proof of Theorem 1, it remains just to apply the following

THEOREM (Tikhonov [39]). *A continuous operator from a compact convex subset in a locally convex linear topological space into itself has at least one fixed point.*

Proof of Theorem 1. By Tikhonov's theorem, $G_{T'}$ has a fixed point such that

$$G_{T'}(v, \theta) = (\hat{v}, \hat{\theta}) = (v, \theta), \qquad ((v, \theta)\in S_{T'}). \tag{5. 69}$$

Moreover,

$$\rho_{(v)} \in \tilde{B}_{T'}^{1+\alpha} \equiv \{\rho\colon |\rho|_{T'}^{(0)} + |\nabla\rho|_{T'}^{(\alpha)} + |D_t\rho|_{T'}^{(\alpha)}\} < +\infty. \tag{5. 70}$$

We can consider (1. 9)^3 as a linear parabolic equation in θ. The fact that

$(v, \theta, \rho)\in H_{T'}^{2+\alpha}\times H_{T'}^{2+\alpha}\times B_{T'}^{1+\alpha}$, and that $\theta(x, 0)=\theta_0(x)\geqq 0$ and $\Psi\geqq 0$, guarantees the non-negativity of $\theta(x, t)$, (cf. [9], CH. II, § 4). Thus, we have Theorem 1. Q.E.D.

§ 6. The problem of uniqueness.

Proof of Theorem 2. Let (v_1, θ_1, ρ_1) and (v_2, θ_2, ρ_2) satisfy (1. 9)-(1. 10) and the assumption of this theorem. Then, the differences,

$$w=v_1-v_2, \qquad \theta=\theta_1-\theta_2, \tag{6. 1}$$

satisfy

$$\begin{cases} \dfrac{\partial w}{\partial t}-\dfrac{\mu}{\rho_{(v_1)}}P_0(D_x)w=\tilde{N}_1(x, t; v_1, \theta_1; v_2, \theta_2), & w(x, 0)=0; \\[2ex] \dfrac{\partial \theta}{\partial t}-\dfrac{\kappa}{C_v\rho_{(v_1)}}\Delta\theta=\tilde{N}_2(x, t; v_1, \theta_1; v_2, \theta_2), & \theta(x, 0)=0. \end{cases} \tag{6. 2}$$

It is obvious that

$$(w, \theta)\in H_T^{2+\alpha}\times H_T^{2+\alpha}, \qquad \tilde{N}_1 \text{ and } \tilde{N}_2\in H_T^{\alpha}. \tag{6. 3}$$

Therefore, w and θ can be expressed by

$$\begin{cases} w(x, t)=\displaystyle\int_0^t d\tau\int_{R^3}\Gamma_{(v_1)}(x, t; \xi, \tau)\tilde{N}_1(\xi, \tau)d\xi, \\[2ex] \theta(x, t)=\displaystyle\int_0^t d\tau\int_{R^3}{}'\Gamma_{(v_1)}(x, t; \xi, \tau)\tilde{N}_2(\xi, \tau)d\xi. \end{cases} \tag{6. 4}$$

Hence, if $T_0\in[0, T]$, then

$$\begin{cases} \langle w\rangle_{T_0}\leqq C(\langle v_1\rangle_T, T_0)\|\tilde{N}_1\|_{T_0}^{(\alpha)}, & (C\searrow 0 \text{ as } T_0\searrow 0, \text{ cf. } (5. 36)), \\ (\theta)_{T_0}\leqq{}'C(\langle v_1\rangle_T, T_0)\|\tilde{N}_2\|_{T_0}^{(\alpha)}, & ('C\searrow 0 \text{ as } T_0\searrow 0), \\ \langle w\rangle'_{T_0}\leqq C(\langle v_1\rangle_T, T_0)\|\tilde{N}_1\|_{T_0}^{(\alpha)}, & (C\searrow C_0(>0) \text{ as } T_0\searrow 0). \end{cases} \tag{6. 5}$$

i) We put

$$M\equiv\max\,[\langle\!\langle v_1\rangle\!\rangle_T, \langle\!\langle v_2\rangle\!\rangle_T]. \tag{6. 6}$$

In such a way as we obtained (5. 52), we have

$$\begin{aligned} &|\rho_{(v_1)}-\rho_{(v_2)}|_{T_0}^{(0)} \\ &\leqq E_1(M, T_0)\langle v_1-v_2\rangle_{T_0}, \qquad (E_1 \text{ decreases montonically as } T_0\searrow 0). \end{aligned} \tag{6. 7}$$

