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ON THE EIGHTH COEFFICIENT OF UNIVALENT FUNCTIONS, II

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0. Let $f(z)$ be a normalized regular function univalent in the unit circle $|z|<1$

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n.$$

For the eighth coefficient a_8 several authors had proved its local maximality at the Koebe function $z/(1-z)^2$ [1], [4], [6]. One of the authors had improved the method in [4] to a great extent in [6]. Obrock [5] and Schiffer [8] proved a general result independently, which can be formulated in the present case in the following manner: If a_2, a_3 and a_4 are real, then $\Re a_8 \leqq 8$ with equality holding only for $z/(1-z)^2$. In [7] we have given the following fact: If a_2 is real non-negative, then $\Re a_8 \leqq 8$ with equality holding only for $z/(1-z)^2$.

In this paper we shall prove the following theorem:

THEOREM. *If* $a_3-3a_2^2/4$ *and* $a_4-3a_2a_3/2+5a_2^3/8$ *are real and* $|\arg a_2| \leqq \pi/7$, *then* $\Re a_8 \leqq 8$. *Equality occurs only for the Koebe function* $z/(1-z)^2$.

By the well known rotation this theorem implies the result due to Obrock and Schiffer as a simple corollary. Our original motivation in [7] and in this paper lies to investigate the status of the general a_8 problem. So the theorems are only byproducts of our original intention. We believe at the present time that the status became almost clear.

Section 1 is devoted to several preparatory lemmas and inequalities from which we start. Section 2 to 9 are concerned with the case $1 \leqq \Re a_2 \leqq 2$. The main part in this paper consists of sections 2, 4, 6 and 8. Section 10 is concerned with the case $0 \leqq \Re a_2 \leqq 1$. Sections 3, 5, 7, 9 and 10 are rather trivial parts and easy to handle.

1. We make use of the same notations as in [4]. By our assumption $y'=\eta'=0$ and $|x'/p| \leqq \tan(\pi/7)$.

First we shall give here several lemmas, which will be used later on.

LEMMA 1. $11(\tau^2+\tau'^2)+9(\varphi^2+\varphi'^2)+7(\xi^2+\xi'^2)+5\eta^2+3y^2+x'^2 \leqq 4x-x^2$.

Proof. This is a simple consequence of the area theorem for $f(1/z^2)^{-1/2}$.

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LEMMA 2. $$\eta+\left(2\beta-\frac{1}{2}p\right)y\leqq(2-p)\beta^2+\frac{1}{12}(8-p^3)+\frac{1}{4}px'^2.$$

Proof. By Grunsky's inequality

$$|b_{11}x_1^2+6b_{13}x_1x_3+3b_{33}x_3^2|\leqq|x_1|^2+3|x_3|^2.$$

Here we take $x_3=1/3$ and put $x_1=\beta$. Taking the real part

$$\eta-\frac{1}{2}py+\frac{1}{12}p^3-\frac{1}{4}px'^2+2\beta y+\beta^2p\leqq\frac{2}{3}+2\beta^2,$$

which is just the desired result.

LEMMA 3. $$\frac{1}{2}p^3\left(\eta+\frac{13}{6}py\right)$$

$$\leqq\frac{4}{3}-\frac{1}{48}p^6+\frac{16}{9}(4-p^2)p^2-\frac{2}{3}p^2(4-p^2)y-y^2-3\left(\eta+\frac{5}{6}py\right)^2$$

$$+2p^2x'^2y-\frac{1}{4}x'^4y+\frac{3}{2}px'^2\eta-\frac{3}{4}x'^2y^2-\left(\frac{16}{9}p^2+\frac{1}{16}p^4\right)x'^2-\frac{1}{16}p^2x'^4-\frac{1}{48}x'^6.$$

Proof. By Golusin's inequality we have

$$|b_{11}x_1+b_{31}x_3|^2+3|b_{13}x_1+b_{33}x_3|^2\leqq|x_1|^2+3|x_3|^2.$$

Here we put $x_1=8p/3$, $x_3=2/3$. Then by a simple rearrangement of terms we have the desired result.

Grunsky's inequality with $m=7$, $x_1=\gamma$, $x_3=\delta/3$, $x_5=p/5$, $x_7=1/7$, $x_2=x_4=x_6=0$, $y'=\eta'=0$ leads us to the following inequality

$$\Re a_8\leqq\frac{2}{7}+\frac{2}{5}p^2+\frac{2}{3}\delta^2+x\gamma^2+\frac{27}{447}p^7-\frac{1}{80}p^7-\frac{1}{12}p^3\delta^2$$

$$+\left(\frac{5}{4}p^2-2\delta\right)\varphi+\left(\frac{11}{8}p^3-p\delta-2\gamma\right)\xi+\left(\frac{7}{8}p^4+\frac{1}{2}p^2\delta-\delta^2-2p\gamma\right)\eta$$

$$+\left(\frac{11}{16}p^5-\frac{1}{2}p^3\delta+\frac{1}{2}p\delta^2-2\delta\gamma\right)y+\left(\frac{3}{2}p^3+\frac{1}{2}p\delta\right)y^2+\frac{9}{4}p\eta^2$$

$$+\left(\frac{27}{8}p^2+2\delta\right)y\eta+\frac{9}{2}py\xi+4\eta\xi+3y\varphi+\frac{9}{8}py^3$$

$$+\left(-\frac{73}{64}p^5+\frac{1}{4}p\delta^2\right)x'^2+\left(-\frac{29}{8}p^2-\delta\right)x'\xi'-\frac{7}{2}px'\varphi'-2x'\tau'-\frac{3}{2}x'y\xi'$$

$$+\left(-\frac{13}{2}p^3+\frac{1}{2}p\delta\right)x'^2y+\left(-\frac{23}{4}p^2+\frac{1}{2}\delta\right)x'^2\eta-\frac{29}{8}px'^2\xi-\frac{5}{4}x'^2\varphi$$

$$-\frac{21}{4}px'^2y^2-\frac{39}{8}x'^2y\eta+\frac{11}{8}x'^3\xi'+\frac{53}{16}px'^4y+\frac{7}{8}x'^4\eta+\frac{131}{64}p^3x'^4-\frac{27}{64}px'^6.$$

Now we put

$$\delta=\frac{5}{8}p^2+Ay-\frac{5}{8}x'^2,\qquad \gamma=\frac{3}{8}p^3+Bpy+C\eta-\frac{3}{2}px'^2,$$

where A, B and C are constants to be fixed in later parts. Then we have

$$\begin{aligned}
\Re a_8\leqq U&+\left\{\left(\frac{13}{128}-\frac{5}{48}A-\frac{3}{4}B\right)p^5+\frac{3}{2}Bp^4+\frac{5}{6}Ap^2\right.\\
&\left.+\left(-\frac{251}{64}+\frac{5}{12}A+3B\right)p^3x'^2-6Bp^2x'^2-\frac{5}{6}Ax'^2+\left(\frac{169}{128}-\frac{5}{16}A\right)px'^4\right\}y\\
&+\left\{\left(\frac{3}{64}-\frac{3}{4}C\right)p^4+\frac{3}{2}Cp^3+\left(-\frac{63}{32}+3C\right)p^2x'^2-6Cpx'^2+\frac{11}{64}x'^4\right\}\eta\\
&+\left\{\left(\frac{29}{16}-\frac{1}{12}A^2-B^2-\frac{5}{8}A-\frac{5}{4}B\right)p^3+2B^2p^2+\frac{2}{3}A^2\right.\\
&\left.+\left(-\frac{89}{16}+\frac{1}{4}A^2+\frac{23}{8}A+\frac{5}{4}B\right)px'^2\right\}y^2 \qquad \text{(A)}\\
&+\left\{\left(\frac{37}{8}-2BC-\frac{3}{4}A-2B-\frac{5}{4}C\right)p^2+4BCp+\left(-\frac{49}{8}+\frac{7}{4}A+\frac{5}{4}C\right)x'^2\right\}y\eta\\
&+\left\{\left(\frac{9}{4}-C^2-2C\right)p+2C^2\right\}\eta^2+\left(\frac{9}{2}-A-2B\right)py\xi+(4-2C)\eta\xi+(3-2A)y\varphi\\
&+\left(\frac{1}{2}A^2-2AB+\frac{1}{2}A+\frac{9}{8}\right)py^3+(-A^2-2AC+2A)y^2\eta\\
&+\left(\frac{113}{64\cdot 12}p^6-\frac{9}{4}p^4-\frac{25}{48}p^2-\frac{331}{64\cdot 12}p^3x'^2+\frac{9}{2}p^2x'^2+\frac{25}{96}x'^2-\frac{83}{256}px'^4\right)x'^2\\
&+\left(-\frac{17}{4}p^2+2x'^2\right)x'\xi'-\frac{7}{2}px'\varphi'-2x'\tau'-\left(A+\frac{3}{2}\right)x'y\xi',
\end{aligned}$$

$$U=\frac{2}{7}+\frac{2}{5}p^2+\frac{27}{448}p^7-\frac{1}{80}p^7+\frac{9}{64}p^6x+\frac{25}{64\cdot 12}(8-p^3)p^4.$$

This is our starting inequality. U is represented as the following polynomial of x

$$8-\frac{31}{4}x-\frac{81}{8}x^2+\frac{1111}{48}x^3-\frac{863}{48}x^4+\frac{2291}{320}x^5$$

$$-\left(\frac{133}{128}+\frac{25}{96}+\frac{7}{40}\right)x^6+\left(\frac{9}{112}+\frac{25}{64\cdot 12}+\frac{1}{80}\right)x^7.$$

2. Case $1\leqq p\leqq 2$, $x'^2/p^2\leqq 1/40$ and $y\geqq 0$. In this case we put $A=1$, $B=5/4$, $C=1$ in (A). Then we have

$$\Re a_8\leqq U+\left(-\frac{1}{384}p^5+\frac{15}{16}p^4x+\frac{5}{6}p^2-\frac{673}{192}p^3x'^2-\frac{15}{4}p^2xx'^2-\frac{5}{6}x'^2+\frac{129}{128}px'^4\right)y$$

$$+\left(\frac{3}{64}p^4+\frac{3}{4}p^3x-\frac{63}{32}p^2x'^2-3pxx'^2+\frac{11}{64}x'^4\right)\eta$$

$$+\left(-\frac{97}{48}p^3+\frac{25}{8}p^2+\frac{2}{3}-\frac{7}{8}px'^2\right)y^2+\left(-\frac{19}{8}p^2+5p-\frac{25}{8}x'^2\right)y\eta$$

$$+\left(-\frac{3}{4}p+2\right)\eta^2+py\xi+2\eta\xi+y\varphi-\frac{3}{8}py^3-y^2\eta$$

$$+\left(\frac{113}{64\cdot 12}p^5-\frac{9}{4}p^4-\frac{25}{48}p^2-\frac{331}{64\cdot 12}p^3x'^2+\frac{9}{2}p^2x'^2+\frac{25}{96}x'^2-\frac{83}{256}px'^4\right)x'^2$$

$$+\left(-\frac{17}{4}p^2+2x'^2\right)x'\xi'-\frac{7}{2}px'\varphi'-2x'\tau'-\frac{5}{2}x'y\xi'.$$

Let $\mathfrak{A}$ be a constant, to be fixed later, such that $3p^2-\mathfrak{A}x'^2\geqq 0$. Applying Lemma 3 to $(3p^4/64-\mathfrak{A}p^2x'^2/64)(\eta+13py/6)$ and Lemma 2 to $(3p^3x/4)(\eta+5py/4)$ we have

$$\Re a_8\leqq\frac{2}{7}+\frac{1}{8}p+\frac{2}{5}p^2+\frac{5}{3}p^3-\frac{23}{96}p^4+\frac{409}{192}p^5-\frac{155}{64}p^6+\frac{9}{64}p^6x$$

$$+\left(\frac{27}{448}-\frac{1}{80}-\frac{25}{64\cdot 12}-\frac{1}{32\cdot 16}+\frac{163}{256}\right)p^7+\frac{1}{24}p^2(10+2p+p^2)x\mathfrak{y}$$

$$+\left\{\left(\frac{5}{384}\mathfrak{A}+\frac{83}{192}\right)p^3-\frac{15}{2}p^2+\frac{\mathfrak{A}}{12}p-\frac{5}{6}+\left(\frac{63}{64}-\frac{1}{16}\mathfrak{A}\right)px'^2\right.$$

$$\left.+\frac{1}{128}\mathfrak{A}\frac{x'^4}{p}\right\}x'^2y$$

$$+\left\{\left(\frac{1}{64}\mathfrak{A}+\frac{75}{64}\right)p^2-6p+\left(\frac{11}{64}-\frac{3}{64}\mathfrak{A}\right)x'^2\right\}x'^2\eta$$

$$+\left\{-\frac{851}{384}p^3+\frac{25}{8}p^2-\frac{3}{32}p+\frac{2}{3}+\left(\frac{25}{32\cdot 12}\mathfrak{A}-\frac{121}{32\cdot 4}\right)px'^2\right.$$

$$\left.+\frac{\mathfrak{A}}{32}\frac{x'^2}{p}+\frac{3}{128}\mathfrak{A}\frac{x'^4}{p}\right\}y^2$$

$$+\left\{-\frac{91}{32}p^2+5p+\left(\frac{5}{32}\mathfrak{A}-\frac{25}{8}\right)x'^2\right\}y\eta+\left(-\frac{33}{32}p+2+\frac{3}{32}\mathfrak{A}\frac{x'^2}{p}\right)\eta^2$$

$$+py\xi+2\eta\xi+y\varphi-\frac{3}{8}py^3-y^2\eta$$

$$+\left\{\left(\frac{1}{32\cdot 48}\mathfrak{A}-\frac{71}{32\cdot 48}\right)p^5-\frac{45}{24}p^4+\left(\frac{1}{18}\mathfrak{A}-\frac{1}{6}\right)p^3-\frac{25}{48}p^2-\frac{2}{9}\mathfrak{A}p\right.$$

$$-\frac{1}{24}\mathfrak{A}\frac{1}{p}$$

$$+\left(\frac{1}{64\cdot 8}\mathfrak{A}-\frac{671}{64\cdot 24}\right)p^3x'^2+\frac{9}{2}p^2x'^2+\frac{1}{18}\mathfrak{A}px'^2+\frac{25}{96}x'^2$$

$$\left.+\left(\frac{\mathfrak{A}}{32\cdot 16}-\frac{167}{32\cdot 16}\right)px'^4+\frac{\mathfrak{A}}{48\cdot 32}\frac{x'^6}{p}\right\}x'^2$$

$$+\left(-\frac{17}{4}p^2+2x'^2\right)x'\xi'-\frac{7}{2}px'\varphi'-2x'\tau'-\frac{5}{2}x'y\xi'.$$

We apply the following trivial inequality

$$2AXY\leqq|A|\alpha X^2+|A|\alpha' Y^2,\qquad \alpha'=1/\alpha,\qquad \alpha>0$$

to the term

$$\frac{1}{24}p^2(10+2p+p^2)xy.$$

Then we have

$$\Re a_8\leqq 8-\frac{x}{96}\{160-(346+144\alpha)x+(2659+192\alpha)x^2-(3383.5+92\alpha)x^3$$

$$+(1845.95+20\alpha)x^4-(479.075+2\alpha)x^5\}$$

$$+\left\{\left(\frac{5}{384}\mathfrak{A}+\frac{83}{192}\right)p^3-\frac{15}{2}p^2+\frac{1}{12}\mathfrak{A}p-\frac{5}{6}+\left(\frac{63}{64}-\frac{1}{16}\mathfrak{A}\right)px'^2\right.$$

$$\left.+\frac{1}{128}\mathfrak{A}\frac{x'^4}{p}\right\}x'^2y$$

$$+\left\{\left(\frac{1}{64}\mathfrak{A}+\frac{75}{64}\right)p^2-6p+\left(\frac{11}{64}-\frac{3}{64}\mathfrak{A}\right)x'^2\right\}x'^2\eta$$

$$-\frac{1}{96}\left[\left\{-\frac{2}{\alpha}p^4+\left(212.75-\frac{4}{\alpha}\right)p^3-\left(300+\frac{20}{\alpha}\right)p^2+9p-64\right.\right.$$

$$\text{(B)}\qquad \left.+(90.75-6.25\mathfrak{A})px'^2-3\mathfrak{A}\frac{x'^2}{p}-2.25\mathfrak{A}\frac{x'^4}{p}\right\}y^2$$

$$+\{273p^2-480p+(300-15\mathfrak{A})x'^2\}y\eta+\left(99p-192-9\mathfrak{A}\frac{x'^2}{p}\right)\eta^2$$

$$-96py\xi-192\eta\xi-96y\varphi\Big]$$

$$-\frac{1}{96}\Big[\Big\{(4.4375-0.0625\mathfrak{A})p^5+180p^4+(16-5.3334\mathfrak{A})p^3+50p^2$$

$$+21.3333\mathfrak{A}p+4\mathfrak{A}\frac{1}{p}$$

$$+(41.9375-0.1875\mathfrak{A})p^3x'^2-432p^2x'^2-5.3334\mathfrak{A}px'^2-25x'^2$$

$$+(31.3125-0.1875\mathfrak{A})px'^4-0.0625\mathfrak{A}\frac{x'^6}{p}\Big\}x'^2$$

$$+(408p^2-192x'^2)x'\xi'+336px'\varphi'+192x'\tau'\Big]-\frac{3}{8}py^3-y^2\eta-\frac{5}{2}x'y\xi'.$$

Case i) $1.8\leqq p\leqq 2$.

We start from (B) with $\mathfrak{A}=45$, $\alpha=1/2$. We remark the following facts:

$$\left(\frac{391}{384}p^3-\frac{15}{2}p^2+\frac{15}{4}p-\frac{5}{6}-\frac{117}{64}px'^2+\frac{45}{128}\frac{x'^4}{p}\right)x'^2y+\frac{15}{8}p^3x'^2y\leqq 0,$$

and by Lemma 1

$$-\frac{15}{8}p^3x'^2y-\frac{5}{2}x'y\xi'\leqq\frac{5}{6p^3}\sqrt{\frac{4-p^2}{3}}\xi'^2\leqq\frac{6.91}{96}\xi'^2$$

$$x'^2\eta\left(\frac{15}{8}p^2-6p-\frac{31}{16}x'^2\right)\leqq\left(-\frac{15}{8}p^2+6p+\frac{31}{16}x'^2\right)\sqrt{\frac{4-p^2}{5}}x'^2$$

$$\leqq\frac{1}{96}(-70.182p^2+224.5824p+72.5214x'^2)x'^2,$$

$$-\frac{3}{8}py^3-y^2\eta\leqq\frac{2}{3p}y\eta^2\leqq\frac{2}{3p}\sqrt{\frac{4-p^2}{3}}\eta^2\leqq\frac{17.904}{96}\eta^2.$$

Making use of these remarks we have

$$\Re a_8\leqq 8-\frac{x}{96}(160-418x+2755x^2-3429.5x^3+1855.95x^4-480.075x^5)$$

$$-\frac{1}{96}\Big\{\left(-4p^4+204.75p^3-340p^2+9p-64-190.5px'^2-135\frac{x'^2}{p}-101.25\frac{x'^4}{p}\right)y^2$$

$$+(273p^2-480p-375x'^2)y\eta+\left(99p-209.904-405\frac{x'^2}{p}\right)\eta^2-96py\xi-192\eta\xi-96y\varphi\Big\}$$

$$-\frac{1}{96}\Big\{\Big(1.625p^5+180p^4-224.003p^3+120.182p^2+735.416p+180\frac{1}{p}$$

$$+33.5p^3x'^2-432p^2x'^2-240.003px'^2-97.522x'^2+22.875px'^4-2.813\frac{x'^6}{p}\Big)x'^2$$

$$+(408p^2-192x'^2)x'\xi'-6.91\xi'^2+336px'\varphi'+192x'\tau'\Big\}.$$

Applying Lemma 1 to $-145.2x/96$ we have

$$\Re a_8\leqq 8-\frac{x}{96}P(x)-\frac{1}{96}Q-\frac{1}{96}R,$$

$$P(x)=14.8-381.7x+2755x^2-3429.5x^3+1855.95x^4-480.075x^5,$$

$$Q=\left(-4p^4+204.75p^3-340p^2+9p+44.9-190.5px'^2-135\frac{x'^2}{p}-101.25\frac{x'^4}{p}\right)y^2$$

$$+(273p^2-480p-375x'^2)y\eta+\left(99p-28.404-405\frac{x'^2}{p}\right)\eta^2$$

$$-96py\xi-192\eta\xi+254.1\xi^2-96y\varphi+326.7\varphi^2,$$

$$R=\Big(1.625p^5+180p^4-224.003p^3+120.182p^2+735.416p+36.3+180\frac{1}{p}$$

$$+33.5p^3x'^2-432p^2x'^2-240.003px'^2-97.522x'^2+22.875px'^4-2.813\frac{x'^6}{p}\Big)x'^2$$

$$+(408p^2-192x'^2)x'\xi'+247.19\xi'^2+336px'\varphi'+326.7\varphi'^2+192x'\tau'+399.3\tau'^2.$$

It is easy to prove that $P'(x)$ is monotone increasing for $0\leqq x\leqq 0.2$ and $P'(0.08)<0$, $P'(0.081)>0$. Let λ be the root of $P'(x)=0$, $0.08<\lambda<0.081$. Construct $N(x)=5P(x)-xP'(x)$. Then $N(x)$ is monotone decreasing for $0\leqq x\leqq 0.1$. Further $N(0.081)>0$. Hence $N(x)>0$ for $0\leqq x\leqq 0.081$. Especially $N(\lambda)>0$, which implies $P(\lambda)>0$. Therefore $P(x)>0$ for $0\leqq x\leqq 0.2$.

Next we prove the positive definiteness of Q. Since

$$7.06y^2-96y\varphi+326.7\varphi^2\geqq 0,$$

we may consider $Q^*=Q-(7.06y^2-96y\varphi+326.7\varphi^2)$. We make the symmetric matrix associated with Q^*:

$$\begin{pmatrix} -4p^4+204.75p^3-340p^2+9p+37.84 & 136.5p^2-240p-187.5x'^2 & -48p \\ -190.5px'^2-135\dfrac{x'^2}{p}-101.25\dfrac{x'^4}{p} & & \\ 136.5p^2-240p-187.5x'^2 & 99p-28.404-405\dfrac{x'^2}{p} & -96 \\ -48p & -96 & 254.1 \end{pmatrix}$$

Its principal diagonal minor determinants are

$$254.1,$$

$$254.1\left(99p-28.404-405\frac{x'^2}{p}\right)-(-96)^2,$$

$$\begin{aligned}&-100623.6p^5+481949.6256p^4+5760763.8021p^3-10968778.908p^2+803998.1484p\\&\quad-621841.990176+411642p^3x'^2-12856380.075p^2x'^2+14456263.4442px'^2\\&\quad-4322241x'^2-1675616.706\frac{x'^2}{p}+8124212.25x'^4+1663887.4605\frac{x'^4}{p}\\&\quad+13892917.5\frac{x'^4}{p^2}+10419688.125\frac{x'^6}{p^2}.\end{aligned}$$

All of them are positive for $1.8\leqq p\leqq 2$, $x'^2/p^2\leqq 1/40$. This implies the non-negativity of Q^* there. Thus Q is positive definite there.

