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ON A SECONDARY TURNING POINT PROBLEM

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§ 1. Introduction.

1. 1. Aspects of the problem. In order to solve the so-called turning point problem, there may be several methods. One of them is the stretching-matching method. This method has been improved by some authors, say, Wasow, Iwano and Sibuya. Especially Wasow has shown that this method is very useful for the problem in his papers [11] and [12]. Iwano [2] and Iwano-Sibuya [3] showed explicitly how to decide the stretching transformations for the stretching-matching method, but did not prove how to match. On the other hand, Wasow [11] showed implicitly both how to stretch and how to match for the special case, and later Nishimoto [6]-[8] and Nakano [4] extended results of Wasow to various cases by applying the theory of Iwano and Sibuya. However, all the cases considered by them assume that a characteristic polygon for respective differential equations consists of one segment. (Definition of a characteristic polygon will be given in the following paragraph.) In this paper we will make a one-segment condition weaker and analize the case satisfying a two segment condition.

1. 2. The characteristic polygon. This paper studies the following type of the differential equations:

$$\varepsilon^{2\sigma}\frac{d^2y}{dx^2}-a(x,\varepsilon)y=0,$$

or in the vectoral representation

$$\varepsilon^{\sigma}\frac{dY}{dx}=A(x,\varepsilon)Y,\qquad A(x,\varepsilon)=\begin{bmatrix}0 & 1\\ a(x,\varepsilon) & 0\end{bmatrix},\tag{1}$$

where ε is a small parameter, σ is an arbitrary positive integer and $a(x,\varepsilon)$ is holomorphic in both variables x and ε in the region

$$D:\ |x|\leqq x_0,\qquad 0<|\varepsilon|\leqq\varepsilon_0,\qquad |\arg\varepsilon|\leqq\varepsilon_1,$$

and asymptotically expansible such that

$$a(x,\varepsilon)\sim x^{\nu}+\sum_{r=1}^{\infty}\left(\sum_{h=m_r}^{\infty}a_{rh}x^h\right)\varepsilon^r$$

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as ε tends to zero uniformly on D, with ν and m_r positive integers.

Since ν is an integer, two eigenvalues of the coefficient matrix of (1) independent of ε are identical for $x=0$ and differ for $x\neq 0$. That is to say, the origin $x=0$ is a turning point of the differential equation (1).

We plot following points:

$$R=(\sigma, -1), \qquad P_0=\left(0, \frac{\nu}{2}\right), \qquad P_r=\left(\frac{r}{2}, \frac{m_r}{2}\right), \qquad r=1, 2, \cdots,$$

in the (X, Y)-plane with the orthogonal coordinate system.

We construct a polygon, convex downward, in such a way that its vertices are some of R and P_r and that all the points R and P_r are situated either on or over the polygon. The polygon thus defined is called a *characteristic* one for the differential equation (1).[1] If the characteristic polygon consists of one segment, we say that the differential equation (1) satisfies the one-segement condition, and if it consists of two segments we say that the differential equation (1) satisfies the two-segment condition, and so on.

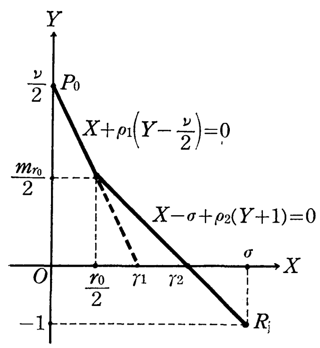

Fig. 1. Characteristic Polygon.

Since we shall consider the case which the equation (1) satisfies the two-segment condition, we show that this condition is represented by a simple inequality.

Without loss of generality, we can assume that the characteristic polygon snaps at a point $P_{r_0}=(\alpha_1, \beta_1)$ whose Y coordinate is situated between $\nu/2$ and 0 and whose X coordinate is situated between σ and 0. The equations of the straight lines between P_0 and P_{r_0} and between P_{r_0} and R are $X+\rho_1(Y-\nu/2)=0$, $\rho_1=r_0/(\nu-m_{r_0})$ and $X-\sigma+\rho_2(Y+1)=0$, $\rho_2=(2\sigma-r_0)/(m_{r_0}+2)$ respectively. Therefore the condition which the characteristic polygon consists of two segments is

$$\frac{r+2}{-\sigma}\frac{r_0}{4}+\frac{\nu}{2}>\frac{m_{r_0}}{2}, \qquad r=1, 2, \cdots.$$

All the points P_r except for P_0 and P_{r_0} are situated over the characteristic polygon. For the sake of simplicity, we consider the case $\sigma=1$ and so $r_0=1$, and we assume that the points P_r except for P_0 and P_{r_0} do not exist. Then the two-segment condition becomes simply $\nu>2m_1+2$, or by changing notations it becomes

$$a(x, \varepsilon)=x^{\nu}+cx^{\mu}\varepsilon, \qquad \nu>2\mu+2, \tag{2}$$

since we take only the principal part of the coefficient of (1).

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1) For the general systems, see Iwano [2].

1. 3. Division of the region D. For the sake of convenience, we divide the region D into several subregions in each of which the given equation (1) is reduced to a special form. In the followings M_i and x_i designate large and small constants respectively, and $\Lambda(\alpha, \beta)=\operatorname{diag}[1, \varepsilon^{\alpha}x^{\beta}]$ with α and β real.