(6. 8)
$$\begin{aligned}
&|(\rho_{(v_1)}-\rho_{(v_2)})(x,t)-(\rho_{(v_1)}-\rho_{(v_2)})(x',t)| \\
&=\left|\int_0^1 \frac{d}{d\lambda}\{\rho_{(v_1)}(x_\lambda,t)-\rho_{(v_2)}(x_\lambda,t)\}d\lambda\right| \\
&\leqq\left|\int_0^1\left\{\frac{\partial\rho_{(v_1)}}{\partial x_{\lambda,i}}(x_\lambda,t)-\frac{\partial\rho_{(v_2)}}{\partial x_{\lambda,i}}(x_\lambda,t)\right\}d\tau\right|\cdot|x_i-x_i'| \\
&\leqq|\nabla\rho_{(v_1)}-\nabla\rho_{(v_2)}|_{T_0}^{(0)}\cdot|x-x'|, \\
&\qquad(x_\lambda=\lambda x+(1-\lambda)x';\ t\in[0,T_0]).
\end{aligned}$$

Therefore, noting that, for a and $b\geqq 0$ and for $\gamma\in[0,1]$,

(6. 8)′
$$a^\gamma b^{1-\gamma}\leqq\max[a,b]\leqq a+b,$$

we have

(6. 8)″
$$\begin{aligned}
&|(\rho_{(v_1)}-\rho_{(v_2)})(x,t)-(\rho_{(v_1)}-\rho_{(v_2)})(x',t)| \\
&\leqq 2^{1-\alpha}|x-x'|^\alpha\{|\nabla\rho_{(v_1)}-\nabla\rho_{(v_2)}|_{T_0}^{(0)}+|\rho_{(v_1)}-\rho_{(v_2)}|_{T_0}^{(0)}\}.
\end{aligned}$$

We can estimate $|\nabla\rho_{(v_1)}-\nabla\rho_{(v_2)}|_{T_0}^{(0)}$ almost in the same way as in (5. 55)′, if we replace $v, v_n, \bar{x}, \bar{x}_n, T'$, and $|\nabla\nabla v|_{x,T'}^{(\alpha)}|(\bar{x}-\bar{x}_n)(\tau_0;\xi,\tau)|$, in (5. 55)′ by $v_1, v_2, \bar{x}_1, \bar{x}_2, T_0$, and $|\nabla\nabla\nabla v_1|_{T_0}^{(0)}|(\bar{x}_1-\bar{x}_2)(\tau_0;\xi,\tau)|$, respectively. As a result, we have

(6. 9)
$$|\nabla\rho_{(v_1)}-\nabla\rho_{(v_2)}|_{T_0}^{(0)}\leqq F_1(M,\bar{M},T_0)\langle v_1-v_2\rangle_{T_0},$$
$$(F_1\searrow 0 \text{ as } T_0\searrow 0,\ \bar{M}=\max[|\nabla\nabla\nabla v_1|_{T_0}^{(0)}, |\nabla\nabla\nabla v_2|_{T_0}^{(0)}]).$$

Thus,

(6. 9)′
$$|\rho_{(v_1)}-\rho_{(v_2)}|_{x,T_0}^{(\alpha)}\leqq 2^{1-\alpha}\{E_1(T_0)+F_1(T_0)|\langle v_1-v_2\rangle_{T_0}.$$

Next, for t and $t'\in[0,T_0]$,

(6. 10)
$$\begin{aligned}
&|(\rho_{(v_1)}-\rho_{(v_2)}(x,t)-(\rho_{(v_1)}-\rho_{(v_2)})(x,t')| \\
&\leqq\left|\int_0^1\left\{\frac{\partial}{\partial t_\lambda}\rho_{(v_1)}(x,t_\lambda)-\frac{\partial}{\partial t_\lambda}\rho_{(v_2)}(x,t_\lambda)\right\}d\lambda\right|\cdot|t-t'| \\
&\leqq\int_0^1|\{\mathrm{div}\,\rho_{(v_1)}(x,t_\lambda)v_1(x,t_\lambda)-\mathrm{div}\,\rho_{(v_2)}(x,t_\lambda)v_2(x,t_\lambda)\}|d\lambda\cdot|t-t'| \\
&\leqq T_0^{1-(\alpha/2)}|t-t'|^{\alpha/2}\{M(E_1(T_0)+F_1(T_0))+\|\rho_{(v_2)}\|_{T_0}^{(1)}\}\langle v_1-v_2\rangle_{T_0}, \\
&\qquad(t_\lambda\equiv\lambda t+(1-\lambda)t').
\end{aligned}$$