We prove the positive definiteness of R. Since

$$23.09x'^2+192x'\tau'+399.3\tau'^2\geqq 0,$$

we may consider $R^*=R-(23.09x'^2+192x'\tau'+399.3\tau'^2)$. We make the symmetric matrix associated with R^*:

$$\begin{pmatrix} 1.625p^5+180p^4-224.003p^3+120.182p^2+735.416p & 204p^2-96x'^2 & 168p \\ +13.21+180\dfrac{1}{p}+33.5p^3x'^2-432p^2x'^2-240.003px'^2 & & \\ -97.522x'^2+22.875px'^4-2.813\dfrac{x'^6}{p} & & \\ 204p^2-96x'^2 & 247.19 & 0 \\ 168p & 0 & 326.7 \end{pmatrix}.$$

Its principal diagonal minor determinants are

$$326.7,$$

$$247.19\cdot 326.7,$$

$$131230.081125p^5+940307.94p^4-18089804.222919p^3+2728843.969086p^2$$
$$+59389970.055768p+1066799.61333+14536255.14\frac{1}{p}+2705358.5955p^3x'^2$$
$$-22090826.736p^2x'^2-19381915.790919px'^2-7875581.520906x'^2$$
$$+1847315.757375px'^4-3010867.2x'^4-227169.365049\frac{x'^6}{p}.$$

All of them are positive for $1.8\leqq p\leqq 2$, $x'^2/p^2\leqq 1/40$. Thus R is positive definite there.

Therefore we have $\Re a_8\leqq 8$ for $1.8\leqq p\leqq 2$, $x'^2/p^2\leqq 1/40$ and $y\geqq 0$. Equality occurs only for $p=2$, that is, for the Koebe function.

Case ii) $1.7\leqq p\leqq 1.8$.

In this case we start from (B) with $\mathfrak{A}=39$, $\alpha=1/2$. We remark the following facts:

$$\left(\frac{361}{384}p^3-\frac{15}{2}p^2+\frac{13}{4}p-\frac{5}{6}-\frac{93}{64}px'^2+\frac{39}{128}\frac{x'^4}{p}\right)x'^2y+2p^3x'^2y\leqq 0,$$

and by Lemma 1

$$-2p^3x'^2y-\frac{5}{2}x'y\xi'\leqq\frac{25}{32p^3}y\xi'^2\leqq\frac{25}{32p^3}\sqrt{\frac{4-p^2}{3}}\xi'^2\leqq\frac{9.286}{96}\xi'^2,$$

$$\left(\frac{57}{32}p^2-6p-\frac{53}{32}x'^2\right)x'^2\eta\leqq\left(-\frac{57}{32}p^2+6p+\frac{53}{32}x'^2\right)\sqrt{\frac{4-p^2}{5}}x'^2$$
$$\leqq\frac{1}{96}(-80.5752p^2+271.4112p+74.9208x'^2)x'^2,$$

$$-\frac{3}{8}py^3-y^2\eta\leqq\frac{2}{3p}y\eta^2\leqq\frac{2}{3p}\sqrt{\frac{4-p^2}{3}}\eta^2\leqq\frac{22.906}{96}\eta^2.$$

Making use of these remarks and applying Lemma 1 to the term $-160x/96-52.52x^2/96$ we have

$$\Re a_8 \leqq 8 - \frac{x^2}{96} P(x) - \frac{1}{96} Q - \frac{1}{96} R,$$

$$P(x) = -430.52 + 2768.13x - 3429.5x^2 + 1855.95x^3 - 480.075x^4,$$

$$Q = \left(-4p^4 + 204.75p^3 - 340p^2 - 30.39p + 134.78 - 153px'^2 - 117\frac{x'^2}{p} - 87.75\frac{x'^4}{p}\right)y^2$$

$$+(273p^2 - 480p - 285x'^2)y\eta + \left(33.35p + 116.394 - 351\frac{x'^2}{p}\right)\eta^2 - 96py\xi - 192\eta\xi$$

$$+(463.82 - 91.91p)\xi^2 - 96y\varphi + (596.34 - 118.17p)\varphi^2,$$

$$R = \left(2p^5 + 180p^4 - 192.003p^3 + 130.575p^2 + 547.457p + 66.26 + 156\frac{1}{p}\right.$$

$$\left. + 34.625p^3x'^2 - 432p^2x'^2 - 208.003px'^2 - 99.921x'^2 + 24px'^4 - 2.438\frac{x'^6}{p}\right)x'^2$$

$$+(408p^2 - 192x'^2)x'\xi' + (454.534 - 91.91p)\xi'^2 + 336px'\varphi' + (596.34 - 118.17p)\varphi'^2$$

$$+192x'\tau' + (728.86 - 144.43p)\tau'^2.$$

$P(x)$ is monotone increasing for $0.2 \leqq x \leqq 0.3$ and $P(0.2) > 0$. Hence $P(x) > 0$ for $0.2 \leqq x \leqq 0.3$.

Since $(2.6 + 1.9p)y^2 - 96y\varphi + (596.34 - 118.17p)\varphi^2 \geqq 0$ for $1.7 \leqq p \leqq 1.8$ we may consider $Q^* = Q - (2.6 + 1.9p)y^2 + 96y\varphi - (596.34 - 118.17p)\varphi^2$. We make the symmetric matrix associated with Q^*:

$$\begin{pmatrix} -4p^4 + 204.75p^3 - 340p^2 - 32.29p + 132.18 - 153px'^2 - 117\dfrac{x'^2}{p} - 87.75\dfrac{x'^4}{p} & 136.5p^2 - 240p - 142.5x'^2 & -48p \\ 136.5p^2 - 240p - 142.5x'^2 & 33.35p + 116.394 - 351\dfrac{x'^2}{p} & -96 \\ -48p & -96 & 463.82 - 91.91p \end{pmatrix}.$$

We can prove the positive definiteness of this matrix for $1.7 \leqq p \leqq 1.8$, $x'^2/p^2 \leqq 1/40$ by taking its principal diagonal minor determinants. Thus Q is positive definite for $1.7 \leqq p \leqq 1.8$, $x'^2/p^2 \leqq 1/40$.

Since $(8.428 + 6.259p)x'^2 + 192x'\tau' + (728.86 - 144.43p)\tau'^2 \geqq 0$ for $1.7 \leqq p \leqq 1.8$ we may consider $R^* = R - (8.428 + 6.259p)x'^2 - 192x'\tau' - (728.86 - 144.43p)\tau'^2$. We make the symmetric matrix associated with R^*:

$$\begin{pmatrix} 2p^5+180p^4-192.003p^3+130.575p^2+541.198p & 204p^2-96x'^2 & 168p \\ +57.832+156\dfrac{1}{p}+34.625p^3x'^2-432p^2x'^2 & & \\ -208.003px'^2-99.921x'^2+24px'^4-2.438\dfrac{x'^6}{p} & & \\ 204p^2-96x'^2 & 454.534-91.91p & 0 \\ 168p & 0 & 596.34-118.17p \end{pmatrix}.$$

By an easy calculation we can prove the positive definiteness of this matrix for $1.7\leqq p\leqq 1.8$, $x'^2/p^2\leqq 1/40$. Thus R is positive definite there.

Therefore we have $\Re a_8<8$ for $1.7\leqq p\leqq 1.8$, $x'^2/p^2\leqq 1/40$, $y\geqq 0$.

Case iii) $1.5\leqq p\leqq 1.7$.

In this case we start from (B) with $\mathfrak{A}=29$, $\alpha=1/2$. We remark the following facts:

$$\left(\frac{311}{384}p^3-\frac{15}{2}p^2+\frac{29}{12}p-\frac{5}{6}-\frac{53}{64}px'^2+\frac{29}{128}\frac{x'^4}{p}\right)x'^2y+2.93p^3x'^2y\leqq 0,$$

and by Lemma 1

$$-2.93p^3x'^2y-\frac{5}{2}x'y\xi'\leqq\frac{25}{46.88p^3}y\xi'^2\leqq\frac{25}{46.88p^3}\sqrt{\frac{4-p^2}{3}}\xi'^2\leqq\frac{11.59}{96}\xi'^2,$$

$$\left(\frac{13}{8}p^2-6p-\frac{19}{16}x'^2\right)x'^2\eta\leqq\left(-\frac{13}{8}p^2+6p+\frac{19}{16}x'^2\right)\sqrt{\frac{4-p^2}{5}}x'^2$$

$$\leqq\frac{1}{96}(-92.3052p^2+340.8192p+67.4538x'^2)x'^2,$$

$$-\frac{3}{8}py^3-y^2\eta\leqq\frac{2}{3p}y\eta^2\leqq\frac{2}{3p}\sqrt{\frac{4-p^2}{3}}\eta^2\leqq\frac{32.592}{96}\eta^2.$$

Making use of these remarks and applying Lemma 1 to the term $-160x/96$ $-201.08x^2/96$ we have

$$\Re a_8\leqq 8-\frac{x^2}{96}P(x)-\frac{1}{96}Q-\frac{1}{96}R,$$

$$P(x)=-579.08+2805.27x-3429.5x^2+1855.95x^3-480.075x^4,$$

$$Q=\left(-4p^4+204.75p^3-340p^2-141.81p+357.62-90.5px'^2-87\frac{x'^2}{p}-65.25\frac{x'^4}{p}\right)y^2$$

$$+(273p^2-480p-135x'^2)y\eta+\left(-152.35p+478.108-261\frac{x'^2}{p}\right)\eta^2-96py\xi-192\eta\xi$$

$$+(983.78-351.89p)\xi^2-96y\varphi+(1264.86-452.43p)\varphi^2,$$

$$R=\Big(2.625p^5+180p^4-138.669p^3+142.305p^2+227.576p+140.54+116\frac{1}{p}$$

$$+36.5p^3x'^2-432p^2x'^2-154.669px'^2-92.454x'^2+25.875px'^4-1.813\frac{x'^6}{p}\Big)x'^2$$

$$+(408p^2-192x'^2)x'\xi'+(972.19-351.89p)\xi'^2+336px'\varphi'$$

$$+(1264.86-452.43p)\varphi'^2+192x'\tau'+(1545.94-552.97p)\tau'^2.$$

$P(x)$ is monotone increasing for $0.3\leqq x\leqq 0.5$ and $P(0.3)>0$. Hence $P(x)>0$ for $0.3\leqq x\leqq 0.5$.

Since $(4.9p-3.4)y^2-96y\varphi+(1264.86-452.43p)\varphi^2\geqq 0$ for $1.5\leqq p\leqq 1.7$, we may consider $Q^*=Q-(4.9p-3.4)y^2+96y\varphi-(1264.86-452.43p)\varphi^2$. We make the symmetric matrix associated with Q^*:

$$\begin{pmatrix} -4p^4+204.75p^3-340p^2-146.71p+361.02 & 136.5p^2-240p-67.5x'^2 & -48p \\ -90.5px'^2-87\frac{x'^2}{p}-65.25\frac{x'^4}{p} & & \\ 136.5p^2-240p-67.5x'^2 & -152.35p+478.108-261\frac{x'^2}{p} & -96 \\ -48p & -96 & 983.78-351.89p \end{pmatrix}.$$

It is easy to prove that this matrix is positive definite for $1.5\leqq p\leqq 1.7$, $x'^2/p^2\leqq 1/40$. Hence Q is positive definite there.

Since $(16.166p-11.385)x'^2+192x'\tau'+(1545.94-552.97p)\tau'^2\geqq 0$ for $1.5\leqq p\leqq 1.7$, we may consider $R^*=R-(16.166p-11.385)x'^2-192x'\tau'-(1545.94-552.97p)\tau'^2$. We make the symmetric matrix associated with R^*:

$$\begin{pmatrix} 2.625p^5+180p^4-138.669p^3+142.305p^2 & 204p^2-96x'^2 & 168p \\ +211.41p+151.925+116\frac{1}{p}+36.5p^3x'^2 & & \\ -432p^2x'^2-154.669px'^2-92.454x'^2 & & \\ +25.875px'^4-1.813\frac{x'^6}{p} & & \\ 204p^2-96x'^2 & 972.19-351.89p & 0 \\ 168p & 0 & 1264.86-452.43p \end{pmatrix}.$$

It is easy to prove that this matrix is positive definite for $1.5\leqq p\leqq 1.7$, $x'^2/p^2\leqq 1/40$. Hence R is positive definite there.

Therefore we have $\Re a_8<8$ for $1.5\leqq p\leqq 1.7$, $x'^2/p^2\leqq 1/40$, $y\geqq 0$.

Case iv) $1.3\leqq p\leqq 1.5$.

In this case we start from (B) with $\mathfrak{A}=60$, $\alpha=1/2$. We remark the following facts:

$$\left(\frac{233}{192}p^3-\frac{15}{2}p^2+5p-\frac{5}{6}-\frac{177}{64}px'^2+\frac{15}{32}\frac{x'^4}{p}\right)x'^2y+\frac{8}{5}p^3x'^2y\leqq 0,$$

$$-\frac{8}{5}p^3x'^2y-\frac{5}{2}x'y\xi'\leqq\frac{125}{128p^3}y\xi'^2\leqq\frac{125}{128p^3}\sqrt{\frac{4-p^2}{3}}\xi'^2\leqq\frac{37.445}{96}\xi'^2,$$

$$\left(\frac{135}{64}p^2-6p-\frac{169}{64}x'^2\right)x'^2\eta\leqq\left(-\frac{135}{64}p^2+6p+\frac{169}{64}x'^2\right)\sqrt{\frac{4-p^2}{5}}x'^2$$

$$\leqq\frac{1}{96}(-137.6595p^2+391.5648p+172.3293x'^2)x'^2,$$

$$-\frac{3}{8}py^3-y^2\eta\leqq\frac{2}{3p}y\eta^2\leqq\frac{2}{3p}\sqrt{\frac{4-p^2}{3}}\eta^2\leqq\frac{43.2}{96}\eta^2.$$

Making use of these remarks and applying Lemma 1 to the term $-160x/96-393.2x^2/96$ we have

$$\Re a_8\leqq 8-\frac{x^2}{96}P(x)-\frac{1}{96}Q-\frac{1}{96}R,$$

$$P(x)=-771.2+2853.3x-3429.5x^2+1855.95x^3-480.75x^4,$$

$$Q=\left(-4p^4+204.75p^3-340p^2-285.9p+645.8-284.25px'^2-180\frac{x'^2}{p}-135\frac{x'^4}{p}\right)y^2$$
$$+(273p^2-480p-600x'^2)y\eta+\left(-392.5p+947.8-540\frac{x'^2}{p}\right)\eta^2-96py\xi$$
$$-192\eta\xi+(1656.2-688.1p)\xi^2-96y\varphi+(2129.4-884.7p)\varphi^2,$$

$$R=\left(0.6875p^5+180p^4-304.004p^3+187.659p^2+790.133p+236.6+240\frac{1}{p}\right.$$
$$\left.+30.687p^3x'^2-432p^2x'^2-320.004px'^2-197.33x'^2+20.062px'^4-3.75\frac{x'^6}{p}\right)x'^2$$
$$+(408p^2-192x'^2)x'\xi'+(1618.755-688.1p)\xi'^2+336px'\varphi'$$
$$+(2129.4-884.7p)\varphi'^2+192x'\tau'+(2602.6-1081.3p)\tau'^2.$$

$P(x)$ is monotone increasing for $0.5\leqq x\leqq 0.7$ and $P(0.5)>0$. Hence $P(x)>0$ for $0.5\leqq x\leqq 0.7$.

Since $(5.7p-5)y^2-96y\varphi+(2129.4-884.7p)\varphi^2\geqq 0$ for $1.3\leqq p\leqq 1.5$, we may consider $Q^*=Q-(5.7p-5)y^2+96y\varphi-(2129.4-884.7p)\varphi^2$. We make the symmetric matrix associated with Q^*:

$$\begin{pmatrix} -4p^4+204.75p^3-340p^2-291.6p+650.8 & 136.5p^2-240p-300x'^2 & -48p \\ -284.25px'^2-180\dfrac{x'^2}{p}-135\dfrac{x'^4}{p} & & \\ 136.5p^2-240p-300x'^2 & -392.5p+947.8-540\dfrac{x'^2}{p} & -96 \\ -48p & -96 & 1656.2-688.1p \end{pmatrix}.$$

It is easy to prove that this matrix is positive definite for $1.3\leqq p\leqq 1.5$, $x'^2/p^2\leqq 1/40$. Hence Q is positive definite there.

Since $(18.923p-16.899)x'^2+192x'\tau'+(2602.6-1081.3p)\tau'^2\geqq 0$ for $1.3\leqq p\leqq 1.5$, we may consider $R^*=R-(18.923p-16.899)x'^2-192x'\tau'-(2602.6-1081.3p)\tau'^2$. We make the symmetric matrix associated with R^*:

$$\begin{pmatrix} 0.6875p^5+180p^4-304.004p^3+187.659p^2 & 204p^2-96x'^2 & 168p \\ +771.21p+253.499+240\dfrac{1}{p}+30.687p^3x'^2 & & \\ -432p^2x'^2-320.004px'^2-197.33x'^2 & & \\ +20.062px'^4-3.75\dfrac{x'^6}{p} & & \\ 204p^2-96x'^2 & 1618.755-688.1p & 0 \\ 168p & 0 & 2129.4-884.7p \end{pmatrix}.$$

It is easy to prove that this matrix is positive definite for $1.3\leqq p\leqq 1.5$, $x'^2/p^2\leqq 1/40$. Hence R is positive definite there.

Therefore we have $\Re a_8<8$ for $1.3\leqq p\leqq 1.5$, $x'^2/p^2\leqq 1/40$, $y\geqq 0$.

Case v) $1\leqq p\leqq 1.3$.

In this case we start from (B) with $\mathfrak{A}=60$, $\alpha=1/2$. We remark the following facts:

$$\left(\frac{233}{192}p^3-\frac{15}{2}p^2+5p-\frac{5}{6}-\frac{177}{64}px'^2+\frac{15}{32}\frac{x'^4}{p}\right)x'^2y+\frac{8}{5}p^3x'^2y\leqq 0,$$

$$-\frac{8}{5}p^3x'^2y-\frac{5}{2}x'y\xi'\leqq\frac{125}{128p^3}y\xi'^2\leqq\frac{125}{128p^3}\sqrt{\frac{4-p^2}{3}}\xi'^2\leqq\frac{93.75}{96}\xi'^2,$$

$$\left(\frac{135}{64}p^2-6p-\frac{169}{64}x'^2\right)x'^2\eta\leqq\left(-\frac{135}{64}p^2+6p+\frac{169}{64}x'^2\right)\sqrt{\frac{4-p^2}{5}}\,x'^2$$

$$\leqq\frac{1}{96}(-156.8565p^2+446.1696p+196.3611x'^2)x'^2,$$

$$-\frac{3}{8}py^3-y^2\eta\leqq\frac{2}{3p}y\eta^2\leqq\frac{2}{3p}\sqrt{\frac{4-p^2}{3}}\eta^2\leqq\frac{64}{96}\eta^2.$$

Making use of these remarks and applying Lemma 1 to the term $-160x/96$ $-428x^2/96$ we have

$$\Re a_8\leqq 8-\frac{x^2}{96}P(x)-\frac{1}{96}Q-\frac{1}{96}R,$$

$$P(x)=-806+2862x-3429.5x^2+1855.95x^3-480.075x^4,$$

$$Q=\left(-4p^4+204.75p^3-340p^2-312p+698-284.25px'^2-180\frac{x'^2}{p}-135\frac{x'^4}{p}\right)y^2$$
$$+(273p^2-480p-600x'^2)y\eta+\left(-436p+1014-540\frac{x'^2}{p}\right)\eta^2-96py\xi-192\eta\xi$$
$$+(1778-749p)\xi^2-96y\varphi+(2286-963p)\varphi^2,$$

$$R=\left(0.6875p^5+180p^4-304.004p^3+206.856p^2+726.828p+254+240\frac{1}{p}\right.$$
$$\left.+30.687p^3x'^2-432p^2x'^2-320.004px'^2-221.361x'^2+20.062px'^4-3.75\frac{x'^6}{p}\right)x'^2$$
$$+(408p^2-192x'^2)x'\xi'+(1684.25-749p)\xi'^2+336px'\varphi'+(2286-963p)\varphi'^2$$
$$+192x'\tau'+(2794-1177p)\tau'^2.$$

$P'(x)$ is monotone decreasing for $0.7\leqq x\leqq 1$ and $P'(0.7)>0$, $P'(1)<0$. Further $P(0.7)>0$, $P(1)>0$. Hence $P(x)>0$ for $0.7\leqq x\leqq 1$.

Since $(4.6p-2.8)y^2-96y\varphi+(2286-963p)\varphi^2\geqq 0$ for $1\leqq p\leqq 1.3$, we may consider $Q^*=Q-(4.6p-2.8)y^2+96y\varphi-(2286-963p)\varphi^2$. We make the symmetric matrix associated with Q^*:

$$\begin{pmatrix} -4p^4+204.75p^3-340p^2-316.6p+700.8 & 136.5p^2-240p-300x'^2 & -48p \\ -284.25px'^2-180\frac{x'^2}{p}-135\frac{x'^4}{p} & & \\ 136.5p^2-240p-300x'^2 & -436p+1014-540\frac{x'^2}{p} & -96 \\ -48p & -96 & 1778-749p \end{pmatrix}.$$

It is easy to prove that this matrix is positive definite for $1\leqq p\leqq 1.3$, $x'^2/p^2\leqq 1/40$. Hence Q is positive definite there.

Since $(15.24p-9.53)x'^2+192x'\tau'+(2794-1177p)\tau'^2\geqq 0$ for $1\leqq p\leqq 1.3$, we may consider $R^*=R-(15.24p-9.53)x'^2-192x'\tau'-(2794-1177p)\tau'^2$. We make the symmetric matrix associated with R^*:

$$\begin{pmatrix} \begin{array}{l} 0.6875p^5+180p^4-304.004p^3+206.856p^2 \\ +711.588p+263.53+240\dfrac{1}{p}+30.687p^3x'^2 \\ -432p^2x'^2-320.004px'^2-221.361x'^2 \\ +20.062px'^4-3.75\dfrac{x'^6}{p} \end{array} & 204p^2-96x'^2 & 168p \\ 204p^2-96x'^2 & 1684.25-749p & 0 \\ 168p & 0 & 2286-963p \end{pmatrix}.$$

It is easy to prove that this matrix is positive definite for $1\leqq p\leqq 1.3$, $x'^2/p^2\leqq 1/40$. Hence R is positive definite there.

