KŌDAI MATH. SEM. REP.
22 (1970), 473—479

CONTINUITY OF MAPPINGS OF VECTOR LATTICES WITH NORMS AND SEMINORMS

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Dirichlet mappings were introduced in [4] and further studied in [1], [5], and [7]. In the present paper we take up Dirichlet p-mappings and give a continuity property which will be of fundamental importance in the problem of characterizing these mappings geometrically.

In §1 we establish a continuity theorem, a generalization of [1], on isomorphisms of vector lattices with norms and seminorms. This theorem has interest in its own right and is the main content of the present paper. As an application we obtain in §2 the norm continuity of Dirichlet p-mappings. A detailed study of these mappings and their geometric characterization will appear elsewhere.

§1. A general continuity theorem.

1. Let X be a *normed vector lattice* over the real number field $\boldsymbol{R}$ such that $\boldsymbol{R}\subset X$, $\|x\|_X\leqq 1$ implies $|x|\leqq 1$, and

$$\alpha\leqq\|(\alpha\vee x)\wedge\beta\|_X\leqq\beta \tag{1}$$

for every $\alpha, \beta\in\boldsymbol{R}$ with $0\leqq\alpha\leqq\beta$ and every $x\in X$.

For a fixed number $p>1$ we consider a seminorm $q_X(\cdot)$ in X with the properties

$$q_X(\alpha)=0 \tag{2}$$

for every $\alpha\in\boldsymbol{R}$,

$$q_X^p(x\wedge\alpha)+q_X^p(x\vee\alpha)=q_X^p(x) \tag{3}$$

for every $x\in X$ and every $\alpha\in\boldsymbol{R}$,

$$\lim_{\alpha'\uparrow\alpha,\beta'\downarrow\beta} q_X((\alpha'\vee x)\wedge\beta')=q_X((\alpha\vee x)\wedge\beta) \tag{4}$$

for every $x\in X$ and $\alpha, \beta\in\boldsymbol{R}$ with $\alpha\leqq\beta$.

A third norm $|||\cdot|||_X$ in X is given by

Received April 13, 1970.

The work was sponsored by the U. S. Army Research Office-Durham, Grant DA-AROD-31-124-G855, University of California, Los Angeles.

$$|||x|||_X = ||x||_X + q_X(x). \tag{5}$$

THEOREM 1. *Let t be an isometric isomorphism of normed vector lattices $(X, ||\cdot||_X)$ and $(Y, ||\cdot||_Y)$, with $t(\alpha)=\alpha$ $(\alpha\in \boldsymbol{R})$. There exists a finite constant $K\geqq 1$ such that*

$$K^{-1}q_X(x)\leqq q_Y(t(x))\leqq Kq_X(x) \tag{6}$$

for every $x\in X$ if and only if there exists a constant $K_0\geqq 1$ with

$$K_0^{-1}|||x|||_X\leqq |||t(x)|||_Y\leqq K_0|||x|||_X \tag{7}$$

for every $x\in X$. The smallest K and K_0 satisfy

$$K=K_0. \tag{8}$$

The proof will be given in 2–6.

2. Since $||t(x)||_Y=||x||_X$, (6) gives (7) with the smallest $K_0\leqq K$. To show that (7) implies (6), set

$$\mathcal{F}=\{x\in X | x\geqq 0,\ ||x_X||\leqq 1\}.$$

From (7) it follows that

$$q_Y(t(x))\leqq K_0(1+q_X(x)) \tag{9}$$

for every $x\in\mathcal{F}$. Fix an arbitrary $x\in\mathcal{F}$ and $n=1, 2, \cdots$, and let

$$x_i=n\left(\left(\frac{i-1}{n}\wedge x\right)\wedge\frac{i}{n}-\frac{i-1}{n}\right)$$

for $i=1, \cdots, n$. By (1), $x_i\in\mathcal{F}$. Since t is an isomorphism of vector lattices with $t(\alpha)=\alpha$ $(\alpha\in\boldsymbol{R})$,

$$t(x_i)=n\left(\left(\frac{i-1}{n}\vee t(x)\right)\wedge\frac{i}{n}-\frac{i-1}{n}\right).$$