ii) We put

$$\rho_1=\rho_{(v_1)},\qquad \rho_2=\rho_{(v_2)}.$$

$$\left(\frac{\nabla_x\rho_1}{\rho_1}-\frac{\nabla_x\rho_2}{\rho_2}\right)(x,t)-\left(\frac{\nabla_x\rho_1}{\rho_1}-\frac{\nabla_x\rho_2}{\rho_2}\right)(x',t)$$

$$=\left[\left\{\frac{\nabla_x\rho_0(x_{1,0}(x,t))}{\rho_0(x_{1,0}(x,t))}-\nabla_x\int_0^t \operatorname{div} v_1(\bar{x}_1(\tau;x,t),\tau)\,d\tau\right\}\right.$$

$$\left.-\left\{\frac{\nabla_x\rho_0(x_{1,0}(x,t))}{\rho_0(x_{2,0}(x,t))}-\nabla_x\int_0^t \operatorname{div} v_2(\bar{x}_2(\tau;x,t),\tau)\,d\tau\right\}\right]$$

$$-\left[\left\{\frac{\nabla_{x'}\rho_0(x_{1,0}(x',t))}{\rho_0(x_{1,0}(x',t))}-\nabla_{x'}\int_0^t \operatorname{div} v_1(\bar{x}_1(\tau;x',t),\tau)\,d\tau\right\}\right.$$

$$\left.-\left\{\frac{\nabla_{x'}\rho_0(x_{2,0}(x',t))}{\rho_0(x_{2,0}(x',t))}-\nabla_{x'}\int_0^t \operatorname{div} v_2(\bar{x}_2(\tau;x',t),\tau)\,d\tau\right\}\right] \tag{6.11}$$

$$=\left[\left\{\frac{\nabla_x\rho_0(x_{1,0}(x,t))-\nabla_x\rho_0(x_{2,0}(x,t))}{\rho_0(x_{1,0}(x,t))}\right.\right.$$

$$\left.\left.+\nabla_x\rho_0(x_{2,0}(x,t))\left(\frac{1}{\rho_0(x_{1,0}(x,t))}-\frac{1}{\rho_0(x_{2,0}(x,t))}\right)\right\}_I-\{x\to x'\}_I\right]_A$$

$$-\left[\int_0^t\{\nabla_{\bar{x}_1}\operatorname{div} v_1(\bar{x}_1(\tau;x,t),\tau)\cdot\nabla_x\bar{x}_1(\tau;x,t)\right.$$

$$\left.-\nabla_{\bar{x}_2}\operatorname{div} v_2(\bar{x}_2(\tau;x,t),\tau)\nabla_x\bar{x}_2(\tau;x,t)\}_{II}d\tau-\int_0^t\{x\to x'\}_{II}\,d\tau\right]_B.$$

$$[\quad]_A=[\{\nabla_x\rho_0(x_{1,0}(x,t))-\nabla_x\rho_0(x_{2,0}(x,t))\}$$

$$-\{\nabla_{x'}\rho_0(x_{1,0}(x',t))-\nabla_{x'}\rho_0(x_{2,0}(x',t))\}]\frac{1}{\rho_0(x_{1,0}(x,t))}$$

$$-\{\nabla_{x'}\rho_0(x_{1,0}(x',t))-\nabla_{x'}\rho_0(x_{2,0}(x',t))\}\left\{\frac{1}{\rho_0(x_{1,0}(x,t))}-\frac{1}{\rho_0(x_{1,0}(x',t))}\right\}$$

$$+\{\nabla_x\rho_0(x_{2,0}(x,t))-\nabla_{x'}\rho_0(x_{2,0}(x',t))\} \tag{6.12}$$

$$\times\{\rho_0(x_{2,0}(x,t))-\rho_0(x_{1,0}(x,t))\}\frac{1}{\rho_0(x_{1,0}(x,t))\rho_0(x_{2,0}(x,t))}$$