Therefore we have $\Re a_8<8$ for $1\leqq p\leqq 1.3$, $x'^2/p^2\leqq 1/40$, $y\geqq 0$.

3. Case $1\leqq p\leqq 2$, $x'^2/p^2\leqq 1/40$ and $y\leqq 0$. In this case we put $A=1/2$, $B=3/2$, $C=1$ in (A).

Applying the trivial inequality $2AXY\leqq|A|X^2+|A|Y^2$ and Lemma 1

$$\left(-\frac{413}{384}p^5+\frac{9}{4}p^4+\frac{5}{12}p^2+\frac{151}{192}p^3x'^2-9p^2x'^2-\frac{5}{12}x'^2+\frac{149}{128}px'^4\right)y-2x'y\xi'$$

$$\leqq -\frac{y}{384}(413p^5-864p^4-160p^2-302p^3x'^2+3456p^2x'^2+160x'^2-447px'^4+384x'^2+384\xi'^2)$$

$$\leqq -\frac{y}{384}(413p^5-864p^4-214.86p^2+219.5-302p^3x'^2+3456p^2x'^2+489.14x'^2-447px'^4)$$

$$\leqq 0$$

for $1\leqq p\leqq 2$, $x'^2/p^2\leqq 1/40$, $y\leqq 0$. Then we have

$$\begin{aligned} \Re a_8\leqq U&+\left(-\frac{45}{64}p^4+\frac{3}{2}p^3+\frac{33}{32}p^2x'^2-6px'^2+\frac{11}{64}x'^4\right)\eta \\ &+\left(-\frac{127}{48}p^3+\frac{9}{2}p^2+\frac{1}{6}-\frac{35}{16}px'^2\right)y^2+(-3p^2+6p-4x'^2)y\eta+\left(-\frac{3}{4}p+2\right)\eta^2 \\ &+py\xi+2\eta\xi+2y\varphi-\frac{1}{4}y^2\eta \end{aligned} \tag{C}$$

$$+\left(\frac{113}{64\cdot 12}p^5-\frac{9}{4}p^4-\frac{25}{48}p^2-\frac{331}{64\cdot 12}p^3x'^2+\frac{9}{2}p^2x'^2+\frac{25}{96}x'^2-\frac{83}{256}px'^4\right)x'^2$$

$$+\left(-\frac{17}{4}p^2+2x'^2\right)x'\xi'-\frac{7}{2}px'\varphi'-2x'\tau'.$$

Case i) $\eta\geqq 0$. By Lemma 2

$$\eta\leqq-\frac{7}{48}p^3+\frac{1}{8}p^2+\frac{2}{3}+\frac{1}{4}px'^2,$$

whence we have

$$\left(-\frac{45}{64}p^4+\frac{3}{2}p^3+\frac{33}{32}p^2x'^2-6px'^2+\frac{11}{64}x'^4\right)\eta$$

$$\leqq\left\{\left(\frac{3}{64}p^4-\frac{60}{32}p^2x'^2\right)+\left(\frac{3}{4}p^3x-3pxx'^2\right)\right\}\eta$$

$$\leqq\frac{105}{64\cdot 16}p^7-\frac{157}{64\cdot 8}p^6+\frac{3}{16}p^5-\frac{15}{32}p^4+p^3-\frac{87}{64\cdot 4}p^5x'^2+\frac{89}{64}p^4x'^2$$

$$-\frac{3}{4}p^3x'^2+\frac{3}{4}p^2x'^2-4px'^2+\frac{9}{32}p^3x'^4-\frac{3}{2}p^2x'^4.$$

By Lemma 1

$$-\frac{654}{96}x\leqq-\frac{163.5}{96}(x^2+x'^2+3y^2+5\eta^2+7\xi^2+7\xi'^2+9\varphi^2+9\varphi'^2+11\tau'^2).$$

Making use of these inequalities and (C) we have

$$\Re a_8\leqq 8-\frac{x^2}{96}P_0(x)-\frac{1}{96}Q_0-\frac{1}{96}R_0,$$

$$P_0(x)=649.5-963.5x+601x^2-195.675x^3+33.175x^4-2.1956x^5,$$

$$Q_0=(254p^3-432p^2+474.5+210px'^2)y^2+(288p^2-576p+384x'^2)y\eta$$

$$+(72p+625.5)\eta^2-96py\xi+1144.5\xi^2-192\eta\xi-192y\varphi+1471.5\varphi^2,$$

$$R_0=(18.5p^5+82.5p^4+72p^3-22p^2+384p+163.5+14.375p^3x'^2$$

$$-288p^2x'^2-25x'^2+31.125px'^4)x'^2$$

$$+(408p^2-192x'^2)x'\xi'+1144.5\xi'^2+336px'\varphi'+1471.5\varphi'^2+192x'\tau'+1798.5\tau'^2.$$

$P_0(x)$ is monotone decreasing for $0\leqq x\leqq 1$ and $P_0(1)>0$. Hence $P_0(x)>0$ for $0\leqq x\leqq 1$.

Q_0 is positive definite for $1\leqq p\leqq 2$, $x'^2/p^2\leqq 1/40$. Indeed we have

$$6.3y^2-192y\varphi+1471.5\varphi^2\geqq 0,$$

$$29.4\eta^2-192\eta\xi+314\xi^2\geqq 0,$$

$$2.8p^2y^2-96py\xi+830.5\xi^2\geqq 0$$

and

$$(254p^3-434.8p^2+468.2+210px'^2)y^2+(288p^2-576p+384x'^2)y\eta+(72p+596.1)\eta^2\geqq 0$$

for $1\leqq p\leqq 2$, $x'^2/p^2\leqq 1/40$. This implies the positive definiteness of Q_0 for $1\leqq p\leqq 2$, $x'^2/p^2\leqq 1/40$.

R_0 is positive definite for $1\leqq p\leqq 2$, $x'^2/p^2\leqq 1/40$. Indeed we have

$$5.13x'^2+192x'\tau'+1798.5\tau'^2\geqq 0,$$

$$19.2p^2x'^2+336px'\varphi'+1471.5\varphi'^2\geqq 0$$

and

$$(18.5p^5+82.5p^4+72p^3-41.2p^2+384p+158.37+14.375p^3x'^2-288p^2x'^2-25x'^2$$
$$+31.125px'^4)x'^2+(408p^2-192x'^2)x'\xi'+1144.5\xi'^2\geqq 0$$

for $1\leqq p\leqq 2$, $x'^2/p^2\leqq 1/40$, which imply the desired result.

Thus we have $\Re a_8\leqq 8$ in the present case with equality holding only for $p=2$.

Case ii) $\eta\leqq 0$. Applying Lemma 1

$$\left(-\frac{45}{64}p^4+\frac{3}{2}p^3+\frac{33}{32}p^2x'^2-6px'^2+\frac{11}{64}x'^4\right)\eta-\frac{1}{4}y^2\eta$$

$$=\left\{\left(\frac{3}{64}p^4-\frac{60}{32}p^2x'^2\right)+\left(\frac{3}{4}p^3x-3pxx'^2-\frac{1}{4}y^2\right)+\frac{11}{64}x'^4-\frac{3}{32}p^2x'^2\right\}\eta$$

$$\leqq\left(\frac{11}{64}x'^2-\frac{3}{32}p^2\right)x'^2\eta\leqq\frac{1}{96}(6.972p^2-12.78x'^2)x'^2.$$

By Lemma 1

$$-\frac{744}{96}x\leqq-\frac{186}{96}(x^2+x'^2+3y^2+5\eta^2+7\xi^2+7\xi'^2+9\varphi^2+9\varphi'^2+11\tau'^2).$$

Making use of these inequalities and (C) we have

$$\Re a_8\leqq 8-\frac{x^2}{96}P_1(x)-\frac{1}{96}Q_1-\frac{1}{96}R_1,$$

$$P_1(x)=1158-2222x+1726x^2-687.3x^3+141.55x^4-12.04x^5,$$

$$Q_1=(254p^3-432p^2+542+210px'^2)y^2+(288p^2-576p+384x'^2)y\eta$$
$$+(72p+738)\eta^2-96py\xi-192\eta\xi+1302\xi^2-192y\varphi+1674\varphi^2,$$

$$R_1=(-14.125p^5+216p^4+43.028p^2+186+41.375p^3x'^2-432p^2x'^2$$
$$-12.22x'^2+31.125px'^4)x'^2+(408p^2-192x'^2)x'\xi'+1302\xi'^2+336px'\varphi'$$
$$+1674\varphi'^2+192x'\tau'+2046\tau'^2.$$

$P_1(x)$ is monotone decreasing for $0\leqq x\leqq 1$ and $P_1(1)>0$. Hence $P_1(x)>0$ for $0\leqq x\leqq 1$.

Since $Q_1\geqq Q_0$ for $1\leqq p\leqq 2$, $x'^2/p^2\leqq 1/40$, Q_1 is positive definite for $1\leqq p\leqq 2$, $x'^2/p^2\leqq 1/40$.

R_1 is positive definite for $1\leqq p\leqq 2$, $x'^2/p^2\leqq 1/40$. Indeed we have

$$4.6x'^2+192x'\tau'+2046\tau'^2\geqq 0,$$
$$16.9p^2x'^2+336px'\varphi'+1674\varphi'^2\geqq 0$$

and

$$(-14.125p^5+216p^4+26.128p^2+181.4+41.375p^3x'^2-432p^2x'^2-12.22x'^2$$
$$+31.125px'^4)x'^2+(408p^2-192x'^2)x'\xi'+1302\xi'^2\geqq 0$$

for $1\leqq p\leqq 2$, $x'^2/p^2\leqq 1/40$, which imply the desired result.

Thus we have $\Re a_8\leqq 8$ in the present case with equality holding only for $p=2$.

4. Case $1\leqq p$, $1/40\leqq x'^2/p^2\leqq 1/20$ and $y\geqq 0$. In this case we put $A=1$, $B=5/4$, $C=3/5$ in (A). Then we have

$$\Re a_8\leqq U+\left(-\frac{1}{384}p^5+\frac{15}{16}p^4x+\frac{5}{6}p^2-\frac{673}{192}p^3x'^2-\frac{15}{4}p^2xx'^2-\frac{5}{6}x'^2+\frac{129}{128}px'^4\right)y$$
$$+\left(\frac{3}{64}p^4+\frac{9}{20}p^3x-\frac{27}{160}p^2x'^2-\frac{18}{5}px'^2+\frac{11}{64}x'^4\right)\eta$$
$$+\left(-\frac{97}{48}p^3+\frac{25}{8}p^2+\frac{2}{3}-\frac{7}{8}px'^2\right)y^2+\left(-\frac{7}{8}p^2+3p-\frac{29}{8}x'^2\right)y\eta$$
$$+\left(\frac{69}{100}p+\frac{18}{25}\right)\eta^2+py\xi+\frac{14}{5}\eta\xi+y\varphi-\frac{3}{8}py^3-\frac{1}{5}y^2\eta$$
$$+\left(\frac{113}{64\cdot 12}p^5-\frac{9}{4}p^4-\frac{25}{48}p^2-\frac{331}{64\cdot 12}p^3x'^2+\frac{9}{2}p^2x'^2+\frac{25}{96}x'^2-\frac{83}{256}px'^4\right)x'^2$$
$$+\left(-\frac{17}{4}p^2+2x'^2\right)x'\xi'-\frac{7}{2}px'\varphi'-2x'\tau'-\frac{5}{2}x'y\xi'.$$

Let $\mathfrak{A}$ and $\mathfrak{B}$ be constants, to be fixed later, such that $3p^2-\mathfrak{A}x'^2\geqq 0$ and $9p^2-\mathfrak{B}x'^2\geqq 0$. Applying Lemma 3 to $(3p^4/64-\mathfrak{A}p^2x'^2/64)(\eta+13py/6)$ and Lemma 2 to $(9p^3x/20-\mathfrak{B}pxx'^2/20)(\eta+3py/2)$ we have

$$
\begin{aligned}
\Re a_8 \leqq\ & \frac{2}{7}+\frac{1}{8}p+\frac{2}{5}p^2+\frac{19}{15}p^3-\frac{19}{480}p^4+\frac{49}{30}p^5-\frac{51}{32}p^6+\left(\frac{19}{40}+\frac{27}{448}-\frac{269}{96\cdot 16}\right)p^7 \\
& +\left\{-\frac{73}{240}p^5+\frac{21}{40}p^4-\frac{1}{4}p^3+\frac{5}{6}p^2+\left(\frac{5}{384}\mathfrak{A}-\frac{3}{40}\mathfrak{B}+\frac{83}{192}\right)p^3x'^2+\left(\frac{3}{20}\mathfrak{B}\right.\right. \\
& \left.\left.-\frac{15}{2}\right)p^2x'^2+\frac{1}{12}\mathfrak{A}px'^2-\frac{5}{6}x'^2+\left(\frac{63}{64}-\frac{1}{16}\mathfrak{A}\right)px'^4+\frac{\mathfrak{A}}{128}\frac{x'^6}{p}\right\}y \\
& +\left\{\left(\frac{1}{64}\mathfrak{A}-\frac{1}{20}\mathfrak{B}-\frac{9}{320}\right)p^2+\left(\frac{1}{10}\mathfrak{B}-\frac{18}{5}\right)p+\left(\frac{11}{64}-\frac{3}{64}\mathfrak{A}\right)x'^2\right\}x'^2\eta \\
& +\left\{-\frac{851}{384}p^3+\frac{25}{8}p^2-\frac{3}{32}p+\frac{2}{3}+\left(\frac{25}{384}\mathfrak{A}-\frac{363}{384}\right)px'^2+\frac{\mathfrak{A}}{32}\frac{x'^2}{p}+\frac{3\mathfrak{A}}{128}\frac{x'^4}{p}\right\}y^2 \\
\text{(D)}\quad & +\left\{-\frac{43}{32}p^2+3p+\left(\frac{5}{32}\mathfrak{A}-\frac{29}{8}\right)x'^2\right\}y\eta+\left(\frac{327}{800}p+\frac{18}{25}+\frac{3\mathfrak{A}}{32}\frac{x'^2}{p}\right)\eta^2 \\
& +py\xi+\frac{14}{5}\eta\xi+y\varphi-\frac{3}{8}py^3-\frac{1}{5}y^2\eta \\
& +\left\{\left(\frac{1}{32\cdot 48}\mathfrak{A}-\frac{13}{5\cdot 48}\mathfrak{B}+\frac{221}{96\cdot 80}\right)p^5+\left(\frac{5}{24}\mathfrak{B}-\frac{81}{40}\right)p^4+\left(\frac{1}{18}\mathfrak{A}-\frac{1}{5}\mathfrak{B}-\frac{1}{6}\right)p^3\right. \\
& +\left(\frac{1}{30}\mathfrak{B}-\frac{25}{48}\right)p^2-\left(\frac{2}{9}\mathfrak{A}+\frac{1}{15}\mathfrak{B}\right)p-\frac{\mathfrak{A}}{24}\frac{1}{p}+\left(\frac{1}{32\cdot 16}\mathfrak{A}+\frac{1}{80}\mathfrak{B}-\frac{671}{96\cdot 16}\right)p^3x'^2 \\
& +\left(\frac{9}{2}-\frac{1}{40}\mathfrak{B}\right)p^2x'^2+\frac{1}{18}\mathfrak{A}px'^2+\frac{25}{96}x'^2+\left(\frac{1}{32\cdot 16}\mathfrak{A}-\frac{167}{32\cdot 16}\right)px'^4 \\
& \left.+\frac{\mathfrak{A}}{96\cdot 16}\frac{x'^6}{p}\right\}x'^2 \\
& +\left(-\frac{17}{4}p^2+2x'^2\right)x'\xi'-\frac{7}{2}px'\varphi'-2x'\tau'-\frac{5}{2}x'y\xi'.
\end{aligned}
$$

Case i) $1.9\leqq p$.

Since $|a_2|\leqq 2$ we have $p<1.98$ in this case. Applying the trivial inequality $2AXY\leqq|A|\alpha X^2+|A|\alpha' Y^2$, $\alpha'=1/\alpha$, $\alpha>0$ we have

$$
\begin{aligned}
& \left(-\frac{73}{240}p^5+\frac{21}{40}p^4+\frac{17}{12}p^3-\frac{5}{2}p^2\right)y=\frac{1}{240}p^2(73p^2+20p-300)xy \\
& \leqq\frac{1}{480}\alpha p^2(73p^2+20p-300)x^2+\frac{1}{480\alpha}p^2(73p^2+20p-300)y^2.
\end{aligned}
$$

Hence from (D) we have

$$
\begin{aligned}
\Re a_8 \leqq 8 - \frac{x}{96}\{&160+(266-25.6\alpha)x+(1244.2+275.2\alpha)x^2-(2064.7\\
&+314.4\alpha)x^3+(1224.95+120.8\alpha)x^4-(331.025+14.6\alpha)x^5+34x^6\}\\
&+\left\{-\frac{5}{3}p^3+\frac{10}{3}p^2+\left(\frac{5}{384}\mathfrak{A}-\frac{3}{40}\mathfrak{B}+\frac{83}{192}\right)p^3x'^2+\left(\frac{3}{20}\mathfrak{B}-\frac{15}{2}\right)p^2x'^2\right.\\
&\qquad\left.+\frac{1}{12}\mathfrak{A}px'^2-\frac{5}{6}x'^2+\left(\frac{63}{64}-\frac{1}{16}\mathfrak{A}\right)px'^4+\frac{\mathfrak{A}}{128}\frac{x'^6}{p}\right\}y\\
&+\left\{\left(\frac{1}{64}\mathfrak{A}-\frac{1}{20}\mathfrak{B}-\frac{9}{320}\right)p^2+\left(\frac{1}{10}\mathfrak{B}-\frac{18}{5}\right)p+\left(\frac{11}{64}-\frac{3}{64}\mathfrak{A}\right)x'^2\right\}x'^2\eta\\
&-\frac{1}{96}\left[\left\{-\frac{73}{5\alpha}p^4+\left(212.75-\frac{4}{\alpha}\right)p^3-\left(300-\frac{60}{\alpha}\right)p^2+9p-64\right.\right.\\
\text{(E)}\qquad&\qquad\left.+(90.75-6.25\mathfrak{A})px'^2-3\mathfrak{A}\frac{x'^2}{p}-2.25\mathfrak{A}\frac{x'^4}{p}\right\}y^2\\
&\qquad+\{129p^2-288p+(348-15\mathfrak{A})x'^2\}y\eta+\left(-39.24p-69.12-9\mathfrak{A}\frac{x'^2}{p}\right)\eta^2\\
&\qquad\left.-96py\xi-268.8\eta\xi-96y\varphi\right]\\
&-\frac{1}{96}\left[\left\{(-0.0625\mathfrak{A}+5.2\mathfrak{B}-2.7625)p^5+(194.4-20\mathfrak{B})p^4+(16-5.3333\mathfrak{A}+19.2\mathfrak{B})p^3\right.\right.\\
&\qquad+(50-3.2\mathfrak{B})p^2+(21.3334\mathfrak{A}+6.4\mathfrak{B})p+4\mathfrak{A}\frac{1}{p}\\
&\qquad+(41.9375-0.1875\mathfrak{A}-1.2\mathfrak{B})p^3x'^2+(2.4\mathfrak{B}-432)p^2x'^2-5.3333\mathfrak{A}px'^2\\
&\qquad\left.-25x'^2+(31.3125-0.1875\mathfrak{A})px'^4-0.0625\mathfrak{A}\frac{x'^6}{p}\right\}x'^2\\
&\qquad\left.+(408p^2-192x'^2)x'\xi'+336px'\varphi'+192x'\tau'\right]-\frac{3}{8}py^3-\frac{1}{5}y^2\eta-\frac{5}{2}x'y\xi'.
\end{aligned}
$$

Here we put $\mathfrak{A}=-70$, $\mathfrak{B}=180$, $\alpha=16$. We remark the following facts:

$$
\begin{aligned}
&\left(-\frac{5}{3}p^3+\frac{10}{3}p^2-\frac{671}{48}p^3x'^2+\frac{39}{2}p^2x'^2-\frac{35}{6}px'^2-\frac{5}{6}x'^2+\frac{343}{64}px'^4-\frac{35}{64}\frac{x'^6}{p}\right)y\\
&\qquad+\frac{23}{5}p^3x'^2y\leqq 0,
\end{aligned}
$$

$$
\left(-\frac{3239}{320}p^2+\frac{72}{5}p+\frac{221}{64}x'^2\right)x'^2\eta\leqq\left(\frac{3239}{320}p^2-\frac{72}{5}p-\frac{221}{64}x'^2\right)\sqrt{\frac{4-p^2}{5}}\,x'^2
$$

$$\leqq \frac{1}{96}(271.39581p^2-386.10432p-92.58795x'^2)x'^2,$$

$$-\frac{23}{5}p^3x'^2y-\frac{5}{2}x'y\xi'\leqq\frac{125}{368p^3}y\xi'^2\leqq\frac{125}{368p^3}\sqrt{\frac{4-p^2}{3}}\xi'^2\leqq\frac{1.72}{96}\xi'^2,$$

$$-\frac{3}{8}py^3-\frac{1}{5}y^2\eta\leqq\frac{2}{75p}y\eta^2\leqq\frac{2}{75p}\sqrt{\frac{4-p^2}{3}}\eta^2\leqq\frac{0.49}{96}\eta^2.$$

Making use of these remarks and applying Lemma 1 to the term $-160x/96$ we have

$$\Re a_8\leqq 8-\frac{x^2}{96}P(x)-\frac{1}{96}Q-\frac{1}{96}R,$$

$$P(x)=-103.6+5647.4x-7095.1x^2+3157.75x^3-564.625x^4+34x^5,$$

$$Q=\left(-0.9125p^4+212.5p^3-296.25p^2+9p+56+528.25px'^2+210\frac{x'^2}{p}+157.5\frac{x'^4}{p}\right)y^2$$
$$+(129p^2-288p+1398x'^2)y\eta+\left(-39.24p+130.39+630\frac{x'^2}{p}\right)\eta^2$$
$$-96py\xi-268.8\eta\xi+280\xi^2-96y\varphi+360\varphi^2,$$

$$R=\left(937.6125p^5-3405.6p^4+3845.331p^3-797.396p^2+44.766p+40-280\frac{1}{p}\right.$$
$$\left.-160.938p^3x'^2+373.331px'^2+67.587x'^2+44.437px'^4+4.375\frac{x'^6}{p}\right)x'^2$$
$$+(408p^2-192x'^2)x'\xi'+278.28\xi'^2+336px'\varphi'+360\varphi'^2+192x'\tau'+440\tau'^2.$$

$P(x)$ is monotone increasing for $0.02\leqq x\leqq 0.1$ and $P(0.02)>0$. Hence $P(x)>0$ for $0.02\leqq x\leqq 0.1$.