In view of (2)

$$q_X(x_i)=nq_X\left(\left(\frac{i-1}{n}\vee x\right)\wedge\frac{i}{n}\right),$$

$$q_Y(t(x_i))=nq_Y\left(\left(\frac{i-1}{n}\vee t(x)\right)\wedge\frac{i}{n}\right).$$

By repeated use of (3) we obtain

$$q_X^p(x)=\sum_{i=1}^{n}q_X^p\left(\left(\frac{i-1}{n}\vee x\right)\wedge\frac{i}{n}\right),$$

$$q_Y^p(t(x)) = \sum_{i=1}^{n} q_Y^p\left(\left(\frac{i-1}{n} \vee t(x)\right) \wedge \frac{i}{n}\right),$$

and therefore

(10) $$n^p q_X^p(x) = \sum_{i=1}^{n} q_X^p(x_i), \qquad n^p q_Y^p(t(x)) = \sum_{i=1}^{n} q_Y^p(t(x_i)).$$

Since $x_i \in \mathscr{F}$, (9) yields $q_Y(t(x_i)) \leqq K_0(1+q_X(x_i))$ and

(11) $$q_Y^p(t(x_i)) \leqq K_0^p(1+q_X(x_i))^p.$$

3. We digress to make an elementary observation we shall make use of. Let $\varphi(\tau)=(1+\tau)^p-(1+\tau^p)$ $(\tau \geqq 0)$. Since $\varphi(0)=0$ and $\varphi'(\tau)>0$ $(\tau \geqq 0)$,

(12) $$0 \leqq \varphi(\tau_1) \leqq \varphi(\tau_2) \qquad (0 \leqq \tau_1 \leqq \tau_2).$$

For $\sigma>0$ and $\psi_\tau(\sigma)=[(\sigma+\tau)^p-(\sigma^p+\tau^p)]/\sigma$, we have

$$\psi_\tau(\sigma) = \frac{\sigma^p}{\sigma}\,\varphi\left(\frac{\tau}{\sigma}\right).$$

We therefore obtain by (12)

(13) $$0 \leqq \psi_{\tau_1}(\sigma) \leqq \psi_{\tau_2}(\sigma)$$

for every $\sigma>0$. On the other hand

$$\lim_{\sigma \downarrow 0} \psi_\tau(\sigma) = \lim_{\sigma \downarrow 0} \frac{\dfrac{d}{d\sigma}[(\sigma+\tau)^p-(\sigma^p+\tau^p)]}{\dfrac{d}{d\sigma}\sigma}$$

$$= \lim_{\sigma \downarrow 0} (p(\sigma+\tau)^{p-1} - p\sigma^{p-1}) = p\tau^{p-1}.$$

It follows by (13) that

(14) $$0 \leqq \lim_{\sigma \downarrow 0} \psi_\tau(\sigma) \leqq \lim_{\sigma \downarrow 0} \psi_{\tau_0}(\sigma) = p\tau_0^{p-1}$$

for every τ in $[0, \tau_0]$. Since $\varphi(n\tau)/n^{p-1} = \psi_\tau(1/n)$ we conclude that

(15) $$0 \leqq \lim_{n \to \infty} \frac{1}{n^{p-1}}\,\varphi(n\tau) \leqq p\tau_0^{p-1}$$

for every τ in $[0, \tau_0]$.

4. We return to (11). In terms of φ it takes on the form

$$q_Y^p(t(x_i)) \leqq K_0^p(1+q_X^p(x_i)+\varphi(q_X(x_i))).$$

If we write

$$z_i=\left(\frac{i-1}{n}\vee x\right)\wedge\frac{i}{n}$$

then $x_i=nz_i-(i-1)$. By (2), $q_X(x_i)=nq_X(z_i)$. Thus

$$q_Y^p(t(x_i))\leqq K_0^p(1+q_X^p(x_i)+\varphi(nq_X(z_i))).$$

On summing with respect to i and using (10) we obtain

$$n^p q_Y^p(t(x))\leqq K_0^p\left(n+n^p q_X^p(x)+\sum_{i=1}^n \varphi(nq_X(z_i))\right)$$

and a fortiori

$$q_Y^p(t(x))\leqq K_0^p q_X^p(x)+\frac{K_0^p}{n^{p-1}}+\frac{K_0^p}{n}\sum_{i=1}^n\frac{\varphi(nq_X(z_i))}{n^{p-1}}. \tag{16}$$