$$+\nabla_{x'}\rho_0(x_{2,0}(x',t))\left\{\frac{\rho_0(x_{2,0}(x,t))-\rho_0(x_{1,0}(x,t))}{\rho_0(x_{1,0}(x,t))\rho_0(x_{2,0}(x,t))}-\frac{\rho_0(x_{2,0}(x',t))-\rho_0(x_{1,0}(x',t))}{\rho_0(x_{1,0}(x',t))\rho_0(x_{2,0}(x',t))}\right\}$$

$$\equiv S_1+S_2+S_3+S_4.$$

For example, the first term is estimated in the following way:

$$|S_1| \leqq (\bar{\rho}_0)^{-1} |\{\nabla_x \rho_0(x_{1,0}(x,t)) - \nabla_x \rho_0(x_{2,0}(x,t))\}_C - \{x \to x'\}_C|^{(1-\alpha)+\alpha}$$

$$\leqq (\bar{\rho}_0)^{-1} 2^{1-\alpha} \{|\nabla x_{1,0}|_T^{(0)} |\nabla\nabla\rho_0|^{(0)} |x_{1,0} - x_{2,0}|_0^{(0)} + |\nabla\rho_0|^{(0)} |\nabla x_{1,0} - \nabla x_{2,0}|_{T_0}^{(0)}\}^{1-\alpha}$$

$$(6.12)' \qquad \times \{|\nabla x_{1,0}|_T^{(0)} |\nabla\rho_0|^{(0)} |\nabla\nabla(x_{1,0} - x_{2,0})|_{T_0}^{(0)} + |\nabla\nabla x_{2,0}|_T^{(0)} |\nabla\rho_0|^{(0)} |\nabla(x_{1,0} - x_{2,0})|_{T_0}^{(0)}$$

$$+ |\nabla\nabla x_{2,0}|_T^{(0)} |\nabla x_{2,0}|_T^{(0)} |\nabla\nabla\rho_0|^{(0)} |x_{1,0} - x_{2,0}|_{T_0}^{(0)} + (|\nabla x_{1,0}|_T^{(0)})^2 |\nabla\nabla\rho_0|^{(L)} |x_{1,0} - x_{2,0}|_{T_0}^{(0)}$$

$$+ |\nabla\nabla\rho_0|^{(0)} (|\nabla x_{1,0}|_T^{(0)} + |\nabla x_{2,0}|_T^{(0)}) |\nabla(x_{1,0} - x_{2,0})|_{T_0}^{(0)}\}^\alpha |x - x'|^\alpha.$$

In this way, by making use of (6.8)$'$, we have

$$(6.12)'' \qquad \begin{aligned} |[\quad]_A| &\leqq F_2(M, T; (\bar{\rho}_0)^{-1}, \|\rho_0\|^{(2+L)}) \{|x_{1,0} - x_{2,0}|_{T_0}^{(0)} + |\nabla(x_{1,0} - x_{2,0})|_{T_0}^{(0)}\} \\ &\quad + |\nabla\nabla(x_{1,0} - x_{2,0})|_{T_0}^{(0)} |x - x'|^\alpha, \end{aligned}$$

where F_2 is a non-negative constant depending on M, T, $(\bar{\rho}_0)^{-1}$, and $\|\rho_0\|^{(2+L)}$. Next,

$$(6.13) \qquad \begin{aligned} [\quad]_B &= \int_0^t [\{\nabla \operatorname{div} v_1(\bar{x}_1(\tau; x, t), \tau) - \nabla \operatorname{div} v_1(\bar{x}_1(\tau; x', t), \tau)\}_D - \{v_1 \to v_2\}_D] \nabla_x \bar{x}_1(\tau; x, t) d\tau \\ &+ \int_0^t \{\nabla \operatorname{div} v_2(\bar{x}_2(\tau; x, t), \tau) - \nabla \operatorname{div} v_2(\bar{x}_2(\tau; x', t), \tau)\} \times \{\nabla_x \bar{x}_1(\tau; x, t) - \nabla_x \bar{x}_2(\tau; x, t)\} d\tau \\ &- \int_0^t \{\nabla \operatorname{div} v_1(\bar{x}_1(\tau; x'\, t), \tau) - \nabla \operatorname{div} v_2(\bar{x}_2(\tau; x', t), \tau)\} \times \{\nabla_x \bar{x}_1(\tau; x, t) - \nabla_{x'} \bar{x}_2(\tau; x', t)\} d\tau \\ &- \int_0^t \nabla \operatorname{div} v_2(\bar{x}_2(\tau; x', t), \tau) [\{\nabla_x \bar{x}_1(\tau; x, t) - \nabla_{x'} \bar{x}_1(\tau; x', t)\}_E - \{\bar{x}_1 \to \bar{x}_2\}_E] d\tau. \end{aligned}$$