Since $6.4y^2-96y\varphi+360\varphi^2\geqq 0$ and $20.95x'^2+192x'\tau'+440\tau'^2\geqq 0$ we consider $Q^*=Q-6.4y^2+96y\varphi-360\varphi^2$ and $R^*=R-20.95x'^2-192x'\tau'-440\tau'^2$. We can prove the positive definiteness of the symmetric matrices of Q^* and R^* for $1.9\leqq p\leqq 1.98$, $1/40\leqq x'^2/p^2\leqq 1/20$ by taking their principal diagonal minor determinants. Hence Q and R are positive definite there.

Thus we have $\Re a_8<8$ for $1.9\leqq p\leqq 1.98$, $1/40\leqq x'^2/p^2\leqq 1/20$, $y\geqq 0$.

Case ii) $1.7\leqq p\leqq 1.9$.

Applying the trivial inequality we have

$$\left(-\frac{73}{240}p^5+\frac{21}{40}p^4+\frac{71}{60}p^3-\frac{61}{30}p^2\right)y=\frac{1}{240}p^2(73p^2+20p-244)xy$$

$$\leqq\frac{1}{480}\alpha p^2(73p^2+20p-244)x^2+\frac{1}{480\alpha}p^2(73p^2+20p-244)y^2.$$

Hence from (D) we have

$$\begin{aligned}
\Re a_8 \leqq 8 &- \frac{x}{96}\{160+(266-70.4\alpha)x+(1244.2+320\alpha)x^2-(2064.7\\
&+325.6\alpha)x^3+(1224.95+120.8\alpha)x^4-(331.025+14.6\alpha)x^5+34x^6\}\\
&+\left\{-\frac{43}{30}p^3+\frac{43}{15}p^2+\left(\frac{5}{384}\mathfrak{A}-\frac{3}{40}\mathfrak{B}+\frac{83}{192}\right)p^3x'^2+\left(\frac{3}{20}\mathfrak{B}-\frac{15}{2}\right)p^2x'^2\right.\\
&\left.+\frac{1}{12}\mathfrak{A}px'^2-\frac{5}{6}x'^2+\left(\frac{63}{64}-\frac{1}{16}\mathfrak{A}\right)px'^4+\frac{\mathfrak{A}}{128}\frac{x'^6}{p}\right\}y\\
&+\left\{\left(\frac{1}{64}\mathfrak{A}-\frac{1}{20}\mathfrak{B}-\frac{9}{320}\right)p^2+\left(\frac{1}{10}\mathfrak{B}-\frac{18}{5}\right)p+\left(\frac{11}{64}-\frac{3}{64}\mathfrak{A}\right)x'^2\right\}x'^2\eta\\
&-\frac{1}{96}\left[\left\{-\frac{73}{5\alpha}p^4+\left(212.75-\frac{4}{\alpha}\right)p^3-\left(300-\frac{244}{5\alpha}\right)p^2+9p-64\right.\right.\\
&\left.+(90.75-6.25\mathfrak{A})px'^2-3\mathfrak{A}\frac{x'^2}{p}-2.25\mathfrak{A}\frac{x'^4}{p}\right\}y^2\\
\text{(F)}\qquad &+\{129p^2-288p+(348-15\mathfrak{A})x'^2\}y\eta+\left(-39.24p-69.12-9\mathfrak{A}\frac{x'^2}{p}\right)\eta^2\\
&\left.-96py\xi-268.8\eta\xi-96y\varphi\right]\\
&-\frac{1}{96}\left[\left\{(-0.0625\mathfrak{A}+5.2\mathfrak{B}-2.7625)p^5+(194.4-20\mathfrak{B})p^4+(16-5.3333\mathfrak{A}+19.2\mathfrak{B})p^3\right.\right.\\
&+(50-3.2\mathfrak{B})p^2+(21.3334\mathfrak{A}+6.4\mathfrak{B})p+4\mathfrak{A}\frac{1}{p}\\
&+(41.9375-0.1875\mathfrak{A}-1.2\mathfrak{B})p^3x'^2+(2.4\mathfrak{B}-432)p^2x'^2-5.3333\mathfrak{A}px'^2\\
&\left.-25x'^2+(31.3125-0.1875\mathfrak{A})px'^4-0.0625\mathfrak{A}\frac{x'^6}{p}\right\}x'^2\\
&\left.+(408p^2-192x'^2)x'\xi'+336px'\varphi'+192x'\tau'\right]\\
&-\frac{3}{8}py^3-\frac{1}{5}y^2\eta-\frac{5}{2}x'y\xi'.
\end{aligned}$$

Here we put $\mathfrak{A}=-30$, $\mathfrak{B}=180$, $\alpha=4$. We remark the following facts:

$$\left(-\frac{43}{30}p^3+\frac{43}{15}p^2-\frac{323}{24}p^3x'^2+\frac{39}{2}p^2x'^2-\frac{5}{2}px'^2-\frac{5}{6}x'^2+\frac{183}{64}px'^4\right.$$

$$-\frac{15}{64}\frac{x'^6}{p}\Big)y+3p^3x'^2y\leqq 0,$$

$$\left(-\frac{3039}{320}p^2+\frac{72}{5}p+\frac{101}{64}x'^2\right)x'^2\eta\leqq\left(\frac{3039}{320}p^2-\frac{72}{5}p-\frac{101}{64}x'^2\right)\sqrt{\frac{4-p^2}{5}}x'^2$$

$$\leqq\frac{1}{96}(429.59304p^2-651.38688p-71.3868x'^2)x'^2,$$

$$-3p^3x'^2y-\frac{5}{2}x'y\xi'\leqq\frac{25}{48p^3}y\xi'^2\leqq\frac{25}{48p^3}\sqrt{\frac{4-p^2}{3}}\xi'^2\leqq\frac{6.2}{96}\xi'^2,$$

$$-\frac{3}{8}py^3-\frac{1}{5}y^2\eta\leqq\frac{2}{75p}y\eta^2\leqq\frac{0.92}{96}\eta^2.$$

Using these remarks and applying Lemma 1 to the term $-160x/96-248x^2/96$ we have

$$\Re a_8\leqq 8-\frac{x^2}{96}P(x)-\frac{1}{96}Q-\frac{1}{96}R,$$

$$P(x)=-223.6+2586.2x-3367.1x^2+1708.15x^3-389.425x^4+34x^5,$$

$$Q=\left(-3.65p^4+211.75p^3-287.8p^2-177p+428+278.25px'^2+90\frac{x'^2}{p}+67.5\frac{x'^4}{p}\right)y^2$$

$$+(129p^2-288p+798x'^2)y\eta+\left(-349.24p+749.96+270\frac{x'^2}{p}\right)\eta^2$$

$$-96py\xi-268.8\eta\xi+(1148-434p)\xi^2-96y\varphi+(1476-558p)\varphi^2,$$

$$R=\Big(935.1125p^5-3405.6p^4+3631.999p^3-955.594p^2+1101.384p+164-120\frac{1}{p}$$

$$-168.4375p^3x'^2+159.999px'^2+46.386x'^2+36.937px'^4+1.875\frac{x'^6}{p}\Big)x'^2$$

$$+(408p^2-192x'^2)x'\xi'+(1141.8-434p)\xi'^2+336px'\varphi'+(1476-558p)\varphi'^2$$

$$+192x'\tau'+(1804-682p)\tau'^2.$$

$P(x)$ is monotone increasing for $0.1\leqq x\leqq 0.3$ and $P(0.1)>0$. Hence $P(x)>0$ for $0.1\leqq x\leqq 0.3$.

Since $(6.7p-7)y^2-96y\varphi+(1476-558p)\varphi^2\geqq 0$ and $(22.2p-23.4)x'^2+192x'\tau'+(1804-682p)\tau'^2\geqq 0$ for $1.7\leqq p\leqq 1.9$ we consider $Q^*=Q-(6.7p-7)y^2+96y\varphi-(1476-558p)\varphi^2$ and $R^*=R-(22.2p-23.4)x'^2-192x'\tau'-(1804-682p)\tau'^2$. We can prove the positive definiteness of the symmetric matrices of Q^* and R^* for $1.7\leqq p\leqq 1.9$, $1/40\leqq x'^2/p^2\leqq 1/20$ by taking their principal diagonal minor determinants. Hence Q and R are positive definite there.

Thus we have $\Re a_8<8$ for $1.7\leqq p\leqq 1.9$, $1/40\leqq x'^2/p^2\leqq 1/20$, $y\geqq 0$.

Case iii) $1.4\leqq p\leqq 1.7$.

Applying the trivial inequality we have

$$\left(-\frac{73}{240}p^5+\frac{21}{40}p^4-\frac{1}{4}p^3+\frac{5}{6}p^2\right)y=\frac{1}{240}p^2(73p^2+20p+100)xy$$

$$\leqq\frac{1}{480}\alpha p^2(73p^2+20p+100)x^2+\frac{1}{480\alpha}p^2(73p^2+20p+100)y^2.$$

Hence from (D) we have

$$\begin{aligned}
\Re a_8\leqq 8-\frac{x}{96}&\{160+(266-345.6\alpha)x+(1244.2+595.2\alpha)x^2-(2064.7+394.4\alpha)x^3\\
&+(1224.95+120.8\alpha)x^4-(331.025+14.6\alpha)x^5+34x^6\}\\
&+\left\{\left(\frac{5}{384}\mathfrak{A}-\frac{3}{40}\mathfrak{B}+\frac{83}{192}\right)p^3+\left(\frac{3}{20}\mathfrak{B}-\frac{15}{2}\right)p^2+\frac{1}{12}\mathfrak{A}p-\frac{5}{6}\right.\\
&\quad\left.+\left(\frac{63}{64}-\frac{1}{16}\mathfrak{A}\right)px'^2+\frac{\mathfrak{A}}{128}\frac{x'^4}{p}\right\}x'^2y\\
&+\left\{\left(\frac{1}{64}\mathfrak{A}-\frac{1}{20}\mathfrak{B}-\frac{9}{320}\right)p^2+\left(\frac{1}{10}\mathfrak{B}-\frac{18}{5}\right)p+\left(\frac{11}{64}-\frac{3}{64}\mathfrak{A}\right)x'^2\right\}x'^2\eta\\
&-\frac{1}{96}\left[\left\{-\frac{73}{5\alpha}p^4+\left(212.75-\frac{4}{\alpha}\right)p^3-\left(300+\frac{20}{\alpha}\right)p^2+9p-64\right.\right.\\
&\qquad\left.+(90.75-6.25\mathfrak{A})px'^2-3\mathfrak{A}\frac{x'^2}{p}-2.25\mathfrak{A}\frac{x'^4}{p}\right\}y^2\\
&\qquad+\{129p^2-288p+(348-15\mathfrak{A})x'^2\}y\eta+\left(-39.24p-69.12-9\mathfrak{A}\frac{x'^2}{p}\right)\eta^2\\
&\qquad\left.-96py\xi-268.8\eta\xi-96y\varphi\right]\\
&-\frac{1}{96}\left[\left\{(-0.0625\mathfrak{A}+5.2\mathfrak{B}-2.7625)p^5+(194.4-20\mathfrak{B})p^4+(16-5.3333\mathfrak{A}+19.2\mathfrak{B})p^3\right.\right.\\
&\qquad+(50-3.2\mathfrak{B})p^2+(21.3334\mathfrak{A}+6.4\mathfrak{B})p+4\mathfrak{A}\frac{1}{p}\\
&\qquad+(41.9375-0.1875\mathfrak{A}-1.2\mathfrak{B})p^3x'^2+(2.4\mathfrak{B}-432)p^2x'^2-5.3333\mathfrak{A}px'^2\\
&\qquad\left.-25x'^2-(0.1875\mathfrak{A}-31.3125)px'^4-0.0625\mathfrak{A}\frac{x'^6}{p}\right\}x'^2
\end{aligned}\tag{G}$$

$$+(408p^2-192x'^2)x'\xi'+336px'\varphi'+192x'\tau'\Big]$$

$$-\frac{3}{8}py^3-\frac{1}{5}y^2\eta-\frac{5}{2}x'y\xi'.$$

Here we put $\mathfrak{A}=-25$, $\mathfrak{B}=40$, $\alpha=1.1$. We remark the following facts:

$$\left(-\frac{1111}{384}p^3-\frac{3}{2}p^2-\frac{25}{12}p-\frac{5}{6}+\frac{163}{64}px'^2-\frac{25}{128}\frac{x'^4}{p}\right)x'^2y+\frac{7}{10}p^3x'^2y\leqq 0,$$

$$\left(-\frac{387}{160}p^2+\frac{2}{5}p+\frac{43}{32}x'^2\right)x'^2\eta\leqq\left(\frac{387}{160}p^2-\frac{2}{5}p-\frac{43}{32}x'^2\right)\sqrt{\frac{4-p^2}{5}}x'^2$$

$$\leqq\frac{1}{96}(148.32936p^2-24.52992p-82.4052x'^2)x'^2,$$

$$-\frac{7}{10}p^3x'^2y-\frac{5}{2}x'y\xi'\leqq\frac{125}{56p^3}y\xi'^2\leqq\frac{125}{56p^3}\sqrt{\frac{4-p^2}{3}}\xi'^2\leqq\frac{64.4}{96}\xi'^2,$$

$$-\frac{3}{8}py^3-\frac{1}{5}y^2\eta\leqq\frac{2}{75p}y\eta^2\leqq\frac{2}{75p}\sqrt{\frac{4-p^2}{3}}\eta^2\leqq\frac{1.51}{96}\eta^2.$$

Making use of these remarks and applying Lemma 1 to the term $-160x/96-329.2x^2/96$ we have

$$\mathfrak{R}a_8\leqq 8-\frac{x^2}{96}P(x)-\frac{1}{96}Q-\frac{1}{96}R,$$

$$P(x)=-403.36+1981.22x-2498.54x^2+1357.83x^3-347.085x^4+34x^5,$$

$$Q=\left(-13.28p^4+209.11p^3-318.19p^2-237.9p+549.8+247px'^2+75\frac{x'^2}{p}+56.25\frac{x'^4}{p}\right)y^2$$

$$+(129p^2-288p+723x'^2)y\eta+\left(-450.74p+952.37+225\frac{x'^2}{p}\right)\eta^2-96py\xi$$

$$-268.8\eta\xi+(1432.2-576.1p)\xi^2-96y\varphi+(1841.4-740.7p)\varphi^2,$$

$$R=\Big(206.8p^5-605.6p^4+917.3325p^3-226.33p^2-335.106p+204.6-100\frac{1}{p}$$

$$-1.375p^3x'^2-336p^2x'^2+133.3325px'^2+57.4052x'^2+36px'^4+1.5625\frac{x'^6}{p}\Big)x'^2$$

$$+(408p^2-192x'^2)x'\xi'+(1367.8-576.1p)\xi'^2+336px'\varphi'$$

$$+(1841.4-740.7p)\varphi'^2+192x'\tau'+(2250.6-905.3p)\tau'^2.$$

$P(x)$ is monotone increasing for $0.3\leqq x\leqq 0.6$ and $P(0.3)>0$. Hence $P(x)>0$ for

$0.3 \leqq x \leqq 0.6$.

Since $(5.8p-5.2)y^2-96y\varphi+(1841.4-740.7p)\varphi^2 \geqq 0$ and $(19.3p-17.6)x'^2+192x'\tau'+(2250.6-905.3p)\tau'^2 \geqq 0$ for $1.4 \leqq p \leqq 1.7$ we consider $Q^*=Q-(5.8p-5.2)y^2+96y\varphi-(1841.4-740.7p)\varphi^2$ and $R^*=R-(19.3p-17.6)x'^2-192x'\tau'-(2250.6-905.3p)\tau'^2$. We can prove the positive definiteness of the symmetric matrices of Q^* and R^* for $1.4 \leqq p \leqq 1.7$, $1/40 \leqq x'^2/p^2 \leqq 1/20$ by taking their principal diagonal minor determinants. Hence Q and R are positive definite there.

Thus we have $\Re a_8<8$ for $1.4 \leqq p \leqq 1.7$, $1/40 \leqq x'^2/p^2 \leqq 1/20$, $y \geqq 0$.

Case iv) $1 \leqq p \leqq 1.4$.

In this case we start from (G) with $\mathfrak{A}=0$, $\mathfrak{B}=60$, $\alpha=1.1$. We remark the following facts:

$$\left(-\frac{781}{192}p^3+\frac{3}{2}p^2-\frac{5}{6}+\frac{63}{64}px'^2\right)x'^2y+\frac{29}{10}p^3x'^2y \leqq 0,$$

$$\left(-\frac{969}{320}p^2+\frac{12}{5}p+\frac{11}{64}x'^2\right)x'^2\eta \leqq \left(\frac{969}{320}p^2-\frac{12}{5}p-\frac{11}{64}x'^2\right)\sqrt{\frac{4-p^2}{5}}\,x'^2$$

$$\leqq \frac{1}{96}(225.17622p^2-178.46784p-12.7809x'^2)x'^2,$$

$$-\frac{29}{10}p^3x'^2y-\frac{5}{2}x'y\xi' \leqq \frac{125}{232p^3}y\xi'^2 \leqq \frac{125}{232p^3}\sqrt{\frac{4-p^2}{3}}\,\xi'^2 \leqq \frac{51.8}{96}\xi'^2,$$

$$-\frac{3}{8}py^3-\frac{1}{5}y^2\eta \leqq \frac{2}{75p}y\eta^2 \leqq \frac{2}{75p}\sqrt{\frac{4-p^2}{3}}\,\eta^2 \leqq \frac{2.56}{96}\eta^2.$$

Making use of these remarks and applying Lemma 1 to the term $-160x/96-490x^2/96$ we have

$$\Re a_8 \leqq 8-\frac{x^2}{96}P(x)-\frac{1}{96}Q-\frac{1}{96}R,$$

$$P(x)=-564.16+2021.42x-2498.54x^2+1357.83x^3-347.085x^4+34x^5,$$

$$Q=(-13.28p^4+209.11p^3-318.19p^2-358.5p+791+90.75x'^2)y^2$$
$$+(129p^2-288p+348x'^2)y\eta+(-651.74p+1353.32)\eta^2-96py\xi$$
$$-268.8\eta\xi+(1995-857.5p)\xi^2-96y\varphi+(2565-1102.5p)\varphi^2,$$

$$R=(309.2375p^5-1005.6p^4+1168p^3-367.177p^2+439.967p+285$$
$$-30.063p^3x'^2-288p^2x'^2-12.22x'^2+31.3125px'^4)x'^2$$
$$+(408p^2-192x'^2)x'\xi'+(1943.2-857.5p)\xi'^2+336px'\varphi'$$
$$+(2565-1102.5p)\varphi'^2+192x'\tau'+(3135-1347.5p)\tau'^2.$$

$P'(x)$ is monotone decreasing for $0.6\leqq x\leqq 1$ and $P'(0.6)>0$, $P'(1)<0$. Further $P(0.6)>0$, $P(1)>0$. Hence $P(x)>0$ for $0.6\leqq x\leqq 1$.

Since $(4.8p-3.2)y^2-96y\varphi+(2565-1102.5p)\varphi^2\geqq 0$ and $(15.8p-10.6)x'^2+192x'\tau'+(3135-1347.5p)\tau'^2\geqq 0$ for $1\leqq p\leqq 1.4$ we consider $Q^*=Q-(4.8p-3.2)y^2+96y\varphi-(2565-1102.5p)\varphi^2$ and $R^*=R-(15.8p-10.6)x'^2-192x'\tau'-(3135-1347.5p)\tau'^2$. We can prove the positive definteness of the symmetric matrices of Q^* and R^* for $1\leqq p\leqq 1.4$, $1/40\leqq x'^2/p^2\leqq 1/20$ by taking their principal diagonal minor determinants. Hence Q and R are positive definite there.

Thus we have $\Re a_8<8$ for $1\leqq p\leqq 1.4$, $1/40\leqq x'^2/p^2\leqq 1/20$, $y\geqq 0$.