Let $z_{i(n)}$ be such that $q_X(z_{i(n)})=\max_{1\leqq i\leqq n}q_X(z_i)$. By (12),

$$\frac{1}{n}\sum_{i=1}^n\frac{\varphi(nq_X(z_i))}{n^{p-1}}\leqq\frac{\varphi(nq_X(z_{i(n)}))}{n^{p-1}}. \tag{17}$$

Here by (3), $q_X(z_i)\leqq q_X(x)$ and therefore

$$a_m\equiv\sup_{n\geqq m}q_X(z_{i(n)})\leqq q_X(x)$$

for $m=1, 2, \cdots$. Since $q_X(z_{i(n)})\leqq a_m$ $(n\geqq m)$, (15) implies by (17) that

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^n\frac{\varphi(nq_X(z_i))}{n^{p-1}}\leqq\limsup_{n\to\infty}\frac{\varphi(nq_X(z_{i(n)}))}{n^{p-1}}\leqq pa_m^{p-1} \tag{18}$$

for every $m=1, 2, \cdots$. On letting $n\to\infty$ in (16) and using (18) we conclude that

$$q_Y^p(t(x))\leqq K_0^p q_X^p(x)+K_0^p pa_m^{p-1}. \tag{19}$$

5. We assert that

$$\lim_{m\to\infty}a_m=0. \tag{20}$$

If this were not true there would exist an increasing sequence $\{n(k)\}$ of integers >0 such that

$$\varepsilon=\lim_{k\to\infty}q_X(z_{i(n(k))})>0.$$

By again selecting a subsequence, if necessary, we may assume that

$$\lim_{k\to\infty}\frac{i(n(k))}{n(k)}=c$$

exists. In view of (4),

$$\lim_{l\uparrow\infty} q_X\left(\left(\left(c-\frac{1}{l}\right)\vee x\right)\wedge\left(c+\frac{1}{l}\right)\right)=q_X((c\vee x)\wedge c)$$

$$=q_X(c)=0.$$

Therefore we can find an $l>0$ such that

$$q_X\left(\left(\left(c-\frac{1}{l}\right)\vee x\right)\wedge\left(c+\frac{1}{l}\right)\right)<\varepsilon.$$

For all sufficiently large k,

$$c-\frac{1}{l}<\frac{i(n(k))-1}{n(k)}<\frac{i(n(k))}{n(k)}<c+\frac{1}{l}.$$

It follows by (3) that

$$q_X(z_{i(n(k))})\leqq q_X\left(\left(\left(c-\frac{1}{l}\right)\vee x\right)\wedge\left(c+\frac{1}{l}\right)\right)<\varepsilon,$$

a contradiction. Thus (20) holds.

6. On combining (19) and (20) we see that

$$q_Y(t(x))\leqq K_0 q_X(x)$$

as soon as $x\in\mathcal{F}$.

For an arbitrary $x\in X$, $x\neq 0$, let $y=x/\|x\|_X$. Then

$$x=\|x\|_X(y\vee 0+y\wedge 0)=\|x\|_X(y\vee 0-(-y)\vee 0).$$

Since $y\vee 0$, $-(y\wedge 0)\in\mathcal{F}$ and

$$q_X^p(x)=\|x\|_X^p\ (q_X^p(y\vee 0)+q_X^p((-y)\vee 0)),$$

$$q_Y^p(t(x))=\|x\|_X^p(q_Y^p(t(y)\vee 0)+q_Y^p(t(-y)\vee 0)),$$

we obtain

$$q_Y^p(t(x))\leqq\|x\|_X^p(K_0^p q_X^p(y\vee 0)+K_0^p q_X^p((-y)\vee 0))$$

$$\leqq K_0^p[\|x\|_X^p(q_X^p(y\vee 0)+q_X^p((-y)\vee 0))]=K_0^p q_X^p(x).$$

Thus (6) is valid, and the smallest possible K in (6) satisfies

$$K\leqq K_0.$$

The proof of Theorem 1 is herewith complete.

§ 2. Dirichlet p-mappings.