Hence,

$$(6.14) \qquad \begin{aligned} |[\quad]_B| &\leqq \int_0^t |\{\nabla \operatorname{div} v_1(\bar{x}_1(\tau; x, t), \tau) - \nabla \operatorname{div} v_2(\bar{x}_1(\tau; x, t), \tau)\}_F - \{x \to x'\}_F| \cdot |\nabla_x \bar{x}_1(\tau; x, t)| d\tau \\ &+ \int_0^t |\{\nabla \operatorname{div} v_2(\bar{x}_1(\tau; x, t), \tau) - \nabla \operatorname{div} v_2(\bar{x}_2(\tau; x, t), \tau)\}_G - \{x \to x'\}_G|^{1-\alpha} \\ &\times |\{\nabla \operatorname{div} v_2(\bar{x}_1(\tau; x, t), \tau) - \nabla \operatorname{div} v_2(\bar{x}_2(\tau; x, t), \tau)\}_G - \{x \to x'\}_G|^\alpha \cdot |\nabla_x \bar{x}_1(\tau; x, t)| d\tau \\ &+ \int_0^t |\nabla \operatorname{div} v_2|_{x,T}^{(\alpha)} (|\nabla_x \bar{x}_2|_{T_0}^{(0)} |x - x'|)^\alpha \cdot |\nabla_x \bar{x}_1 - \nabla_x \bar{x}_2|_{T_0}^{(0)} d\tau \\ &+ \int_0^t \{|\nabla\nabla\nabla v_1|_T^{(0)} |\bar{x}_1 - \bar{x}_2|_{T_0}^{(0)} + |\nabla\nabla(v_1 - v_2)|_{T_0}^{(0)}\} \times (2|\nabla_x \bar{x}_1|_{T_0}^{(0)})^{1-\alpha} (|\nabla\nabla \bar{x}_1|_{T_0}^{(0)})^\alpha |x - x'|^\alpha d\tau \\ &+ \int_0^t |\nabla\nabla v_2|_T^{(0)} \left\{\int_0^1 |\nabla\nabla \bar{x}_1(\tau; x_\lambda, t) - \nabla\nabla \bar{x}_2(\tau; x_\lambda, t)| \cdot |x - x'|^\alpha d\lambda\right\}^\alpha \{2|\nabla_x(\bar{x}_1 - \bar{x}_2)|_{T_0}^{(0)}\}^{1-\alpha} d\tau \end{aligned}$$

$$(|\bar{x}_1|^{(0)}_{T_0} \equiv \sup_{\substack{0\le\tau\le t\le T_0\\ x\in R^3}} |\bar{x}_1(\tau; x, t)|, \text{ etc.})$$

$$\leqq [|\nabla\nabla(v_1-v_2)|^{(\alpha)}_{x,T}(|\nabla\bar{x}_1|^{(0)}_{T_0})^{1+\alpha}\cdot T_0 + T_0|\nabla\bar{x}_1|^{(0)}_T\{2|\nabla\nabla\nabla v_2|^{(0)}_T|\bar{x}_1-\bar{x}_2|^{(0)}_{T_0}\}^{1-\alpha}$$

$$\cdot\{|\nabla\nabla\nabla v_2|^{(L)}_{x,T}|\bar{x}_1-\bar{x}_2|^{(0)}_{T_0}|\nabla\bar{x}_1|^{(0)}_{T_0}+|\nabla\nabla\nabla v_2|^{(0)}_T|\nabla(\bar{x}_1-\bar{x}_2)|^{(0)}_{T_0}\}^{\alpha}$$

$$+T_0(|\nabla\nabla v_2|^{(\alpha)}_{x,T}(|\nabla_x\bar{x}_2|^{(0)}_{T_0})^{\alpha}|\nabla_x(\bar{x}_1-\bar{x}_2)|^{(0)}_{T_0})+T_0\{|\nabla\nabla\nabla v_2|^{(0)}_T|\bar{x}_1-\bar{x}_2|^{(0)}_{T_0}+|\nabla\nabla(v_1-v_2)|^{(0)}_{T_0}\}$$