5. Case $1\leqq p$, $1/40\leqq x'^2/p^2\leqq 1/20$ and $y\leqq 0$. In this case we put $A=1/2$, $B=3/2$, $C=3/5$ in (A).

Applying the trivial inequality $2AXY\leqq|A|X^2+|A|Y^2$ and Lemma 1 we have

$$\left(-\frac{413}{384}p^5+\frac{934}{384}p^4+\frac{5}{12}p^2+\frac{151}{192}p^3x'^2-9p^2x'^2-\frac{5}{12}x'^2+\frac{149}{128}px'^4\right)y-2x'y\xi'$$

$$\leqq-\frac{y}{384}(413p^5-934p^4-160p^2-302p^3x'^2+3456p^2x'^2+160x'^2+384x'^2-447px'^4+384\xi'^2)$$

$$\leqq-\frac{y}{384}(413p^5-934p^4-214.86p^2+219.44-302p^3x'^2+3456p^2x'^2+489.14x'^2$$

$$-447px'^4)\leqq 0$$

for $1\leqq p\leqq 1.98$, $1/40\leqq x'^2/p^2\leqq 1/20$, $y\leqq 0$. Then we have

$$\begin{aligned}\Re a_8\leqq\, & U-\frac{35}{192}p^4y+\left(-\frac{129}{320}p^4+\frac{9}{10}p^3-\frac{27}{160}p^2x'^2-\frac{18}{5}px'^2+\frac{11}{64}x'^4\right)\eta\\ &+\left(-\frac{127}{48}p^3+\frac{9}{2}p^2+\frac{1}{6}-\frac{35}{16}px'^2\right)y^2+\left(-\frac{13}{10}p^2+\frac{18}{5}p-\frac{9}{2}x'^2\right)y\eta\\ &+\left(\frac{69}{100}p+\frac{18}{25}\right)\eta^2+py\xi+\frac{14}{5}\eta\xi+2y\varphi+\frac{3}{20}y^2\eta \qquad \text{(H)}\\ &+\left(\frac{113}{64\cdot 12}p^5-\frac{9}{4}p^4-\frac{25}{48}p^2-\frac{331}{64\cdot 12}p^3x'^2+\frac{9}{2}p^2x'^2+\frac{25}{96}x'^2-\frac{83}{256}px'^4\right)x'^2\\ &+\left(-\frac{17}{4}p^2+2x'^2\right)x'\xi'-\frac{7}{2}px'\varphi'-2x'\tau'-2x'y\xi'.\end{aligned}$$

Case i) $\eta\geqq 0$. By Lemma 1,

$$\left(-\frac{129}{320}p^4+\frac{111}{320}p^3-\frac{27}{160}p^2x'^2-\frac{18}{5}px'^2+\frac{11}{64}x'^4\right)\eta+\frac{3}{20}y^2\eta$$

$$\leqq\frac{\eta}{320}(-129p^4+111p^3-54p^2x'^2-1152px'^2+55x'^4+64-16p^2-16x'^2)\leqq 0.$$

By Lemma 2

$$\frac{177}{320}p^3\eta-\frac{35}{192}p^4y\leqq\frac{9}{16}p^3\left(\eta-\frac{1}{3}py\right)\leqq-\frac{13}{256}p^6+\frac{1}{128}p^5+\frac{3}{8}p^3+\frac{9}{64}p^4x'^2.$$

Further by Lemma 1

$$-\frac{300}{96}x\leqq-\frac{75}{96}(x^2+x'^2+3y^2+5\eta^2+7\xi^2+7\xi'^2+9\varphi^2+9\varphi'^2+11\tau'^2),$$

$$-\frac{538.8}{96}x^2\leqq-\frac{134.7}{96}x(x^2+x'^2+3y^2+5\eta^2+7\xi^2+7\xi'^2+9\varphi^2+9\varphi'^2+11\tau'^2).$$

Making use of these inequalities and (H) we have

$$\Re a_8\leqq 8-\frac{x^2}{96}P_0(x)-\frac{1}{96}Q_0-\frac{1}{96}R_0,$$

$$P_0(x)=1402.2-2801.3x+2011x^2-745.05x^3+146.425x^4-13x^5,$$

$$\begin{aligned}Q_0=&(254p^3-432p^2-404.1p+1017.2+210px'^2)y^2+(124.8p^2-345.6p+432x'^2)y\eta\\&+(-739.74p+1652.88)\eta^2-96py\xi-268.8\eta\xi+(2410.8-942.9p)\xi^2\\&-192y\varphi+(3099.6-1212.3p)\varphi^2,\end{aligned}$$

$$\begin{aligned}R_0=&(-14.125p^5+202.5p^4+50p^2-134.7p+344.4+41.375p^3x'^2\\&\qquad -432p^2x'^2-25x'^2+31.125px'^4)x'^2\\&+(408p^2-192x'^2)x'\xi'+(2410.8-942.9p)\xi'^2+336px'\varphi'\\&+(3099.6-1212.3p)\varphi'^2+192x'\tau'+(3788.4-1481.7p)\tau'^2.\end{aligned}$$

$P_0(x)$ is monotone decreasing for $0.02\leqq x\leqq 1$ and $P_0(1)>0$. Hence $P_0(x)>0$ for $0.02\leqq x\leqq 1$.

Q_0 is positive definite for $1\leqq p\leqq 1.98$, $1/40\leqq x'^2/p^2\leqq 1/20$. Indeed we have

$$(8.77p-3.88)y^2-192y\varphi+(3099.6-1212.3p)\varphi^2\geqq 0,$$

$$51.61\eta^2-268.8\eta\xi+350\xi^2\geqq 0,$$

$$(-2.45p+57.71)y^2-96py\xi+(2060.8-942.9p)\xi^2\geqq 0$$

and

$$\begin{aligned}&(254p^3-432p^2-410.42p+963.37+210px'^2)y^2\\&+(124.8p^2-345.6p+432x'^2)y\eta+(-739.74p+1601.27)\eta^2\geqq 0\end{aligned}$$

for $1\leqq p\leqq 1.98$, $1/40\leqq x'^2/p^2\leqq 1/20$. This implies the positive definiteness of Q_0 for

$1 \leqq p \leqq 1.98$, $1/40 \leqq x'^2/p^2 \leqq 1/20$.

R_0 is positive definite for $1 \leqq p \leqq 1.98$, $1/40 \leqq x'^2/p^2 \leqq 1/20$. Indeed we have

$$(7.18p-3.18)x'^2+192x'\tau'+(3788.4-1481.7p)\tau'^2 \geqq 0,$$

$$(-23.29p+213.9)x'^2+336px'\varphi'+(3099.6-1212.3p)\varphi'^2 \geqq 0$$

and

$$(-14.125p^5+202.5p^4+50p^2-118.59p+133.68+41.375p^3x'^2-432p^2x'^2$$
$$-25x'^2+31.125px'^4)x'^2+(408p^2-192x'^2)x'\xi'+(2410.8-942.9p)\xi'^2 \geqq 0$$

for $1 \leqq p \leqq 1.98$, $1/40 \leqq x'^2/p^2 \leqq 1/20$, which imply the desired result.

Thus we have $\Re a_8 < 8$ in the present case.

Case ii) $\eta \leqq 0$. By Lemma 1

$$\left(-\frac{129}{320}p^4+\frac{129}{160}p^3-\frac{27}{160}p^2x'^2-\frac{18}{5}px'^2+\frac{11}{64}x'^4\right)\eta$$
$$\leqq -\frac{\eta}{320}(54p^2+1152p-55x'^2)$$
$$\leqq \frac{1}{96}(12.549p^2+267.702p-12.78x'^2)x'^2.$$

By Lemma 2

$$\frac{3}{32}p^3\eta-\frac{70}{384}p^4y \leqq \frac{3}{32}p^3(\eta-2py)$$
$$\leqq -\frac{31}{512}p^6+\frac{27}{256}p^5+\frac{1}{16}p^3+\frac{3}{128}p^4x'^2.$$

Further by Lemma 1

$$-\frac{510}{96}x \leqq -\frac{127.5}{96}(x^2+x'^2+3y^2+5\eta^2+7\xi^2+7\xi'^2+9\varphi^2+9\varphi'^2+11\tau'^2),$$

$$-\frac{357.6}{96}x^2 \leqq -\frac{89.4}{96}x(x^2+x'^2+3y^2+5\eta^2+7\xi^2+7\xi'^2+9\varphi^2+9\varphi'^2+11\tau'^2).$$

Making use of these remarks and (H) we have

$$\Re a_8 \leqq 8-\frac{x^2}{96}P_1(x)-\frac{1}{96}Q_1-\frac{1}{96}R_1,$$

$$P_1(x)=1290.9-2651.6x+1973.5x^2-746.925x^3+147.3625x^4-13x^5,$$

$$Q_1=(254p^3-432p^2-268.2p+902.9+210px'^2)y^2+(124.8p^2-345.6p+432x'^2)y\eta$$
$$+(-513.24p+1462.38)\eta^2-96py\xi-268.8\eta\xi+(2144.1-625.8p)\xi^2$$
$$-192y\varphi+(2756.7-804.6p)\varphi^2,$$

$$R_1=(-14.125p^5+213.75p^4+37.451p^2-357.102p+306.3$$
$$+41.375p^3x'^2-432p^2x'^2-12.22x'^2+31.125px'^4)x'^2$$
$$+(408p^2-192x'^2)x'\xi'+(2144.1-625.8p)\xi'^2+336px'\varphi'$$
$$+(2756.7-804.6p)\varphi'^2+192x'\tau'+(3369.3-983.4p)\tau'^2.$$

$P_1(x)$ is monotone decreasing for $0.02\leqq x\leqq 1$ and $P_1(1)>0$. Hence $P_1(x)>0$ for $0.02\leqq x\leqq 1$.

Since $Q_1\geqq Q_0$ for $1\leqq p\leqq 1.98$, $1/40\leqq x'^2/p^2\leqq 1/20$, Q_1 is positive definite there. Here Q_0 was defined in Case i) and was positive definite for $1\leqq p\leqq 1.98$, $1/40\leqq x'^2/p^2\leqq 1/20$.

R_1 is positive definite for $1\leqq p\leqq 1.98$, $1/40\leqq x'^2/p^2\leqq 1/20$. Indeed we have

$$(2.71p+1.16)x'^2+192x'\tau'+(3369.3-983.4p)\tau'^2\geqq 0,$$

$$(-35.08p+168.55)x'^2+336px'\varphi'+(2756.7-804.6p)\varphi'^2\geqq 0$$

and

$$(-14.125p^5+213.75p^4+37.451p^2-324.732p+136.59+41.375p^3x'^2-432p^2x'^2$$
$$-12.22x'^2+31.125px'^4)x'^2+(408p^2-192x'^2)x'\xi'+(2144.1-625.8p)\xi'^2\geqq 0$$

for $1\leqq p\leqq 1.98$, $1/40\leqq x'^2/p^2\leqq 1/20$, which imply the desired result.

Thus we have $\Re a_8<8$ in the present case.

6. Case $1\leqq p$, $1/20\leqq x'^2/p^2\leqq 1/10$ and $y\geqq 0$.

Case i) $1.9\leqq p$.

Since $|a_2|\leqq 2$ we have $p<1.952$ in this case. We start from (E) with $\mathfrak{A}=-21$, $\mathfrak{B}=90$, $\alpha=15$. We remark the following facts:

$$\left(-\frac{5}{3}p^3+\frac{10}{3}p^2-\frac{1243}{192}p^3x'^2+6p^2x'^2-px'^2-\frac{5}{6}x'^2+\frac{111}{64}px'^4\right.$$
$$\left.-\frac{3}{32}\frac{x'^6}{p}\right)y+\frac{61}{20}p^3x'^2y\leqq 0,$$

$$\left(-\frac{1509}{320}p^2+\frac{27}{5}p+\frac{47}{64}x'^2\right)x'^2\eta\leqq\left(\frac{1509}{320}p^2-\frac{27}{5}p-\frac{47}{64}x'^2\right)\sqrt{\frac{4-p^2}{5}}x'^2$$
$$\leqq\frac{1}{96}(126.43911p^2-144.78912p-19.69065x'^2)x'^2,$$

$$-\frac{61}{20}p^3x'^2y-\frac{5}{2}x'y\xi'\leqq\frac{125}{244p^3}y\xi'^2\leqq\frac{125}{244p^3}\sqrt{\frac{4-p^2}{3}}\xi'^2\leqq\frac{7.8}{96}\xi'^2,$$

$$-\frac{3}{8}py^3-\frac{1}{5}y^2\eta\leqq\frac{2}{75p}y\eta^2\leqq\frac{0.49}{96}\eta^2.$$

Making use of these remarks and applying Lemma 1 to the term $-160x/96-164x^2/96$ we have

$$\Re a_8\leqq 8-\frac{x^2}{96}P(x)-\frac{1}{96}Q-\frac{1}{96}R,$$

$$P(x)=-242+5413.2x-6780.7x^2+3036.95x^3-550.025x^4+34x^5,$$

$$Q=\left(-0.98p^4+212.48p^3-296p^2-114p+302+165.75px'^2+36\frac{x'^2}{p}+27\frac{x'^4}{p}\right)y^2$$
$$+(129p^2-288p+528x'^2)y\eta+\left(-244.24p+540.39+108\frac{x'^2}{p}\right)\eta^2$$
$$-96py\xi-268.8\eta\xi+(854-287p)\xi^2-96y\varphi+(1098-369p)\varphi^2,$$

$$R=\left(465.9875p^5-1605.6p^4+1807.9996p^3-364.4392p^2+423.7883p+122-48\frac{1}{p}\right.$$
$$\left.-63.8125p^3x'^2-216p^2x'^2+63.9996px'^2-5.3094x'^2+33.5625px'^4+0.75\frac{x'^6}{p}\right)x'^2$$
$$+(408p^2-192x'^2)x'\xi'+(846.2-287p)\xi'^2+336px'\varphi'$$
$$+(1098-369p)\varphi'^2+192x'\tau'+(1342-451p)\tau'^2.$$

$P(x)$ is monotone increasing for $0.048\leqq x\leqq 0.1$ and $P(0.048)>0$. Hence $P(x)>0$ for $0.048\leqq x\leqq 0.1$.

Since $(5.9p-5.4)y^2-96y\varphi+(1098-369p)\varphi^2\geqq 0$ and $(19.5p-18)x'^2+192x'\tau'+(1342-451p)\tau'^2\geqq 0$ for $1.9\leqq p<1.952$ we consider $Q^*=Q-(5.9p-5.4)y^2+96y\varphi-(1098-369p)\varphi^2$ and $R^*=R-(19.5p-18)x'^2-192x'\tau'-(1342-451p)\tau'^2$. We can prove the positive definiteness of the symmetric matrices of Q^* and R^* for $1.9\leqq p<1.952$, $1/20\leqq x'^2/p^2\leqq 1/10$ by taking their principal diagonal minor determinants. Hence Q and R are positive definite there.

Thus we have $\Re a_8<8$ for $1.9\leqq p<1.952$, $1/20\leqq x'^2/p^2\leqq 1/10$, $y\geqq 0$.

Case ii) $1.7\leqq p\leqq 1.9$

We start from (F) with $\mathfrak{A}=0$, $\mathfrak{B}=90$, $\alpha=4$. We remark the following facts:

$$\left(-\frac{43}{30}p^3+\frac{43}{15}p^2-\frac{1213}{192}p^3x'^2+6p^2x'^2-\frac{5}{6}x'^2+\frac{63}{64}px'^4\right)y+\frac{109}{100}p^3x'^2y\leqq 0,$$

$$\left(-\frac{1449}{320}p^2+\frac{27}{5}p+\frac{11}{64}x'^2\right)x'^2\eta\leqq\left(\frac{1449}{320}p^2-\frac{27}{5}p-\frac{11}{64}x'^2\right)\sqrt{\frac{4-p^2}{5}}x'^2$$

$$\leqq\frac{1}{96}(204.83064p^2-244.27008p-7.7748x'^2)x'^2,$$

$$-\frac{109}{100}p^3x'^2y-\frac{5}{2}x'y\xi'\leqq\frac{625}{436p^3}y\xi'^2\leqq\frac{625}{436p^3}\sqrt{\frac{4-p^3}{3}}\xi'^2\leqq\frac{17.04}{96}\xi'^2,$$

$$-\frac{3}{8}py^3-\frac{1}{5}y^2\eta\leqq\frac{2}{75p}y\eta^2\leqq\frac{0.92}{96}\eta^2.$$

Making use of these remarks and applying Lemma 1 to the term $-160x/96$ $-248x^2/96$ we have

$$\Re a_8\leqq 8-\frac{x^2}{96}P(x)-\frac{1}{96}Q-\frac{1}{96}R,$$

$$P(x)=-223.6+2586.2x-3367.1x^2+1708.15x^3-389.425x^4+34x^5,$$

$$\begin{aligned}Q=&(-3.65p^4+211.75p^3-287.8p^2-177p+428+90.75x'^2)y^2\\&+(129p^2-288p+348x'^2)y\eta+(-349.24p+749.96)\eta^2-96py\xi\\&-268.8\eta\xi+(1148-434p)\xi^2-96y\varphi+(1476-558p)\varphi^2,\end{aligned}$$

$$\begin{aligned}R=&(465.2375p^5-1605.6p^4+1744p^3-442.83064p^2+758.27008p+164\\&-66.0625p^3x'^2-216p^2x'^2-17.2252x'^2+31.3125px'^4)x'^2\\&+(408p^2-192x'^2)x'\xi'+(1130.96-434p)\xi'^2+336px'\varphi'\\&+(1476-558p)\varphi'^2+192x'\tau'+(1804-682p)\tau'^2.\end{aligned}$$

$P(x)$ is monotone increasing for $0.1\leqq x\leqq 0.3$ and $P(0.1)>0$. Hence $P(x)>0$ for $0.1\leqq x\leqq 0.3$.

Since $(6.7p-7)y^2-96y\varphi+(1476-558p)\varphi^2\geqq 0$ and $(22.2p-23.4)x'^2+192x'\tau'+(1804-682p)\tau'^2\geqq 0$ for $1.7\leqq p\leqq 1.9$ we consider $Q^*=Q-(6.7p-7)y^2+96y\varphi-(1476-558p)\varphi^2$ and $R^*=R-(22.2p-23.4)x'^2-192x'\tau'-(1804-682p)\tau'^2$. We can prove the positive definiteness of the symmetric matrices of Q^* and R^* for $1.7\leqq p\leqq 1.9$, $1/20\leqq x'^2/p^2\leqq 1/10$ by taking their principal diagonal minor determinants. Hence Q and R are positive definite there.

Thus we have $\Re a_8<8$ for $1.7\leqq p\leqq 1.9$, $1/20\leqq x'^2/p^2\leqq 1/10$, $y\geqq 0$.

Case iii) $1.4\leqq p\leqq 1.7$

In this case we apply the same process as in the case $1.4\leqq p\leqq 1.7$, $1/40\leqq x'^2/p^2\leqq 1/20$, $y\geqq 0$. Then we have

$$\Re a_8 \leqq 8 - \frac{x^2}{96} P(x) - \frac{1}{96} Q - \frac{1}{96} R,$$

where $P(x)$, Q and R are the same as them in the case $1.4 \leqq p \leqq 1.7$, $1/40 \leqq x'^2/p^2 \leqq 1/20$, $y \geqq 0$. It is not so difficult to prove the positive definiteness of Q and R for $1.4 \leqq p \leqq 1.7$, $1/20 \leqq x'^2/p^2 \leqq 1/10$. Thus we have $\Re a_8 < 8$ in the present case.

Case iv) $1 \leqq p \leqq 1.4$.

We apply the same process as in the case $1 \leqq p \leqq 1.4$, $1/40 \leqq x'^2/p^2 \leqq 1/20$, $y \geqq 0$ and we have

$$\Re a_8 \leqq 8 - \frac{x^2}{96} P(x) - \frac{1}{96} Q - \frac{1}{96} R,$$

Where $P(x)$, Q and R are the same as them in the case $1 \leqq p \leqq 1.4$, $1/40 \leqq x'^2/p^2 \leqq 1/20$, $y \geqq 0$. It is not so difficult to prove the positive definiteness of Q and R for $1 \leqq p \leqq 1.4$, $1/20 \leqq x'^2/p^2 \leqq 1/10$. Thus we have $\Re a_8 < 8$ in the present case.

7. Case $1 \leqq p$, $1/20 \leqq x'^2/p^2 \leqq 1/10$ and $y \leqq 0$. In this case we put $A = 1/2$, $B = 3/2$, $C = 3/5$ in (A). Then we have

$$\text{(I)} \quad \begin{aligned} \Re a_8 \leqq U &+ \left(-\frac{413}{384} p^5 + \frac{9}{4} p^4 + \frac{5}{12} p^2 + \frac{151}{192} p^3 x'^2 - 9 p^2 x'^2 - \frac{5}{12} x'^2 + \frac{149}{128} p x'^4 \right) y \\ &+ \left(-\frac{129}{320} p^4 + \frac{9}{10} p^3 - \frac{27}{160} p^2 x'^2 - \frac{18}{5} p x'^2 + \frac{11}{64} x'^4 \right) \eta \\ &+ \left(-\frac{127}{48} p^3 + \frac{9}{2} p^2 + \frac{1}{6} - \frac{35}{16} p x'^2 \right) y^2 + \left(-\frac{13}{10} p^2 + \frac{18}{5} p - \frac{9}{2} x'^2 \right) y\eta \\ &+ \left(\frac{69}{100} p + \frac{18}{25} \right) \eta^2 + p y \xi + \frac{14}{5} \eta \xi + 2 y \varphi + \frac{3}{20} y^2 \eta \\ &+ \left(\frac{113}{64 \cdot 12} p^5 - \frac{9}{4} p^4 - \frac{25}{48} p^2 - \frac{331}{64 \cdot 12} p^3 x'^2 + \frac{9}{2} p^2 x'^2 + \frac{25}{96} x'^2 - \frac{83}{256} p x'^4 \right) x'^2 \\ &+ \left(-\frac{17}{4} p^2 + 2 x'^2 \right) x' \xi' - \frac{7}{2} p x' \varphi' - 2 x' \tau' - 2 x' y \xi'. \end{aligned}$$

Case i) $\eta \geqq 0$. We remark the following facts:

$$\left(-\frac{413}{384} p^5 + \frac{1061}{384} p^4 + \frac{5}{12} p^2 + \frac{151}{192} p^3 x'^2 - 9 p^2 x'^2 - \frac{5}{12} x'^2 + \frac{149}{128} p x'^4 \right) y - 2 x' y \xi' \leqq 0,$$

$$\left(-\frac{129}{320} p^4 + \frac{141}{320} p^3 - \frac{27}{160} p^2 x'^2 - \frac{18}{5} p x'^2 + \frac{11}{64} x'^4 \right) \eta + \frac{3}{20} y^2 \eta \leqq 0.$$

Further by Lemma 2

$$\frac{147}{320}p^3\eta-\frac{197}{384}p^4y\leqq\frac{15}{32}p^3\left(\eta-\frac{10}{9}py\right)$$

$$\leqq-\frac{1145}{13824}p^6+\frac{605}{6912}p^5+\frac{5}{16}p^3+\frac{15}{128}p^4x'^2.$$

Making use of these remarks and applying Lemma 1 to the term $-249.2x/96$ $-591.2x^2/96$ we have, from (I)

$$\Re a_8\leqq 8-\frac{x^2}{96}P_0(x)-\frac{1}{96}Q_0-\frac{1}{96}R_0,$$

$$P_0(x)=1499.2-2980.4x+2119.05x^2-774.32x^3+149.5x^4-12.04x^5,$$

$$Q_0=(254p^3-432p^2-443.4p+1057.7+210px'^2)y^2+(124.8p^2$$
$$-345.6p+432x'^2)y\eta+(-805.24p+1720.38)\eta^2-96py\xi-268.8\eta\xi$$
$$+(2505.3-1034.6p)\xi^2-192y\varphi+(3221.1-1330.2p)\varphi^2,$$

$$R_0=(-14.125p^5+204.75p^4+50p^2-147.8p+357.9+41.375p^3x'^2$$
$$-432p^2x'^2-25x'^2+31.125px'^4)x'^2$$
$$+(408p^2-192x'^2)x'\xi'+(2505.3-1034.6p)\xi'^2+336px'\varphi'$$
$$+(3221.1-1330.2p)\varphi'^2+192x'\tau'+(3936.9-1625.8p)\tau'^2.$$

$P_0(x)$ is monotone decreasing for $0.048\leqq x\leqq 1$ and $P_0(1)>0$. Hence $P_0(x)>0$ for $0.048\leqq x\leqq 1$.