7. Let M be a Riemannian manifold, i.e. an orientable C^1-manifold with covariant tensor $(g_{ij})_{i,j=1}^m$ on M such that for each parametric ball (B, x), the $g_{ij}(x)$ are Borel measurable on B and there exists a finite constant $k_B \geqq 1$ such that

$$k_B^{-1}\sum_{i=1}^m (\xi^i)^2 \leqq \sum_{i,j=1}^m g_{ij}(x)\xi^i\xi^j \leqq k_B \sum_{i=1}^m (\xi^i)^2$$

for almost every $x \in B$ and for every real vector $\xi = (\xi^1, \cdots, \xi^m)$. We use the standard notation $(g^{ij}) = (g_{ij})^{-1}$, $g = \det(g_{ij})$, $dV = \sqrt{g}\,dx^1 \cdots dx^m$, and have

$$|\text{grad } \varphi|^2 = \sum_{i,j=1}^m g^{ij}(x)\frac{\partial\varphi}{\partial x^i}\frac{\partial\varphi}{\partial x^j}.$$

We are interested in the Dirichlet p-integral

$$D_M^p(\varphi) = \int_M |\text{grad } \varphi|^p dV.$$

8. For a region $\Omega \subset E^m$ we denote by $W^{1,p}(\Omega)$ $(p \geqq 1)$ the *Sobolev* $(1, p)$-*space* (cf. Yosida [8]). The *local Sobolev* $(1, p)$-*space* $W_{\text{loc}}^{1,p}(M)$ over a Riemannian manifold M is the space of functions f on M such that $f|\Omega \in W^{1,p}(\Omega)$ for every parametric ball $\Omega \subset M$. We consider the subspace

$$W^p(M) = \left\{ f \,\middle|\, f \in W_{\text{loc}}^{1,p}(M), \quad \int_M |\text{grad } f|^p dV < \infty \right\}.$$

9. Let T be a homemorphism of a Riemannian manifold M_1 onto another M_2 such that $f \in W^p(M_1)$ if and only if $T^*f \in W^p(M_2)$, where $T^*f = f \circ T^{-1}$. We shall call such a T a *Dirichlet p-mapping* $(p \geqq 1)$. From a potential-theoretic viewpoint, an important problem is to characterize such mappings geometrically. For $p = 2$ a complete solution of the problem was given in [2], [3]. To study T for a general $p > 1$, we establish here the following norm continuity, which will play a fundamental role in our problem:

THEOREM 2. *A Dirichlet p-mapping T $(p > 1)$ is norm continuous: there exists a finite constant $K \geqq 1$ such that*

$$K^{-1}D_{M_1}^p(f) \leqq D_{M_2}^p(T^*f) \leqq K D_{M_1}^p(f) \tag{21}$$

for every f in $W^p(M_1)$.

We shall sketch the proof, the details being left to the reader. The case $p = 1$ remains open.

10. The spaces

$$X = W^p(M_1) \cap L^\infty(M_1), \qquad Y = W^p(M_2) \cap L^\infty(M_2)$$

with norms

$$\|\cdot\|_X=\|\cdot\|_{L^\infty(M_1)}, \qquad \|\cdot\|_Y=\|\cdot\|_{L^\infty(M_2)}$$

are seen to be vector lattices under the pointwise operations

$$f\wedge g=\min(f, g), \qquad f\vee g=\max(f, g).$$

Endow X and Y with the seminorms

$$q_X(\cdot)=(D^p_{M_1}(\cdot))^{1/p}, \qquad q_Y(\cdot)=(D^p_{M_2}(\cdot))^{1/p}.$$

It is readily verified that $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, and the isomorphism t given by

$$t(f)=f\circ T^{-1}$$

satisfy all conditions of Theorem 1.

11. The spaces $(X, |||\cdot|||_X)$, $(Y, |||\cdot|||_Y)$ are seen to be Banach algebras. Since t is an algebraic isomorphism of X onto Y, Gelfand's theorem applies (e.g. Rickart [6]): there exists a finite constant $K\geqq 1$ such that

$$K^{-1}|||f|||_X\leqq|||t(f)|||_Y\leqq K|||f|||_X \tag{22}$$

for all $f\in X$. By Theorem 1, (22) is equivalent to

$$K^{-1}q_X(f)\leqq q_Y(t(f))\leqq Kq_X(f) \tag{23}$$

for all $f\in X$. This gives (21).

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