$$\times(2|\nabla_x\bar{x}_1|^{(0)}_{T_0})^{1-\alpha}(|\nabla\nabla\bar{x}_1|^{(0)}_{T_0})^{\alpha}+2^{1-\alpha}|\nabla\nabla v_2|^{(0)}_T\{|\nabla\nabla(\bar{x}_1-\bar{x}_2)|^{(0)}_{T_0}+|\nabla(\bar{x}_1-\bar{x}_2)|^{(0)}_{T_0}]|x-x'|^{\alpha}\}.$$

We need to estimate $|\nabla\nabla(\bar{x}_1-\bar{x}_2)|^{(0)}_{T_0}$. We use the notations of (5. 56) and (5. 56)′.

$$(6.15)\quad \begin{cases} \dfrac{d}{d\tau}\left(\dfrac{\partial}{\partial x_k}\bar{X}_1\right)=\dfrac{\partial}{\partial x_k}(\hat{V}_1\bar{X}_1)=\left(\dfrac{\partial}{\partial x_k}\hat{V}_1\right)\bar{X}_1+\hat{V}_1\left(\dfrac{\partial}{\partial x_k}\bar{X}_1\right), & \dfrac{\partial}{\partial x_k}\bar{X}_1\Big|_{\tau=t}=0,\\ \dfrac{d}{d\tau}\left(\dfrac{\partial}{\partial x_k}\bar{X}_2\right)=\dfrac{\partial}{\partial x_k}(\hat{V}_2\bar{X}_2)=\left(\dfrac{\partial}{\partial x_k}\hat{V}_2\right)\bar{X}_2+\hat{V}_2\left(\dfrac{\partial}{\partial x_k}\bar{X}_2\right), & \dfrac{\partial}{\partial x_k}\bar{X}_2\Big|_{\tau=t}=0\end{cases}$$

Hence,

$$(6.16)\quad \begin{aligned}\left|\frac{\partial}{\partial x_k}(\bar{X}_1-\bar{X}_2)(\tau)\right| &\leqq \int_\tau^t|\hat{V}_1|\cdot\left|\frac{\partial}{\partial x_k}(\bar{X}_1-\bar{X}_2)(\tau)\right|d\tau\\ &+\int_\tau^t\left\{\left|\frac{\partial}{\partial x_k}(\hat{V}_1-\hat{V}_2)\right|\cdot|\bar{X}_1|+\left|\frac{\partial}{\partial x_k}\hat{V}_2\right|\cdot|\bar{X}_1-\bar{X}_2|\right.\\ &\left.+|\hat{V}_1-\hat{V}_2|\left|\frac{\partial}{\partial x_k}\bar{X}_2\right|\right\}dt.\end{aligned}$$

From this, we have

$$(6.16)'\qquad |\nabla\nabla(\bar{x}_1-\bar{x}_2)|^{(0)}_{T_0}\leqq F_3(M, M', T_0)\langle v_1-v_2\rangle_{T_0}, \qquad (F_3\searrow 0 \text{ as } T_0\searrow 0).$$

Thus,

$$(6.17)\quad \begin{cases}|[\quad]_A|\leqq\bar{F}_2(M, T_0)\langle v_1-v_2\rangle_{T_0}|x-x'|^{\alpha}, \qquad (\bar{F}_2\searrow 0 \text{ as } T_0\searrow 0),\\ |[\quad]_B|\leqq[F_{4,1}(M,\bar{M},M',T_0)\langle v_1-v_2\rangle_{T_0}+F_{4,2}(M,T_0)\langle v_1-v_2\rangle'_{T_0}]|x-x'|^{\alpha},\end{cases}$$

$$F_{4,1} \text{ and } F_{4,2}\searrow 0 \text{ as } T_0\searrow 0;\ M'\equiv\max[|\nabla\nabla\nabla v_1|^{(L)}_{x,T}, |\nabla\nabla\nabla v_2|^{(L)}_{x,T}]).$$