Q_0 is positive definite for $1\leqq p\leqq 1.952$, $1/20\leqq x'^2/p^2\leqq 1/10$. Indeed we have

$$(11.56p-6.68)y^2-192y\varphi+(3221.1-1330.2p)\varphi^2\geqq 0,$$

$$60.22\eta^2-268.8\eta\xi+300\xi^2\geqq 0,$$

$$(-2.23p+51.68)y^2-96py\xi+(2205.3-1034.6p)\xi^2\geqq 0$$

and

$$(254p^3-432p^2-452.73p+1012.7+210px'^2)y^2$$
$$+(124.8p^2-345.6p+432x'^2)y\eta+(-805.24p+1660.16)\eta^2\geqq 0$$

for $1\leqq p\leqq 1.952$, $1/20\leqq x'^2/p^2\leqq 1/10$. This implies the positive definiteness of Q_0 for $1\leqq p\leqq 1.952$, $1/20\leqq x'^2/p^2\leqq 1/10$.

R_0 is positive definite for $1\leqq p\leqq 1.952$, $1/20\leqq x'^2/p^2\leqq 1/10$. Indeed we have

$$(9.46p-5.47)x'^2+192x'\tau'+(3936.9-1625.8p)\tau'^2\geqq 0,$$

$$(-21.22p+213.64)x'^2+336px'\varphi'+(3221.1-1330.2p)\varphi'^2\geqq 0,$$

$$(-14.125p^5+204.75p^4+50p^2-136.04p+149.73+41.375p^3x'^2$$
$$-432p^2x'^2-25x'^2+31.125px'^4)x'^2+(408p^2-192x'^2)x'\xi'$$
$$+(2505.3-1034.6p)\xi'^2\geqq 0$$

for $1\leqq p\leqq 1.952$, $1/20\leqq x'^2/p^2\leqq 1/10$, which imply the desired result.
Thus we have $\Re a_8<8$ in the present case.

Case ii) $\eta\leqq 0$, $1.6\leqq p\leqq 1.952$. We remark the following facts:

$$\left(-\frac{413}{384}p^5+\frac{9}{4}p^4+\frac{5}{12}p^2+\frac{151}{192}p^3x'^2-9p^2x'^2-\frac{5}{12}x'^2+\frac{149}{128}px'^4\right)y-2x'y\xi'$$
$$\leqq -\frac{y}{384}\left(3828p^3-5184p^2-1659.46+2194.4\frac{1}{p^2}-447px'^2\right)x'^2$$
$$\leqq \frac{1}{96}\left(663.106p^3-897.998p^2-287.459+380.125\frac{1}{p^2}-77.431px'^2\right)x'^2,$$

$$\left(-\frac{129}{320}p^4+\frac{9}{10}p^3-\frac{27}{160}p^2x'^2-\frac{18}{5}px'^2+\frac{11}{64}x'^4\right)\eta$$
$$\leqq -\frac{\eta}{320}(1344p^2-1728p-55x'^2)x'^2$$
$$\leqq \frac{1}{96}(216.398p^2-278.225p-8.855x'^2)x'^2.$$

Making use of these remarks and applying Lemma 1 to the term $-744x/96-560x^2/96$ we have, from (I)

$$\Re a_8\leqq 8-\frac{x^2}{96}P_1(x)-\frac{1}{96}Q_1-\frac{1}{96}R_1,$$
$$P_1(x)=598-2082x+1726x^2-687.3x^3+141.55x^4-13x^5,$$
$$Q_1=(254p^3-432p^2-420p+1382+210px'^2)y^2+(124.8p^2-345.6p$$
$$+432x'^2)y\eta+(-766.24p+2260.88)\eta^2-96py\xi-268.8\eta\xi$$
$$+(3262-980p)\xi^2-192y\varphi+(4194-1260p)\varphi^2,$$
$$R_1=\Big(-14.125p^5+216p^4-663.106p^3+731.6p^2+138.225p+753.459$$
$$-380.125\frac{1}{p^2}+41.375p^3x'^2-432p^2x'^2+77.431px'^2-33.855x'^2$$

$$+31.125px'^4\Big)x'^2+(408p^2-192x'^2)x'\xi'+(3262-980p)\xi'^2$$

$$+336px'\varphi'+(4194-1260p)\varphi'^2+192x'\tau'+(5126-1540p)\tau'^2.$$

$P_1(x)$ is monotone decreasing for $0.048\leqq x\leqq 0.4$ and $P_1(0.4)>0$. Hence $P_1(x)>0$ for $0.048\leqq x\leqq 0.4$.

Since $Q_1\geqq Q_0$ for $1.6\leqq p\leqq 1.952$, $1/20\leqq x'^2/p^2\leqq 1/10$, Q_1 is positive definite there. Here Q_0 was defined in Case i) and was positive definite for $1\leqq p\leqq 1.952$, $1/20\leqq x'^2/p^2\leqq 1/10$.

R_1 is positive definite for $1.6\leqq p\leqq 1.952$, $1/20\leqq x'^2/p^2\leqq 1/10$. Indeed we have

$$(2.6p-0.69)x'^2+192x'\tau'+(5126-1540p)\tau'^2\geqq 0;$$

$$(-22.4p+105.73)x'^2+336px'\varphi'+(4194-1260p)\varphi'^2\geqq 0$$

and

$$x'^2\Big(-14.125p^5+216p^4-663.106p^3+731.6p^2+158.025p+648.419$$

$$-380.125\frac{1}{p^2}+41.375p^3x'^2-432p^2x'^2+77.431px'^2-33.855x'^2$$

$$+31.125px'^4\Big)+(408p^2-192x'^2)x'\xi'+(3262-980p)\xi'^2\geqq 0$$

for $1.6\leqq p\leqq 1.952$, $1/20\leqq x'^2/p^2\leqq 1/10$. This implies the desired result.

Thus we have $\Re a_8<8$ in the present case.

Case iii) $\eta\leqq 0$, $1\leqq p\leqq 1.6$. We remark the following facts:

$$\Big(-\frac{413}{384}p^5+\frac{924}{384}p^4+\frac{5}{12}p^2+\frac{151}{192}p^3x'^2-9p^2x'^2-\frac{5}{12}x'^2$$

$$+\frac{149}{128}px'^4\Big)y-2x'y\xi'\leqq 0,$$

$$\Big(-\frac{129}{320}p^4+\frac{207}{320}p^3-\frac{27}{160}p^2x'^2-\frac{18}{5}px'^2+\frac{11}{64}x'^4\Big)\eta$$

$$\leqq -\frac{\eta}{160}(27p^2+576p-27.5x'^2)x'^2$$

$$\leqq \frac{1}{96}(12.549p^2+267.702p-12.78x'^2)x'^2.$$

Further by Lemma 2

$$\frac{27}{128}p^3\eta-\frac{5}{32}p^4y=\frac{27}{128}p^3\left(\eta-\frac{20}{27}py\right)$$

$$\leqq-\frac{1141}{55296}p^6+\frac{9}{64}p^3+\frac{169}{27648}p^5+\frac{27}{512}p^4x'^2.$$

Making use of these remarks and applying Lemma 1 to the term $-572.6x/96-292.88x^2/96$, from (I) we have

$$\Re a_8\leqq 8-\frac{x^2}{96}P_2(x)-\frac{1}{96}Q_2-\frac{1}{96}R_2,$$

$$P_2(x)=1169.74-2428.76x+1838.98x^2-710.49x^3+143.53x^4-13x^5,$$

$$\begin{aligned}Q_2=&(254p^3-432p^2-219.66p+852.77+210px'^2)y^2\\&+(124.8p^2-345.6p+432x'^2)y\eta+(-432.34p+1378.83)\eta^2\\&-96py\xi-268.8\eta\xi+(2027.13-512.54p)\xi^2-192y\varphi\\&+(2606.31-658.98p)\varphi^2,\end{aligned}$$

$$\begin{aligned}R_2=&(-14.125p^5+210.9375p^4+37.451p^2-340.922p+289.59\\&+41.375p^3x'^2-432p^2x'^2-12.22x'^2+31.125px'^4)x'^2\\&+(408p^2-192x'^2)x'\xi'+(2027.13-512.54p)\xi'^2+336px'\varphi'\\&+(2606.31-658.98p)\varphi'^2+192x'\tau'+(3185.49-805.42p)\tau'^2.\end{aligned}$$

$P_2(x)$ is monotone decreasing for $0.4\leqq x\leqq 1$ and $P_2(1)=0$. Hence $P_2(x)\geqq 0$ for $0.4\leqq x\leqq 1$.

Since $Q_2\geqq Q_0$ for $1\leqq p\leqq 1.6$, $1/20\leqq x'^2/p^2\leqq 1/10$, Q_2 is positive definite there. Here Q_0 was defined in Case i) and was positive definite for $1\leqq p\leqq 1.952$, $1/20\leqq x'^2/p^2\leqq 1/10$.

R_2 is positive definite for $1\leqq p\leqq 1.6$, $1/20\leqq x'^2/p^2\leqq 1/10$. Indeed we have

$$(1.98p+1.9)x'^2+192x'\tau'+(3185.49-805.42p)\tau'^2\geqq 0,$$

$$(-42.83p+115.09)x'^2+336px'\varphi'+(2606.31-658.98p)\varphi'^2\geqq 0$$

and

$$\begin{aligned}&(-14.125p^5+210.9375p^4+37.451p^2-300.072p+172.6\\&+41.375p^3x'^2-432p^2x'^2-12.22x'^2+31.125px'^4)x'^2\\&+(408p^2-192x'^2)x'\xi'+(2027.13-512.54p)\xi'^2\geqq 0\end{aligned}$$

for $1\leqq p\leqq 1.6$, $1/20\leqq x'^2/p^2\leqq 1/10$, which imply the desired result.

Thus we have $\Re a_8<8$ in the present case.

8. Case $1\leqq p$, $1/10\leqq x'^2/p^2\leqq\tan^2(\pi/7)$ and $y\geqq 0$. It is evident that $p<1.907$ and $x'^2/p^2<0.2379$ in this case.

Case i) Case $1.8\leqq p<1.907$.

We start from (D) with $\mathfrak{A}=-21.2$, $\mathfrak{B}=37.8$, and use the following trivial inequality

$$\left(-\frac{73}{240}p^5+\frac{21}{40}p^4+\frac{13}{10}p^3-\frac{34}{15}p^2\right)y$$

$$=\frac{1}{240}p^2(73p^2+20p-272)xy$$

$$\leqq\frac{1}{80}p^2(73p^2+20p-272)x^2+\frac{1}{2880}p^2(73p^2+20p-272)y^2.$$

Then we have

$$\begin{aligned}\mathfrak{R}a_8\leqq 8&-\frac{x}{96}(160-22x+3029.8x^2-3984.7x^3+1949.75x^4-418.625x^5+34x^6)\\&+\left(-\frac{31}{20}p^3+\frac{31}{10}p^2-\frac{257.16}{96}p^3x'^2-1.83p^2x'^2-\frac{10.6}{6}px'^2-\frac{5}{6}x'^2\right.\\&\qquad\left.+\frac{73.9}{32}px'^4-\frac{5.3}{32}\frac{x'^6}{p}\right)y\\&+\left(-\frac{359.9}{160}p^2+0.18p+\frac{37.3}{32}x'^2\right)x'^2\eta\\&-\frac{1}{96}\left\{\left(-2.434p^4+212.08p^3-290.94p^2+9p-64+223.25px'^2+63.6\frac{x'^2}{p}\right.\right.\\&\qquad\left.+47.7\frac{x'^4}{p}\right)y^2+(129p^2-288p+666x'^2)y\eta+\left(-39.24p-69.12\right.\\&\qquad\left.\left.+190.8\frac{x'^2}{p}\right)\eta^2-96py\xi-268.8\eta\xi-96y\varphi\right\}\\&-\frac{1}{96}\left\{\left(195.1225p^5-561.6p^4+854.8259p^3-70.96p^2-210.349p\right.\right.\\&\qquad-84.8\frac{1}{p}+0.5525p^3x'^2-341.28p^2x'^2+113.065px'^2-25x'^2\\&\qquad\left.\left.+35.287px'^4+1.325\frac{x'^6}{p}\right)x'^2+(408p^2-192x'^2)x'\xi'+336px'\varphi'+192x'\tau'\right\}\\&-\frac{3}{8}py^3-\frac{1}{5}y^2\eta-\frac{5}{2}x'y\xi'.\end{aligned}$$

We remark the following facts:

$$\left(-\frac{31}{20}p^3+\frac{31}{10}p^2-\frac{257.16}{96}p^3x'^2-1.83p^2x'^2-\frac{10.6}{6}px'^2-\frac{5}{6}x'^2\right.$$

$$\left.+\frac{73.9}{32}px'^4-\frac{5.3}{32}\frac{x'^6}{p}\right)y+\frac{33}{10}p^3x'^2y\leqq 0,$$

$$\left(-\frac{359.9}{160}p^2+0.18p+\frac{37.3}{32}x'^2\right)x'^2\eta\leqq\left(\frac{359.9}{160}p^2-0.18p-\frac{37.3}{32}x'^2\right)\sqrt{\frac{4-p^2}{5}}x'^2$$

$$\leqq\frac{1}{96}(84.1951p^2-6.7374p-43.6298x'^2)x'^2,$$

$$-\frac{33}{10}p^3x'^2y-\frac{5}{2}x'y\xi'\leqq\frac{125}{264p^3}y\xi'^2\leqq\frac{125}{264p^3}\sqrt{\frac{4-p^2}{3}}\xi'^2\leqq\frac{3.93}{96}\xi'^2,$$

$$-\frac{3}{8}py^3-\frac{1}{5}y^2\eta\leqq\frac{2}{75p}y\eta^2\leqq\frac{0.72}{96}\eta^2.$$

Making use of these remarks and applying Lemma 1 to the term $-160x/96-272.8x^2/96$ we have

$$\Re a_8\leqq 8-\frac{x^2}{96}P(x)-\frac{1}{96}Q-\frac{1}{96}R,$$

$$P(x)=-254.8+3098x-3984.7x^2+1949.75x^3-418.625x^4+34x^5,$$

$$Q=\left(-2.434p^4+212.08p^3-290.94p^2-195.6p+465.2+223.25px'^2\right.$$

$$\left.+63.6\frac{x'^2}{p}+47.7\frac{x'^4}{p}\right)y^2+(129p^2-288p+666x'^2)y\eta$$

$$+\left(-380.24p+812.16+190.8\frac{x'^2}{p}\right)\eta^2-96py\xi$$

$$-268.8\eta\xi+(1234.8-477.4p)\xi^2-96y\varphi+(1587.6-613.8p)\varphi^2,$$

$$R=\left(195.1225p^5-561.6p^4+854.8259p^3-155.1551p^2-271.8116p\right.$$

$$+176.4-84.8\frac{1}{p}+0.5525p^3x'^2-341.28p^2x'^2$$

$$\left.+113.065px'^2+18.6298x'^2+35.287px'^4+1.325\frac{x'^6}{p}\right)x'^2$$

$$+(408p^2-192x^2)x'\xi'+(1230.87-477.4p)\xi'^2+336px'\varphi'$$

$$+(1587.6-613.8p)\varphi'^2+192x'\tau'+(1940.4-750.2p)\tau'^2.$$

$P(x)$ is monotone increasing for $0.093\leqq x\leqq 0.2$ and $P(0.093)>0$. Hence $P(x)>0$ for $0.093\leqq x\leqq 0.2$.

Since $(8.1p-9.8)y^2-96y\varphi+(1587.6-613.8p)\varphi^2\geqq 0$ and $(26.7p-32.4)x'^2+192x'\tau'+(1940.4-750.2p)\tau'^2\geqq 0$ for $1.8\leqq p<1.907$ we consider $Q^*=Q-(8.1p-9.8)y^2+96y\varphi-(1587.6-613.8p)\varphi^2$ and $R^*=R-(26.7p-32.4)x'^2-192x'\tau'-(1940.4-750.2p)\tau'^2$. We can prove the positive definiteness of the symmetric matrices of Q^* and R^* for $1.8\leqq p<1.907$, $1/10\leqq x'^2/p^2<0.2379$ by taking their principal diagonal minor determinants. Hence Q and R are positive definite there.

Thus we have $\Re a_8<8$ for $1.8\leqq p<1.907$, $1/10\leqq x'^2/p^2\leqq \tan^2(\pi/7)$, $y\geqq 0$.

Case ii) $1.7\leqq p\leqq 1.8$.

In this case we start from (F) with $\mathfrak{A}=0$, $\mathfrak{B}=37.8$, $\alpha=1$. We remark the following facts:

$$\left(-\frac{43}{30}p^3+\frac{43}{15}p^2-\frac{115.33}{48}p^3x'^2-1.83p^2x'^2-\frac{5}{6}x'^2+\frac{63}{64}px'^4\right)y+\frac{5}{2}p^3x'^2y\leqq 0,$$

$$\left(-\frac{306.9}{160}p^2+0.18p+\frac{11}{64}x'^2\right)x'^2\eta\leqq\left(\frac{306.9}{160}p^2-0.18p-\frac{11}{64}x'^2\right)\sqrt{\frac{4-p^2}{5}}\,x'^2$$

$$\leqq\frac{1}{96}(86.766768p^2-8.142336p-7.7748x'^2)x'^2,$$

$$-\frac{5}{2}p^3x'^2y-\frac{5}{2}x'y\xi'\leqq\frac{5}{8p^3}y\xi'^2\leqq\frac{5}{8p^3}\sqrt{\frac{4-p^2}{3}}\,\xi'^2\leqq\frac{7.5}{96}\xi'^2,$$

$$-\frac{3}{8}py^3-\frac{1}{5}y^2\eta\leqq\frac{2}{75p}y\eta^2\leqq\frac{0.92}{96}\eta^2.$$

Making use of these remarks and applying Lemma 1 to the term $-160x/96-484x^2/96$ we have

$$\Re a_8\leqq 8-\frac{x^2}{96}P(x)-\frac{1}{96}Q-\frac{1}{96}R,$$

$$P(x)=-248.4+1685.2x-2390.3x^2+1345.75x^3-345.625x^4+34x^5,$$

$$Q=(-14.6p^4+208.75p^3-251.2p^2-354p+782+90.75px'^2)y^2$$
$$+(129p^2-288p+348x'^2)y\eta+(-644.24p+1339.96)\eta^2-268.8\eta\xi-96py\xi$$
$$+(1974-847p)\xi^2-96y\varphi+(2538-1089p)\varphi^2,$$

$$R=(193.7975p^5-561.6p^4+741.76p^3-157.7268p^2+129.0623p+282$$
$$-3.4225p^3x'^2-341.28p^2x'^2-17.2252x'^2+31.3125px'^4)x'^2$$
$$+(408p^2-192x'^2)x'\xi'+(1966.5-847p)\xi'^2+336px'\varphi'$$
$$+(2538-1089p)\varphi'^2+192x'\tau'+(3102-1331p)\tau'^2.$$

$P(x)$ is monotone increasing for $0.2 \leqq x \leqq 0.3$ and $P(0.2) > 0$. Hence $P(x) > 0$ for $0.2 \leqq x \leqq 0.3$.

Since $(10.14p - 13.88)y^2 - 96y\varphi + (2538 - 1089p)\varphi^2 \geqq 0$ and $(33.3p - 45.6)x'^2 + 192x'\tau' + (3102 - 1331p)\tau'^2 \geqq 0$ for $1.7 \leqq p \leqq 1.8$ we consider $Q^* = Q - (10.14p - 13.88)y^2 + 96y\varphi - (2538 - 1089p)\varphi^2$ and $R^* = R - (33.3p - 45.6)x'^2 - 192x'\tau' - (3102 - 1331p)\tau'^2$. We can prove the positive definiteness of the symmetric matrices of Q^* and R^* for $1.7 \leqq p \leqq 1.8$, $1/10 \leqq x'^2/p^2 \leqq 0.2379$ by taking their principal diagonal minor determinants. Hence Q and R are positive definite there.

Thus we have $\Re a_8 < 8$ for $1.7 \leqq p \leqq 1.8$, $1/10 \leqq x'^2/p^2 \leqq \tan^2(\pi/7)$, $y \geqq 0$.

Case iii) $1.4 \leqq p \leqq 1.7$.

In this case we start from (G) with $\mathfrak{A} = 0$, $\mathfrak{B} = 37.8$, $\alpha = 1.1$. We remark the following facts:

$$\left(-\frac{115.33}{48}p^3 - 1.83p^2 - \frac{5}{6} + \frac{63}{64}px'^2\right)x'^2y + \frac{17}{5}p^3x'^2y \leqq 0,$$

$$\left(-\frac{613.8}{320}p^2 + 0.18p + \frac{11}{64}x'^2\right)x'^2\eta \leqq \left(\frac{613.8}{320}p^2 - 0.18p - \frac{11}{64}x'^2\right)\sqrt{\frac{4-p^2}{5}}\,x'^2$$

$$\leqq \frac{1}{96}(117.628632p^2 - 11.038464p - 10.5402x'^2)x'^2,$$

$$-\frac{17}{5}p^3x'^2y - \frac{5}{2}x'y\xi' \leqq \frac{125}{272p^3}y\xi'^2 \leqq \frac{125}{272p^3}\sqrt{\frac{4-p^2}{3}}\,\xi'^2 \leqq \frac{13.3}{96}\xi'^2,$$

$$-\frac{3}{8}py^3 - \frac{1}{5}y^2\eta \leqq \frac{2}{75p}y\eta^2 \leqq \frac{1.15}{96}\eta^2.$$

Making use of these remarks and applying Lemma 1 to the term $-160x/96 - 329.2x^2/96$ we have

$$\Re a_8 \leqq 8 - \frac{x^2}{96}P(x) - \frac{1}{96}Q - \frac{1}{96}R,$$

$$P(x) = -403.36 + 1981.22x - 2498.54x^2 + 1357.83x^3 - 347.085x^4 + 34x^5,$$

$$Q = (-13.28p^4 + 209.11p^3 - 318.19p^2 - 237.9p + 549.8 + 90.75px'^2)y^2$$
$$+ (129p^2 - 288p + 348x'^2)y\eta + (-450.74p + 952.37)\eta^2 - 96py\xi$$
$$- 268.8\eta\xi + (1432.2 - 576.1p)\xi^2 - 96y\varphi + (1841.4 - 740.7p)\varphi^2,$$

$$R = (193.7975p^5 - 561.6p^4 + 741.76p^3 - 188.5887p^2 + 170.6584p$$
$$+ 204.6 - 3.4225p^3x'^2 - 341.28p^2x'^2 - 14.4598x'^2 + 31.3125px'^4)x'^2$$
$$+ (408p^2 - 192x'^2)x'\xi' + (1418.9 - 576.1p)\xi'^2 + 336px'\varphi'$$
$$+ (1841.4 - 740.7p)\varphi'^2 + 192x'\tau' + (2250.6 - 905.3p)\tau'^2.$$

$P(x)$ is monotone increasing for $0.3 \leqq x \leqq 0.6$ and $P(0.3)>0$. Hence $P(x)>0$ for $0.3 \leqq x \leqq 0.6$.

Since $(5.8p-5.2)y^2-96y\varphi+(1841.4-740.7p)\varphi^2 \geqq 0$ and $(19.3p-17.6)x'^2+192x'\tau'+(2250.6-905.3p)\tau'^2 \geqq 0$ for $1.4 \leqq p \leqq 1.7$ we consider $Q^*=Q-(5.8p-5.2)y^2+96y\varphi-(1841.4-740.7p)\varphi^2$ and $R^*=R-(19.3p-17.6)x'^2-192x'\tau'-(2250.6-905.3p)\tau'^2$. We can prove the positive defineteness of the symmetric matrices of Q^* and R^* for $1.4 \leqq p \leqq 1.7$, $1/10 \leqq x'^2/p^2 < 0.2379$ by taking their principal diagonal minor determinants. Hence Q and R are positive definite there.

Thus we have $\Re a_8 < 8$ for $1.4 \leqq p \leqq 1.7$, $1/10 \leqq x'^2/p^2 \leqq \tan^2(\pi/7)$, $y \geqq 0$.

Case iv) $1 \leqq p \leqq 1.4$.

We start from (G) with $\mathfrak{A}=0$, $\mathfrak{B}=37.8$, $\alpha=1.1$. We remark the following facts:

$$\left(-\frac{115.33}{48}p^3-1.83p^2-\frac{5}{6}+\frac{63}{64}px'^2\right)x'^2y+\frac{37}{10}p^3x'^2y \leqq 0,$$

$$\left(-\frac{613.8}{320}p^2+0.18p+\frac{11}{64}x'^2\right)x'^2\eta$$

$$\leqq \frac{1}{96}(142.634844p^2-13.385088p-12.7809x'^2)x'^2,$$

$$-\frac{37}{10}p^3x'^2y-\frac{5}{2}x'y\xi' \leqq \frac{125}{296p^3}y\xi'^2 \leqq \frac{125}{296p^3}\sqrt{\frac{4-p^2}{3}}\xi'^2 \leqq \frac{40.6}{96}\xi'^2,$$

$$-\frac{3}{8}py^3-\frac{1}{5}y^2\eta \leqq \frac{2}{75p}y\eta^2 \leqq \frac{2}{75p}\sqrt{\frac{4-p^2}{3}}\eta^2 \leqq \frac{2.56}{96}\eta^2.$$

Making use of these remarks and applying Lemma 1 to the term $-160x/96-490x^2/96$ we have

$$\Re a_8 \leqq 8-\frac{x^2}{96}P(x)-\frac{1}{96}Q-\frac{1}{96}R,$$

$$P(x)=-564.16+2021.42x-2498.54x^2+1357.83x^3-347.085x^4+34x^5,$$

$$\begin{aligned} Q=&(-13.28p^4+209.11p^3-318.19p^2-358.5p+791+90.75px'^2)y^2\\ &+(129p^2-288p+348x'^2)y\eta+(-651.74p+1353.32)\eta^2-96py\xi-268.8\eta\xi\\ &+(1995-857.5p)\xi^2-96y\varphi+(2565-1102.5p)\varphi^2, \end{aligned}$$

$$\begin{aligned} R=&(193.7975p^5-561.6p^4+741.76p^3-213.5949p^2+132.805p+285\\ &-3.4225p^3x'^2-341.28p^2x'^2-12.2191x'^2+31.3125px'^4)x'^2\\ &+(408p^2-192x'^2)x'\xi'+(1954.4-857.5p)\xi'^2+336px'\varphi'\\ &+(2565-1102.5p)\varphi'^2+192x'\tau'+(3135-1347.5p)\tau'^2. \end{aligned}$$

$P'(x)$ is monotone decreasing for $0.6 \leqq x \leqq 1$ and $P'(1)<0$, $P'(0.6)>0$. Further $P(1)>0$, $P(0.6)>0$. Hence $P(x)>0$ for $0.6 \leqq x \leqq 1$.

Since $(4.8p-3.2)y^2-96y\varphi+(2565-1102.5p)\varphi^2 \geqq 0$ and $(15.8p-10.6)x'^2+192x'\tau'+(3135-1347.5p)\tau'^2 \geqq 0$ for $1 \leqq p \leqq 1.4$ we consider $Q^*=Q-(4.8p-3.2)y^2+96y\varphi-(2565-1102.5p)\varphi^2$ and $R^*=R-(15.8p-10.6)x'^2-192x'\tau'-(3135-1347.5p)\tau'^2$. We can prove the positive definiteness of the symmetric matrices of Q^* and R^* for $1 \leqq p \leqq 1.4$, $1/10 \leqq x'^2/p^2 < 0.2379$ by taking their principal diagonal minor determinants. Hence Q and R are positive definite there.

Thus we have $\Re a_8 < 8$ for $1 \leqq p \leqq 1.4$, $1/10 \leqq x'^2/p^2 \leqq \tan^2(\pi/7)$, $y \geqq 0$.

9. Case $1 \leqq p$, $1/10 \leqq x'^2/p^2 \leqq \tan^2(\pi/7)$ and $y \leqq 0$. In this case we put $A=1/2$, $B=3/2$, $C=3/5$ in (A).

Applying the trivial inequality $2AXY \leqq |A|\alpha X^2+|A|\alpha' Y^2$, $\alpha'=1/\alpha$, $\alpha>0$ we have

$$\left(-\frac{413}{384}p^5+\frac{9}{4}p^4+\frac{5}{12}p^2+\frac{151}{192}p^3x'^2-9p^2x'^2-\frac{5}{12}x'^2+\frac{149}{128}px'^4\right)y$$

$$\leqq -\frac{1}{384}(1433.839p^3-175.392p^2-512.48-447px'^2)x'^2y$$

$$\leqq \frac{\alpha}{96}(179.23p^3-21.92p^2-64.06-55.87px'^2)y^2$$

$$+\frac{1}{96\alpha}(179.23p^3-21.92p^2-64.06-55.87px'^2)x'^4,$$

$$-2x'y\xi' \leqq \gamma y^2+\frac{1}{\gamma}x'^2\xi'^2, \qquad \gamma>0.$$

Making use of these remarks we have

$$\begin{aligned}
\Re a_8 \leqq\ & 8-\frac{x}{96}(744+972x-2222x^2+1726x^3-687.3x^4+141.55x^5-12.04x^6) \\
&+\left(-\frac{129}{320}p^4+\frac{9}{10}p^3-\frac{27}{160}p^2x'^2-\frac{18}{5}px'^2+\frac{11}{64}x'^4\right)\eta+\frac{3}{20}y^2\eta \\
&-\frac{1}{96}[\{(254-179.23\alpha)p^3-(432-21.92\alpha)p^2-(16-64.06\alpha+96\gamma) \\
&\quad +(210+55.87\alpha)px'^2\}y^2+(124.8p^2-345.6p+432x'^2)y\eta \\
&\quad +(-66.24p-69.12)\eta^2-96py\xi-268.8\eta\xi-192y\varphi] \qquad \text{(J)} \\
&-\frac{1}{96}\left[\left\{-14.125p^5+216p^4+50p^2+\left(41.375-\frac{179.23}{\alpha}\right)p^3x'^2\right.\right.
\end{aligned}$$

$$-\left(432-\frac{21.92}{\alpha}\right)p^2x'^2-\left(25-\frac{64.06}{\alpha}\right)x'^2+\left(31.125+\frac{55.87}{\alpha}\right)px'^4\Bigg\}x'^2$$

$$+(408p^2-192x'^2)x'\xi'-\frac{96}{\gamma}x'^2\xi'^2+336px'\varphi'+192x'\tau'\Bigg].$$

Case i) $1.9\leqq p$.

If $\eta\leqq 0$, we have

$$\left(-\frac{129}{320}p^4+\frac{9}{10}p^3-\frac{27}{160}p^2x'^2-\frac{18}{5}px'^2+\frac{11}{64}x'^4\right)\eta+\frac{3}{20}y^2\eta$$

$$\leqq -\frac{\eta}{320}(596.187p^2-58.464p-55x'^2)x'^2.$$

If $\eta\geqq 0$, by applying Lemma 1 we have

$$\left(-\frac{129}{320}p^4+\frac{9}{10}p^3-\frac{27}{160}p^2x'^2-\frac{18}{5}px'^2+\frac{11}{64}x'^4\right)\eta+\frac{3}{20}y^2\eta$$

$$\leqq \frac{\eta}{320}\left(-1344p^2+1728p-176+640\frac{1}{p^2}+55x'^2\right)x'^2$$

$$\leqq \frac{\eta}{320}(596.187p^2-58.464p-55x'^2)x'^2.$$

Hence we have

$$\left(-\frac{129}{320}p^4+\frac{9}{10}p^3-\frac{27}{160}p^2x'^2-\frac{18}{5}px'^2+\frac{11}{64}x'^4\right)\eta+\frac{3}{20}y^2\eta$$

$$\leqq \frac{\beta}{96}(89.43p^2-8.76p-8.25x'^2)\eta^2+\frac{1}{96\beta}(89.43p^2-8.76p-8.25x'^2)x'^4.$$

Thus, from (J), we have

$$\Re a_8\leqq 8-\frac{x}{96}(744+972x-2222x^2+1726x^3-687.3x^4+141.55x^5-12.04x^6)$$

$$-\frac{1}{96}[\{(254-179.23\alpha)p^3-(432-21.92\alpha)p^2-(16-64.06\alpha+96\gamma)$$

$$+(210+55.87\alpha)px'^2\}y^2+(124.8p^2-345.6p+432x'^2)y\eta$$

$$+\{-89.43\beta p^2+(-66.24+8.76\beta)p-69.12+8.25\beta x'^2\}\eta^2$$

$$-96py\xi-268.8\eta\xi-192y\varphi] \tag{K}$$

$$-\frac{1}{96}\Big[\Big\{-14.125p^5+216p^4+50p^2+\Big(41.375-\frac{179.23}{\alpha}\Big)p^3x'^2$$

$$-\Big(432-\frac{21.92}{\alpha}+\frac{89.43}{\beta}\Big)p^2x'^2+\frac{8.76}{\beta}px'^2-\Big(25-\frac{64.06}{\alpha}\Big)x'^2$$

$$+\Big(31.125+\frac{55.87}{\alpha}\Big)px'^4+\frac{8.25}{\beta}x'^4\Big\}x'^2+(408p^2-192x'^2)x'\xi'$$

$$-\frac{96}{\gamma}x'^2\xi'^2+336px'\varphi'+192x'\tau'\Big].$$

Here we put $\alpha=4/5$, $\beta=12/5$, $\gamma=1/2$. Applying Lemma 1 to the term $-744x/96$ $-976x^2/96$ we have

$$\Re a_8\leqq 8-\frac{x^2}{96}P(x)-\frac{1}{96}Q-\frac{1}{96}R,$$

$$P(x)=182-1978x+1726x^2-687.3x^3+141.55x^4-12.04x^5,$$

$$Q=(110.61p^3-414.47p^2-732p+2009.24+254.69px'^2)y^2$$

$$+(124.8p^2-345.6p+432x'^2)y\eta+(-214.64p^2-1265.22p+3300.88+19.8x'^2)\eta^2$$

$$-96py\xi-268.8\eta\xi-(4718-1708p)\xi^2-192y\varphi+(6066-2196p)\varphi^2,$$

$$R=(-14.125p^5+216p^4+50p^2-244p+674-182.67p^3x'^2-442.17p^2x'^2$$

$$+3.59px'^2+55.07x'^2+100.96px'^4+3.38x'^4)x'^2$$

$$+(408p^2-192x'^2)x'\xi'+(4718-1708p-192x'^2)\xi'^2+336px'\varphi'$$

$$+(6066-2196p)\varphi'^2+192x'\tau'+(7414-2684p)\tau'^2.$$

$P(x)$ is monotone decreasing for $0.093\leqq x\leqq 0.1$ and $P(0.1)>0$. Hence $P(x)>0$ for $0.093\leqq x\leqq 0.1$.

Since for $1.9\leqq p\leqq 1.907$, $1/10\leqq x'^2/p^2\leqq 0.2379$

$$(6.38p-7.25)y^2-192y\varphi+(6066-2196p)\varphi^2\geqq 0,$$

$$22.58\eta^2-268.8\eta\xi+800\xi^2\geqq 0,$$

$$(-1.35p+15.27)y^2-96py\xi+(3918-1708p)\xi^2\geqq 0$$

and

$$(110.61p^3-414.47p^2-737.03p+2001.22+254.69px'^2)y^2$$

$$+(124.8p^2-345.6p+432x'^2)y\eta+(-214.64p^2-1265.22p$$

$$+3278.3+19.8x'^2)\eta^2\geqq 0,$$

Q is positive definite there. Since for $1.9\leqq p\leqq 1.907$, $1/10\leqq x'^2/p^2\leqq 0.2379$

$$(5.23p-5.95)x'^2+192x'\tau'+(7414-2684p)\tau'^2\geqq 0,$$

$$(-12.86p+79.21)x'^2+336px'\varphi'+(6066-2196p)\varphi'^2\geqq 0$$

and

$$\begin{aligned}(-14.125p^5&+216p^4+50p^2-236.37p+600.74-182.67p^3x'^2\\&-442.17p^2x'^2+3.59px'^2+55.07x'^2+100.96px'^4+3.38x'^4)x'^2\\&+(408p^2-192x'^2)x'\xi'+(4718-1708p-192x'^2)\xi'^2\geqq 0,\end{aligned}$$

R is positive definite there.

Thus we have $\Re a_8<8$ for $1.9\leqq p$, $1/10\leqq x'^2/p^2\leqq \tan^2(\pi/7)$, $y\leqq 0$.

Case ii) $1.7\leqq p\leqq 1.9$.

We put $\alpha=3/4$, $\beta=2$, $\gamma=1/2$ in (K). Applying Lemma 1 to the term $-774x/96-680x^2/96$ we have

$$\Re a_8\leqq 8-\frac{x^2}{96}P(x)-\frac{1}{96}Q-\frac{1}{96}R,$$

$$P(x)=478-2052x+1726x^2-687.3x^3+141.55x^4-12.04x^5,$$

$$\begin{aligned}Q=&(119.57p^3-415.56p^2-510p+1562.04+251.9px'^2)y^2\\&+(124.8p^2-345.6p+432x'^2)y\eta+(-178.86p^2-898.72p+2560.88+16.5x'^2)\eta^2\\&-96py\xi-268.8\eta\xi+(3682-1190p)\xi^2-192y\varphi+(4734-1530p)\varphi^2,\end{aligned}$$

$$\begin{aligned}R=&(-14.125p^5+216p^4+50p^2-170p+526-198.8p^3x'^2-447.57p^2x'^2\\&+4.38px'^2+60.19x'^2+105.43px'^4+4.12x'^4)x'^2\\&+(408p^2-192x'^2)x'\xi'+(3682-1190p-192x'^2)\xi'^2+336px'\varphi'\\&+(4734-1530p)\varphi'^2+192x'\tau'+(5786-1870p)\tau'^2.\end{aligned}$$

$P(x)$ is monotone decreasing for $0.1\leqq x\leqq 0.3$ and $P(0.3)>0$. Hence $P(x)>0$ for $0.1\leqq x\leqq 0.3$.

It is not so difficult to prove the positive definiteness of Q and R for $1.7\leqq p\leqq 1.9$, $1/10\leqq x'^2/p^2\leqq 0.2379$.

Therefore we have $\Re a_8<8$ in the present case.

Case iii) $1.4\leqq p\leqq 1.7$.

We put $\alpha=4/5$, $\beta=3$, $\gamma=1/2$ in (K). Applying Lemma 1 to the term $-744x/96-370x^2/96$ we have

$$\Re a_8\leqq 8-\frac{x^2}{96}P(x)-\frac{1}{96}Q-\frac{1}{96}R,$$

$$P(x)=788-2129.5x+1726x^2-687.3x^3+141.55x^4-12.04x^5,$$

$$Q=(110.61p^3-414.47p^2-277.5p+1100.24+254.69px'^2)y^2$$

$$+(124.8p^2-345.6p+432x'^2)y\eta+(-268.29p^2-502.46p+1785.88+24.75x'^2)\eta^2$$

$$-96py\xi-268.8\eta\xi+(2597-647.5p)\xi^2-192y\varphi+(3339-832.5p)\varphi^2,$$

$$R=(-14.125p^5+216p^4+50p^2-92.5p+371-182.67p^3x'^2-434.41p^2x'^2$$

$$+2.92px^2+55.07x'^2+100.96px'^4+2.75x'^4)x'^2$$

$$+(408p^2-192x'^2)x'\xi'+(2597-647.5p-192x'^2)\xi'^2+336px'\varphi'$$

$$+(3339-832.5p)\varphi'^2+192x'\tau'+(4081-1017.5p)\tau'^2.$$

$P(x)$ is monotone decreasing for $0.3\leqq x\leqq 0.6$ and $P(0.6)>0$. Hence $P(x)>0$ for $0.3\leqq x\leqq 0.6$.

It is not so difficult to prove the positive definiteness of Q and R for $1.4\leqq p\leqq 1.7$, $1/10\leqq x'^2/p^2\leqq 0.2379$.

Therefore we have $\Re a_8<8$ in the present case.

Case iv) $1\leqq p\leqq 1.4$.

In this case we put $\alpha=1$, $\gamma=1/2$ in (J). Using the following inequality

$$\left(-\frac{129}{120}p^4+\frac{9}{10}p^3-\frac{27}{160}p^2x'^2-\frac{18}{5}px'^2+\frac{11}{64}x'^4\right)\eta+\frac{3}{20}y^2\eta$$

$$\leqq\frac{|\eta|}{48}(89.43p^2-8.76p+50.7-8.25x'^2)x'^2$$

$$\leqq\frac{1}{32}(89.43p^2-8.76p+50.7-8.25x'^2)\eta^2$$

$$+\frac{1}{144}(89.43p^2-8.76p+50.7-8.25x'^2)x'^4$$

and applying Lemma 1 to the term $-744x/96-138x^2/96$ we have

$$\Re a_8 \leqq 8 - \frac{x^2}{96} P(x) - \frac{1}{96} Q - \frac{1}{96} R,$$

$$P(x) = 1020 - 2187.5x + 1726x^2 - 687.3x^3 + 141.55x^4 - 12.04x^5,$$

$$\begin{aligned} Q = & (74.77p^3 - 410.08p^2 - 103.5p + 765.06 + 265.87px'^2)y^2 \\ & + (124.8p^2 - 345.6p + 432x'^2)y\eta + (-268.29p^2 - 212.46p + 1053.78 + 24.75x'^2)\eta^2 \\ & - 96py\xi - 268.8\eta\xi + (1785 - 241.5p)\xi^2 - 192y\varphi + (2295 - 310.5p)\varphi^2, \end{aligned}$$

$$\begin{aligned} R = & (-14.125p^5 + 216p^4 + 50p^2 - 34.5p + 255 - 137.86p^3x'^2 - 439.89p^2x'^2 \\ & \quad + 2.92px'^2 + 22.16x'^2 + 86.99px'^4 + 2.75x'^4)x'^2 \\ & + (408p^2 - 192x'^2)x'\xi' + (1785 - 241.5p - 192x'^2)\xi'^2 + 336px'\varphi' \\ & + (2295 - 310.5p)\varphi'^2 + 192x'\tau' + (2805 - 379.5p)\tau'^2. \end{aligned}$$

$P(x)$ is monotone decreasing for $0.6 \leqq x \leqq 1$ and $P(1) > 0$. Hence $P(x) > 0$ for $0.6 \leqq x \leqq 1$.

It is not so difficult to prove the positive definiteness of Q and R for $1 \leqq p \leqq 1.4$, $1/10 \leqq x^2/p^2 \leqq 0.2379$.

Therefore we have $\Re a_8 < 8$ in the present case.

10. Case $0 \leqq p \leqq 1$. We divide this case into several subcases.

Case i) $0.8 \leqq p \leqq 1$, $x'^2/p^2 \leqq 1/20$ and $y \geqq 0$.