Accordingly, we have

$$(6.18)\qquad |\nabla\log\rho_{(v_1)}-\nabla\log\rho_{(v_2)}|^{(\alpha)}_{x,T_0}\leqq\{\bar{\bar{F}}_2(T_0)+F_{4,1}(T_0)\}\langle v_1-v_2\rangle_{T_0}+F_{4,2}\langle v_1-v_2\rangle'_{T_0}.$$

iii) For $T_0\geqq t\geqq t'>0$,

$$\{(\nabla_x \log \rho_1 - \nabla_x \log \rho_2)(x,t)\}_{H_0} - \{t \to t'\}_{H_0}$$

$$(6.19)\quad \begin{aligned} &= \Bigg[\frac{\nabla_x \rho_0(x_{1,0}(x,t)) - \nabla_x \rho_0(x_{2,0}(x,t))}{\rho_0(x_{1,0}(x,t))} + \nabla_x \rho_0(x_{2,0}(x,t)) \left\{ \frac{1}{\rho_0(x_{1,0}(x,t))} - \frac{1}{\rho_0(x_{2,0}(x,t))} \right\} \\ &\quad - \Bigg\{ \frac{\nabla_x \rho_0(x_{1,0}(x,t')) - \nabla_x \rho_0(x_{2,0}(x,t'))}{\rho_0(x_{1,0}(x,t'))} \\ &\qquad + \nabla_x \rho_0(x_{2,0}(x,t')) \left(\frac{1}{\rho_0(x_{1,0}(x,t'))} - \frac{1}{\rho_0(x_{2,0}(x,t'))} \right) \Bigg\} \Bigg]_H \\ &\quad - \Bigg[\int_0^t \{\nabla \bar{x}_1 \operatorname{div} v_1(\bar{x}_1(\tau;x,t),\tau) \nabla_x \bar{x}_1(\tau;x,t) - \nabla \bar{x}_2 \operatorname{div} v_2(\bar{x}_2(\tau;x,t),\tau) \nabla_x \bar{x}_2(\tau;x,t)\}_I d\tau \\ &\quad - \int_0^{t'} \{t \to t'\}_I d\tau \Bigg]_J . \end{aligned}$$

We can calculate in a way analogous to that in ii). As a result, we have

$$(6.20)\quad |[\quad]_H| \leqq F_{5,1}(M, T_0; \|\rho_0\|^{(2+L)}) \langle v_1 - v_2 \rangle_{T_0} (t-t')^{\alpha/2}, \qquad (F_{5,1} \searrow 0 \text{ as } T_0 \searrow 0),$$

$$(6.20)'\quad \begin{aligned} |[\quad]_J| &\leqq F_{5,2}(M, \bar{M}, M', T_0) \langle v_1 - v_2 \rangle_{T_0} (t-t')^{\alpha/2} \\ &\quad + F_{5,3}(M, T_0) |\nabla \nabla (v_1 - v_2)|_{x, T_0}^{(\alpha)} (t-t')^{\alpha/2}, \qquad (F_{5,2} \text{ and } F_{5,3} \searrow 0 \text{ as } T_0 \searrow 0). \end{aligned}$$

From (6.18), (6.20), and (6.20)$'$, we have

$$(6.21)\quad \begin{aligned} &|\nabla \log \rho_{(v_1)} - \nabla \log \rho_{(v_2)}|_{T_0}^{(\alpha)} \\ &\leqq \{\bar{F}_2(T_0) + F_{4,1}(T_0) + F_{5,2}(T_0)\} \langle v_1 - v_2 \rangle_{T_0} + \{F_{4,2}(T_0) + F_{5,3}(T_0)\} \langle v_1 - v_2 \rangle'_{T_0} \\ &\equiv F_6(T_0) \langle v_1 - v_2 \rangle_{T_0} + \hat{F}_6(T_0) \langle v_1 - v_2 \rangle'_{T_0}, \\ &\qquad (F_6 \text{ and } \hat{F}_6(T_0) \searrow 0 \text{ as } T_0 \searrow 0). \end{aligned}$$

iv) By making use of the results of i), ii), and iii), it is easy to show that

$$(6.22)\quad \begin{cases} |\tilde{N}_1|_{T_0}^{(\alpha)} \leqq F_{7,1}(T_0) \langle w \rangle'_{T_0} + F_{7,2}(\langle w \rangle_{T_0} + (\theta)_{T_0}), \\ |\tilde{N}_2|_{T_0}^{(\alpha)} \leqq F_{7,3}(T_0)(\langle w \rangle_{T_0} + (\theta)_{T_0}), \end{cases}$$

where $F_{7,1}(T_0) \searrow 0$, $F_{7,2}(T_0) \searrow \text{const}(>0)$, $F_{7,3}(T_0) \searrow \text{const}(>0)$, as $T_0 \searrow 0$. Therefore, from (6.5) we obtain:

$$(6.23)\quad \begin{cases} \langle w \rangle_{T_0} + (\theta)_{T_0} \leqq C(T_0) F_{7,2}(T_0) + {}'C(T_0) F_{7,3}(T_0)(\langle w \rangle_{T_0} + (\theta)_{T_0}) + F_{7,1}(T_0) C(T_0) \langle w \rangle'_{T_0}, \\ \qquad \langle w \rangle'_{T_0} \leqq C(T_0) F_{7,2}(T_0)(\langle w \rangle_{T_0} + (\theta)_{T_0}) + C(T_0) F_{7,1}(T_0) \langle w \rangle'_{T_0}, \end{cases}$$

$$(C \text{ and } {}'C \searrow 0 \text{ as } T_0 \searrow 0).$$

Next, there exists $T_1 \in (0, T]$ such that

$$0 \leqq C(T_1)F_{7,1}(T_1) < 1. \tag{6.24}$$

Thus, for $T_0 \in (0, T_1]$

$$\langle w \rangle'_{T_0} \leqq \frac{C(T_0)F_{7,2}(T_0)}{1-C(T_0)F_{7,1}(T_0)} (\langle w \rangle_{T_0} + (\theta)_{T_0}). \tag{6.25}$$

Hence,

$$\begin{aligned} &(\langle w \rangle_{T_0} + (\theta)_{T_0}) \\ &\leqq \left\{ C(T_0)F_{7,2}(T_0) + {}'C(T_0)F_{7,3}(T_0) + \frac{F_{7,1}(T_0)C(T_0)C(T_0)F_{7,2}(T_0)}{1-C(T_0)F_{7,1}(T_0)} \right\} (\langle w \rangle_{T_0} + (\theta)_{T_0}) \\ &\equiv F_8(T_0)(\langle w \rangle_{T_0} + (\theta)_{T_0}). \end{aligned} \tag{6.26}$$

There exists $T_2 \in (0, T_1]$ such that

$$0 \leqq F_8(T_2) < 1. \tag{6.27}$$

Therefore,

$$0 \leqq (\langle w \rangle_{T_2} + (\theta)_{T_2})(1 - F_8(T_2)) \leqq 0. \tag{6.28}$$

Thus,

$$(w, \theta)(x, t) = 0, \qquad (t \in [0, T_2]). \tag{6.29}$$

We can repeat this procedure, and by virtue of the assumption of the theorem, after a finite number of repetitions, we arrive at the conclusion of the theorem.

Q.E.D.

An outline of the proof of Theorem 3. By the hypothesis of the theorem, for some $T' \in (0, T]$, we can find an operator $G_{T'}$ from a subset of $H_{T'}^{4+\alpha} \times H_{T'}^{3+\alpha}$ into itself. Next, we follow the same procedure as we did in Theorem 1. Finally we apply Tikhonov's fixed point theorem to the results obtained. The uniqueness of the solution is obvious.

The definitions of the notations in Theorem 3 are as follows:

$$\begin{cases} H_{T'}^{n+\alpha} \equiv \left\{ g: \sum_{2r+|m|=0}^{n} |D_t^r D_x^m g|_{T'}^{(0)} + \sum_{2r+|m|=n-1}^{n} |D_t^r D_x^m g|_{t,T'}^{(\alpha/2)} \right. \\ \qquad\qquad \left. + \sum_{2r+|m|=n} |D_t^r D_x^m g|_{x,T'}^{(\alpha)} < +\infty \right\}, \qquad (n=3, 4), \\ H_{T'}^{n+\alpha} \equiv \left\{ g: \sum_{|m|=0}^{n} |D_x^m g|_{T'}^{(0)} + \sum_{|m|=0}^{n} |D_x^m g|_{t,T'}^{(\alpha/2)} + \sum_{|m|=n} |D_x^m g|_{x,T'}^{(\alpha)} < +\infty \right\}, \qquad (n=3, 4). \end{cases} \tag{6.30}$$

In studies to follow, the boundary value and global problems and related theories of abstract analysis will be treated.

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DEPARTMENT OF MATHEMATICS,
KOBE COMMERCIAL UNIVERSITY.