In this case we start from (A) with $A = 1/2$, $B = 3/2$, $C = 3/2$. Then we have

$$\begin{aligned} \Re a_8 \leqq & U(p) + \frac{3}{64}p^4\left(\eta + \frac{13}{6}py\right) + \frac{9}{8}p^3x(\eta + py) + \frac{5}{96}p^2(8 - p^3)y \\ & + \left(\frac{151}{192}p^3 - 9p^2 - \frac{5}{12} + \frac{149}{128}px'^2\right)x'^2y + \left(\frac{81}{32}p^2 - 9p + \frac{11}{64}x'^2\right)x'^2\eta \\ & + \left(-\frac{127}{48}p^3 + \frac{9}{2}p^2 + \frac{1}{6} - \frac{35}{16}px'^2\right)y^2 + \left(-\frac{41}{8}p^2 + 9p - \frac{27}{8}x'^2\right)y\eta \\ & + \left(-3p + \frac{9}{2}\right)\eta^2 + py\xi + \eta\xi + 2y\varphi - \frac{3}{4}y^2\eta \\ & + \left(\frac{113}{64\cdot 12}p^5 - \frac{9}{4}p^4 - \frac{25}{48}p^2 - \frac{331}{64\cdot 12}p^3x'^2 + \frac{9}{2}p^2x'^2 + \frac{25}{96}x'^2 - \frac{83}{256}px'^4\right)x'^2 \\ & + \left(-\frac{17}{4}p^2 + 2x'^2\right)x'\xi' - \frac{7}{2}px'\varphi' - 2x'\tau' - 2x'y\xi'. \end{aligned}$$

Now we remark the following facts:

$$U(p)\leqq U(1)<1.101973,$$

$$\frac{3}{64}p^4\left(\eta+\frac{13}{6}py\right)\leqq\frac{3}{64}p^4\left\{\frac{16}{9}p^2(2-p)+\frac{1}{12}(8-p^3)+\frac{1}{4}px'^2\right\}<0.111264,$$

$$\frac{5}{96}p^2(8-p^3)y\leqq\frac{5}{96}p^2(8-p^3)\sqrt{\frac{4-p^2}{3}}<0.364584,$$

$$\left(\frac{151}{192}p^3-9p^2-\frac{5}{12}+\frac{149}{128}px'^2\right)x'^2y+\frac{857}{100}p^3x'^2y\leqq 0,$$

$$-\frac{857}{100}p^3x'^2y-2x'y\xi'\leqq\frac{100}{857}y\xi'^2\leqq\frac{100}{857}\sqrt{\frac{4-p^2}{3}}\xi'^2\leqq\frac{23.16}{96}\xi'^2,$$

$$\left(\frac{81}{32}p^2-9p+\frac{11}{64}x'^2\right)x'^2\eta\leqq\left(-\frac{81}{32}p^2+9p-\frac{11}{64}x'^2\right)x'^2\sqrt{\frac{4-p^2}{5}}<0.250535.$$

Further if $\eta\geqq 0$

$$\frac{9}{8}p^3x(\eta+py)-\frac{3}{4}y^2\eta\leqq\frac{9}{8}p^3x\left\{\frac{9}{16}p^2(2-p)+\frac{1}{12}(8-p^3)+\frac{1}{4}px'^2\right\}<1.303125.$$

If $\eta\leqq 0$

$$\frac{9}{8}p^3x(\eta+py)-\frac{3}{4}y^2\eta=\frac{3}{8}\eta(3p^3x-2y^2)+\frac{9}{8}p^4xy$$

$$\leqq\frac{3}{8}\sqrt{\frac{4-p^2}{5}}\frac{(9p^3-2p-4)x}{3}+\frac{9}{8}p^4x\sqrt{\frac{4-p^2}{3}}<1.303125.$$

Making use of these remarks we have

$$\Re a_8<8-4.868519$$

$$-\frac{1}{96}\{(254p^3-432p^2-16+210px'^2)y^2+(492p^2-864p+324x'^2)y\eta$$

$$+(288p-432)\eta^2-96py\xi-96\eta\xi-192y\varphi\}$$

$$-\frac{1}{96}\{(-14.125p^5+216p^4+50p^2+41.375p^3x'^2-432p^2x'^2-25x'^2+31.125px'^4)x'^2$$

$$+(408p^2-192x'^2)x'\xi'+336px'\varphi'+192x'\tau'-23.16\xi'^2\}.$$

By applying Lemma 1 we have

$$-4.868 \leqq -1.217(p^2+x'^2+3y^2+5\eta^2+7\xi^2+7\xi'^2+9\varphi^2+9\varphi'^2+11\tau'^2)$$

$$-1.217p^2 \leqq -0.30425(p^4+p^2x'^2+3p^2y^2+5p^2\eta^2+7p^2\xi^2+7p^2\xi'^2 +9p^2\varphi^2+9p^2\varphi'^2+11p^2\tau'^2).$$

Hence we have

$$\Re a_8 < 8-0.304p^4-\frac{1}{96}Q-\frac{1}{96}R,$$

$$Q=(254p^3-344.376p^2+334.496+210px'^2)y^2+(492p^2-864p+324x'^2)y\eta +(146.04p^2+288p+152.16)\eta^2-96py\xi-96\eta\xi+(204.456p^2+817.824)\xi^2 -192y\varphi+(262.872p^2+1051.488)\varphi^2,$$

$$R=(-14.125p^5+216p^4+79.208p^2+116.832 +41.375p^3x'^2-432p^2x'^2-25x'^2+31.125px'^4)x'^2 +(408p^2-192x'^2)x'\xi'+(204.456p^2+794.664)\xi'^2+336px'\varphi' +(262.872p^2+1051.488)\varphi'^2+192x'\tau'+(321.288p^2+1285.152)\tau'^2.$$

Since for $0.8 \leqq p \leqq 1$, $x'^2/p^2 \leqq 1/20$

$$7.556y^2-192y\varphi+(262.872p^2+1051.488)\varphi^2 \geqq 0,$$

$$5.12y^2-96py\xi+450\xi^2 \geqq 0,$$

$$4.621\eta^2-96\eta\xi+(204.456p^2+367.824)\xi^2 \geqq 0$$

and

$$(254p^3-344.376p^2+321.82+210px'^2)y^2+(492p^2-864p +324x'^2)y\eta+(146.04p^2+288p+147.539)\eta^2 \geqq 0,$$

Q is positive definite there. Since for $0.8 \leqq p \leqq 1$, $x'^2/p^2 \leqq 1/20$

$$6.2x'^2+192x'\tau'+(321.288p^2+1285.152)\tau'^2 \geqq 0,$$

$$21.5x'^2+336px'\varphi'+(262.872p^2+1051.488)\varphi'^2 \geqq 0$$

and

$$(-14.125p^5+216p^4+79.208p^2+89.132+41.375p^3x'^2-432p^2x'^2$$
$$-25x'^2+31.125px'^4)x'^2$$
$$+(408p^2-192x'^2)x'\xi'+(204.456p^2+794.664)\xi'^2\geqq 0,$$

R is positive definite there. Thus in the present case $\Re a_8<8$.

Case ii) $0.8\leqq p\leqq 1$, $x'^2/p^2\leqq 1/20$ and $y\leqq 0$.

In this case we start from (A) with $A=0$, $B=3/2$, $C=3/2$. Then we have

$$\Re a_8\leqq U(p)+\frac{3}{64}p^4\left(\eta+\frac{13}{6}py\right)+\frac{9}{8}p^3x(\eta+py)$$
$$+\left(\frac{37}{64}p^3-9p^2+\frac{169}{128}px'^2\right)x'^2y+\left(\frac{81}{32}p^2-9p+\frac{11}{64}x'^2\right)x'^2\eta$$
$$+\left(-\frac{37}{16}p^3+\frac{9}{2}p^2-\frac{59}{16}px'^2\right)y^2+\left(-\frac{19}{4}p^2+9p-\frac{17}{4}x'^2\right)y\eta$$
$$+\left(-3p+\frac{9}{2}\right)\eta^2+\frac{3}{2}py\xi+\eta\xi+3y\varphi$$
$$+\left(\frac{113}{64\cdot 12}p^5-\frac{9}{4}p^4-\frac{25}{48}p^2-\frac{331}{64\cdot 12}p^3x'^2+\frac{9}{2}p^2x'^2+\frac{25}{96}x'^2-\frac{83}{256}px'^4\right)x'^2$$
$$+\left(-\frac{17}{4}p^2+2x'^2\right)x'\xi'-\frac{7}{2}px'\varphi'-2x'\tau'-\frac{3}{2}x'y\xi'.$$

We remark the following facts;

$$U(p)\leqq U(1)<1.101973,$$

$$\frac{3}{64}p^4\left(\eta+\frac{13}{6}py\right)<0.111264,$$

$$\frac{9}{8}p^3x(\eta+py)<1.303125,$$

$$\left(\frac{37}{64}p^3-9p^2+\frac{169}{128}px'^2\right)x'^2y-\frac{3}{2}x'y\xi'$$
$$\leqq\left(\frac{37}{64}p^3-9p^2+\frac{169}{128}px'^2\right)x'^2y+\frac{9}{4}x'^2|y|+\frac{1}{4}|y|\xi'^2$$
$$<0.533594+\frac{25.4016}{96}\xi'^2,$$

$$\left(\frac{81}{32}p^2-9p+\frac{11}{64}x'^2\right)x'^2\eta<0.250535.$$

Making use of these remarks and applying Lemma 1 to -4.699509 we have

$$\Re a_8<8-0.2935p^4-\frac{1}{96}Q-\frac{1}{96}R,$$

$$Q=(222p^3-347.472p^2+338.112+354px'^2)y^2+(456p^2-864p+408x'^2)y\eta$$
$$+(140.88p^2+288p+131.52)\eta^2-144py\xi+(197.232p^2+788.928)\xi^2$$
$$-96\eta\xi-288y\varphi+(253.584p^2+1014.336)\varphi^2,$$

$$R=(-14.125p^5+216p^4+78.176p^2+112.704$$
$$+41.375p^3x'^2-432p^2x'^2-25x'^2+31.125px'^4)x'^2$$
$$+(408p^2-192x'^2)x'\xi'+(197.232p^2+763.576)\xi'^2+336px'\varphi'$$
$$+(253.584p^2+1014.336)\varphi'^2+192x'\tau'+(309.936p^2+1239.744)\tau'^2.$$

It is not so difficult to prove the positive definiteness of Q and R for $0.8\leqq p\leqq 1$, $x'^2/p^2\leqq 1/20$. Hence we have in the present case $\Re a_8<8$.

Case iii) $0.8\leqq p\leqq 1$, $1/20\leqq x'^2/p^2\leqq \tan^2(\pi/7)$, $y\geqq 0$.

We start from (A) with $A=1/2$, $B=3/2$, $C=1$. Then we have

$$\Re a_8\leqq U(p)+\frac{3}{64}p^4\left(\eta+\frac{13}{6}py\right)+\frac{3}{4}p^3x\left(\eta+\frac{3}{2}py\right)+\frac{5}{96}p^2(8-p^3)y$$
$$+\left(\frac{151}{192}p^3-9p^2-\frac{5}{12}+\frac{149}{128}px'^2\right)x'^2y+\left(\frac{33}{32}p^2-6p+\frac{11}{64}x'^2\right)x'^2\eta$$
$$+\left(-\frac{127}{48}p^3+\frac{9}{2}p^2+\frac{1}{6}-\frac{35}{16}px'^2\right)y^2+(-3p^2+6p-4x'^2)y\eta$$
$$+\left(-\frac{3}{4}p+2\right)\eta^2+py\xi+2\eta\xi+2y\varphi-\frac{1}{4}y^2\eta$$
$$+\left(\frac{113}{64\cdot 12}p^5-\frac{9}{4}p^4-\frac{25}{48}p^2-\frac{331}{64\cdot 12}p^3x'^2+\frac{9}{2}p^2x'^2+\frac{25}{96}x'^2-\frac{83}{256}px'^4\right)x'^2$$
$$+\left(-\frac{17}{4}p^2+2x'^2\right)x'\xi'-\frac{7}{2}px'\varphi'-2x'\tau'-2x'y\xi'.$$

We remark the following facts:

$$U(p)\leqq U(1)<1.101973,$$

$$\frac{3}{64}p^4\left(\eta+\frac{13}{6}py\right)<0.113465,$$

$$\frac{5}{96}p^2(8-p^3)y<0.364584,$$

$$\left(\frac{151}{192}p^3-9p^2-\frac{5}{12}+\frac{149}{128}px'^2\right)x'^2y+\frac{167}{20}p^3x'^2y\leqq 0,$$

$$-\frac{167}{20}p^3x'^2y-2x'y\xi'\leqq\frac{20}{167p^3}y\xi'^2\leqq\frac{23.77}{96}\xi'^2,$$

$$\left(\frac{33}{32}p^2-6p+\frac{11}{64}x'^2\right)x'^2\eta<0.95237,$$

$$\frac{3}{4}p^3x\left(\eta+\frac{3}{2}py\right)-\frac{1}{4}y^2\eta<1.232107.$$

Making use of these remarks and applying Lemma 1 to -4.235501 we have

$$\Re a_8\leqq 8-0.2646p^4-\frac{1}{96}Q-\frac{1}{96}R,$$

$$Q=(254p^3-355.797p^2+288.92+210px'^2)y^2+(288p^2-576p+384x'^2)y\eta$$
$$+(127.005p^2+72p+316.2)\eta^2-96py\xi-192\eta\xi+(177.807p^2+711.48)\xi^2$$
$$-192y\varphi+(228.609p^2+914.76)\varphi^2,$$

$$R=(-14.125p^5+216p^4+75.401p^2+101.64$$
$$+41.375p^3x'^2-432p^2x'^2-25x'^2+31.125px'^4)x'^2$$
$$+(408p^2-192x'^2)x'\xi'+(177.807p^2+687.71)\xi'^2+336px'\varphi'$$
$$+(228.609p^2+914.76)\varphi'^2+192x'\tau'+(279.411p^2+1118.04)\tau'^2.$$

It is not so difficult to prove the positive definiteness of Q and R for $0.8\leqq p\leqq 1$, $1/20\leqq x'^2/p^2\leqq 0.2379$. Hence we have in the present case $\Re a_8<8$.

Case iv) $0.8 \leqq p \leqq 1$, $1/20 \leqq x'^2/p^2 \leqq 1/10$ and $y \leqq 0$.

We start from (A) with $A=0$, $B=5/4$, $C=1$. Then we have

$$\Re a_8 \leqq U(p) + \frac{3}{64}p^4\left(\eta + \frac{13}{6}py\right) + \frac{3}{4}p^3x\left(\eta + \frac{5}{4}py\right)$$

$$+\left(-\frac{11}{64}p^3 - \frac{15}{2}p^2 + \frac{169}{128}px'^2\right)x'^2y + \left(\frac{33}{32}p^2 - 6p + \frac{11}{64}x'^2\right)x'^2\eta$$

$$+\left(-\frac{21}{16}p^3 + \frac{25}{8}p^2 - 4px'^2\right)y^2 + \left(-\frac{13}{8}p^2 + 5p - \frac{39}{8}x'^2\right)y\eta$$

$$+\left(-\frac{3}{4}p + 2\right)\eta^2 + 2py\xi + 2\eta\xi + 3y\varphi + \frac{9}{8}py^3$$

$$+\left(\frac{113}{64\cdot 12}p^5 - \frac{9}{4}p^4 - \frac{25}{48}p^2 - \frac{331}{64\cdot 12}p^3x'^2 + \frac{9}{2}p^2x'^2 + \frac{25}{96}x'^2 - \frac{83}{256}px'^4\right)x'^2$$

$$+\left(-\frac{17}{4}p^2 + 2x'^2\right)x'\xi' - \frac{7}{2}px'\varphi' - 2x'\tau' - \frac{3}{2}x'y\xi'.$$

We remark the following facts:

$$U(p) < 1.101973,$$

$$\frac{3}{64}p^4\left(\eta + \frac{13}{6}py\right) < 0.111849,$$

$$\frac{3}{4}p^3x\left(\eta + \frac{5}{4}py\right) < 1.030469,$$

$$\left(-\frac{11}{64}p^3 - \frac{15}{2}p^2 + \frac{169}{128}px'^2\right)x'^2y - \frac{3}{2}x'y\xi'$$

$$\leqq \left(-\frac{11}{64}p^3 - \frac{15}{2}p^2 + \frac{169}{128}px'^2\right)x'^2y + \frac{9}{4}x'^2|y| + \frac{1}{4}|y|\xi'^2$$

$$< 0.985586 + \frac{25.4016}{96}\xi'^2,$$

$$\left(\frac{33}{32}p^2 - 6p + \frac{11}{64}x'^2\right)x'^2\eta < 0.384214.$$

Making use of these remarks and applying Lemma 1 to -4.385909 we have

$$\Re a_8<8-0.274p^4-\frac{1}{96}Q-\frac{1}{96}R,$$

$$\begin{aligned}Q=&(126p^3-221.07p^2+315.72+384px'^2)y^2+(156p^2-480p+468x'^2)y\eta\\&+(131.55p^2+72p+334.2)\eta^2-192py\xi-192\eta\xi+(184.17p^2+736.68)\xi^2\\&-288y\varphi+(236.79p^2+947.16)\varphi^2,\end{aligned}$$

$$\begin{aligned}R=&(-14.125p^5+216p^4+76.31p^2+105.24+41.375p^3x'^2\\&\qquad -432p^2x'^2-25x'^2+31.125px'^4)x'^2\\&+(408p^2-192x'^2)x'\xi'+(184.17p^2+711.27)\xi'^2+336px'\varphi'\\&+(236.79p^2+947.16)\varphi'^2+192x'\tau'+(289.41p^2+1157.64)\tau'^2.\end{aligned}$$

It is not so difficult to prove the positive definiteness of Q and R for $0.8\leqq p\leqq 1$, $1/20\leqq x'^2/p^2\leqq 1/10$. Hence we have in the present case $\Re a_8<8$.

Case v) $0.8\leqq p\leqq 1$, $1/10\leqq x'^2/p^2\leqq \tan^2(\pi/7)$ and $y\leqq 0$.

We start from (A) with $A=0$, $B=5/4$, $C=1$. We remark the following facts;

$$U(p)<1.101973,$$

$$\frac{3}{64}p^4\left(\eta+\frac{13}{6}py\right)<0.113465,$$

$$\frac{3}{4}p^3x\left(\eta+\frac{5}{4}py\right)<1.056325,$$

$$\begin{aligned}&\left(-\frac{11}{64}p^3-\frac{15}{2}p^2+\frac{169}{128}px'^2\right)x'^2y-\frac{3}{2}x'y\xi'\\&\leqq\left(-\frac{11}{64}p^3-\frac{15}{2}p^2+\frac{169}{128}px'^2\right)x'^2y+\frac{3}{8}|y|x'^2+\frac{3}{2}|y|\xi'^2\\&<1.882942+\frac{152.4096}{96}\xi'^2,\end{aligned}$$

$$\left(\frac{33}{32}p^2-6p+\frac{11}{64}x'^2\right)x'^2\eta<0.908094.$$

Making use of these remarks and applying Lemma 1 to -2.937201 we have

$$\Re a_8 < 8 - 0.1835p^4 - \frac{1}{96}Q - \frac{1}{96}R,$$

$$Q = (126p^3 - 247.134p^2 + 211.464 + 384px'^2)y^2 + (156p^2 - 480p + 468x'^2)y\eta$$
$$+ (88.11p^2 + 72p + 160.44)\eta^2 - 192py\xi - 192\eta\xi + (123.354p^2 + 493.416)\xi^2$$
$$- 288y\varphi + (158.598p^2 + 634.392)\varphi^2,$$

$$R = (-14.125p^5 + 216p^4 + 67.622p^2 + 70.488$$
$$+ 41.375p^3x'^2 - 432p^2x'^2 - 25x'^2 + 31.125px'^4)x'^2$$
$$+ (408p^2 - 192x'^2)x'\xi' + (123.354p^2 + 341.006)\xi'^2 + 336px'\varphi'$$
$$+ (158.598p^2 + 634.392)\varphi'^2 + 192x'\tau' + (193.842p^2 + 775.368)\tau'^2.$$

It is not so difficult to prove the positive definiteness of Q and R for $0.8 \leqq p \leqq 1$, $1/10 \leqq x'^2/p^2 \leqq 0.2379$. Hence we have $\Re a_8 < 8$ in the present case.

Case vi) $0 \leqq p \leqq 0.8$, $x'^2/p^2 \leqq 1/20$.

In this case we start from (A) with $A=0$, $B=3/2$, $C=3/2$. We remark the following facts;

$$U(p) \leqq U(0.8) < 0.695811,$$

$$\frac{3}{64}p^4\left(\eta + \frac{13}{6}py\right) < 0.038319,$$

$$\frac{9}{8}p^3x(\eta + py) < 0.73728,$$

$$-\frac{3}{2}x'y\xi' \leqq \frac{3}{2}\sqrt{\frac{1}{20}}p\sqrt{\frac{4-p^2}{3}}\sqrt{\frac{4-p^2}{7}} < 0.292771,$$

$$\left(\frac{81}{32}p^2 - 9p + \frac{11}{64}x'^2\right)x'^2\eta < 0.14642.$$

Further if $y \geqq 0$,

$$\left(\frac{37}{64}p^3 - 9p^2 + \frac{169}{128}px'^2\right)x'^2y + \frac{9}{8}py^3 \leqq \frac{9}{8}py^3 < 1.066868.$$

If $y \leqq 0$,

$$\left(\frac{37}{64}p^3-9p^2+\frac{169}{128}px'^2\right)x'^2y+\frac{9}{8}py^3\leqq\left(\frac{37}{64}p^3-9p^2+\frac{169}{128}px'^2\right)x'^2y$$

$$<0.18506<1.066868.$$

Making use of these remarks and applying Lemma 1 to -5.022531 we have

$$\Re a_8<8-0.3138p^4-\frac{1}{96}Q-\frac{1}{96}R,$$

$$Q=(222p^3-341.604p^2+361.584+354px'^2)y^2+(456p^2-864p+408x'^2)y\eta$$

$$+(150.66p^2+288p+170.64)\eta^2-144py\xi+(210.924p^2+843.696)\xi^2$$

$$-96\eta\xi-288y\varphi+(271.188p^2+1084.752)\varphi^2,$$

$$R=(-14.125p^5+216p^4+80.132p^2+120.528$$

$$+41.375p^3x'^2-432p^2x'^2-25x'^2+31.125px'^4)x'^2$$

$$+(408p^2-192x'^2)x'\xi'+(210.924p^2+843.696)\xi'^2+336px'\varphi'$$

$$+(271.188p^2+1084.752)\varphi'^2+192x'\tau'+(331.452p^2+1325.808)\tau'^2.$$

It is not so difficult to prove the positive definiteness of Q and R for $0\leqq p\leqq 0.8$, $x'^2/p^2\leqq 1/20$. Hence we have in the present case $\Re a_8<8$.

Case vii) $0\leqq p\leqq 0.8$, $1/20\leqq x'^2/p^2\leqq \tan^2(\pi/7)$.

We start from (A) with $A=0$, $B=5/4$, $C=1$. We remark the following facts:

$$U(p)\leqq U(0.8)<0.695811,$$

$$\frac{3}{64}p^4\left(\eta+\frac{13}{6}py\right)<0.03878,$$

$$\frac{3}{4}p^3x\left(\eta+\frac{5}{4}py\right)<0.572523,$$

$$-\frac{3}{2}x'y\xi'<0.536495,$$

$$\left(\frac{33}{32}p^2-6p+\frac{11}{64}x'^2\right)x'^2\eta<0.516068.$$

Further if $y\geqq 0$,

$$\left(-\frac{11}{64}p^3-\frac{15}{2}p^2+\frac{169}{128}px'^2\right)x'^2y+\frac{9}{8}p^3y\leqq\frac{9}{8}p^3y<1.066868.$$

If $y\leqq 0$,

$$\left(-\frac{11}{64}p^3-\frac{15}{2}p^2+\frac{169}{128}px'^2\right)x'^2y+\frac{9}{8}p^3y\leqq\left(-\frac{11}{64}p^3-\frac{15}{2}p^2+\frac{169}{128}px'^2\right)x'^2y$$
$$<0.78343<1.066868.$$

Making use of these remarks and applying Lemma 1 to -4.573455 we have

$$\Re a_8<8-0.2858p^4-\frac{1}{96}Q-\frac{1}{96}R,$$

$$Q=(126p^3-217.686p^2+329.256+384px'^2)y^2+(156p^2-480p+468x'^2)y\eta$$
$$+(137.19p^2+72p+356.76)\eta^2-192py\xi-192\eta\xi$$
$$+(192.066p^2+768.264)\xi^2-288y\varphi+(246.942p^2+987.768)\varphi^2,$$

$$R=(-14.125p^5+216p^4+77.438p^2+109.572$$
$$+41.375p^3x'^2-432p^2x'^2-25x'^2+31.125px'^4)x'^2$$
$$+(408p^2-192x'^2)x'\xi'+(192.066p^2+768.264)\xi'^2+336px'\varphi'$$
$$+(246.942p^2+987.768)\varphi'^2+192x'\tau'+(301.818p^2+1207.272)\tau'^2.$$

It is not so difficult to prove the positive definiteness of Q and R for $0\leqq p\leqq 0.8$, $1/20\leqq x'^2/p^2\leqq 0.2379$. Hence we have in the present case $\Re a_8<8$.

Summing up the results we have completed the proof of our theorem.

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