1° In the region $M_1|\varepsilon|^{\rho_1}\leqq|x|\leqq x_0$, $\rho_1=r_0/(\nu-m_{r_0})$, the differential equation (1) is changed by $Y=\Lambda(0, \beta_0)Z$, $\beta_0=\nu/2$, to

$$(3)\qquad (x^{-(\nu-\mu)}\varepsilon)x^{\nu/2-\mu}\frac{dZ}{dx}=\left\{\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}+(x^{-(\nu-\mu)}\varepsilon)\begin{bmatrix}0 & 0\\ c & -\frac{\nu}{2}x^{(\nu-2\mu-2)/2}\end{bmatrix}\right\}Z$$

2° In the region $x_1|\varepsilon|^{\rho_1}\leqq|x|\leqq M_1|\varepsilon|^{\rho_1}$, the transformations $x=\varepsilon^{\rho_1}t$, $Y=\Lambda(\gamma_1, 0)U$, $\gamma_1=\nu r_0/2(\nu-m_{r_0})$, change (1) to

$$(4)\qquad \varepsilon^{(\nu-2\mu-2)/2(\nu-\mu)}\frac{dU}{dt}=\begin{bmatrix}0 & 1\\ t^{\nu}+ct^{\mu} & 0\end{bmatrix}U.$$

Since the exponent of ε is positive, this equation is in a same form as the original equation (1) and has several turning points which are roots of $t^{\nu}+ct^{\mu}=0$. If $\mu>0$, $t=0$ is a turning point of (4) and it corresponds to the original turning point $x=0$. The other turning points, namely the roots except for $t=0$ of the equation $t^{\nu}+ct^{\mu}=0$, do not explicitly correspond to the original turning point.[2]

3° In the region $M_2|\varepsilon|^{\rho_2}\leqq|x|\leqq x_1|\varepsilon|^{\rho_1}$, $\rho_2=(2\sigma-r_0)/(m_{r_0}+2)$, the transformations like 2° $x=\varepsilon^{\rho_1}s$, $Y=\Lambda(\alpha_1, \beta_1)V$, $\alpha_1=r_0/2$, $\beta_1=m_{r_0}/2$, change (1) to

$$(5)\qquad \begin{aligned}&(s^{-(\mu+2)(\nu-\mu)/(\nu-2\mu-2)}\varepsilon)^{(\nu-2\mu-2)/2(\nu-\mu)}s\frac{dV}{ds}\\ &=\begin{bmatrix}0 & 1\\ s^{\nu-\mu}+c & -\frac{\mu}{2}(s^{-(\mu+2)(\nu-\mu)/(\nu-2\mu-2)}\varepsilon)^{(\nu-2\mu-2)/2(\nu-\mu)}\end{bmatrix}V.\end{aligned}$$

4° At the last, in the region $|x|\leqq M_2|\varepsilon|^{\rho_2}$, we transform (1) by $x=\varepsilon^{\rho_2}r$, $Y=\Lambda(\gamma_2, 0)W$, $\gamma_2=(\sigma m_{r_0}+r_0)/(m_{r_0}+2)$, then (1) becomes

$$(6)\qquad \frac{dW}{dr}=\left\{\begin{bmatrix}0 & 1\\ cr^{\mu} & 0\end{bmatrix}+\begin{bmatrix}0 & 0\\ r^{\nu} & 0\end{bmatrix}\varepsilon^{(\nu-2\mu-2)/(\mu+2)}\right\}W.$$

Sometimes this differential equation is referred to the inner equation and its solutions are called inner ones. Similarly, the reduced differential equation (3) is sometimes called the outer equation and its solutions outer ones. These namings we use in the following.

2) These roots are not turning points of the given differential equation and they are called *secondary turning points* in the Wasow [13].

1.4. The simplest case. In the above case, as already mentioned, the two-segment condition is $\nu>2\mu+2$, and the simplest may be the case $\mu=0$ and $\nu=3$. Because the reduced equations (4) and (5) coincide. In the present paper we will analyse this case. The characteristic polygon in this case snaps just on the X-axis, and the function $a(x, \varepsilon)=x^3-\varepsilon$ by putting $c=-1$. The assumption $c=-1$ does not lose generality but it is only convenient. Thus the above reduced equations are written as:

1° In the region $M_1|\varepsilon|^{1/3}\leqq|x|\leqq x_0$,

$$(x^{-3}\varepsilon)x^{3/2}\frac{dZ}{dx}=\left\{\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}+(x^{-3}\varepsilon)\begin{bmatrix}0 & 0\\ -1 & -\frac{3}{2}x^{1/2}\end{bmatrix}\right\}Z; \tag{7}$$

2° In the region $M_2|\varepsilon|^{1/2}\leqq|x|\leqq M_1|\varepsilon|^{1/3}$,

$$\varepsilon^{1/6}\frac{dU}{dt}=\begin{bmatrix}0 & 1\\ t^3-1 & 0\end{bmatrix}U; \tag{8}$$

and

3° In the region $|x|\leqq M_2|\varepsilon|^{1/2}$,

$$\frac{dV}{ds}=\left\{\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}+\varepsilon^{1/2}\begin{bmatrix}0 & 0\\ s^3 & 0\end{bmatrix}\right\}V. \tag{9}$$

As mentioned in *1.3.4°*, the original turning point $x=0$ corresponds to the point $t=0$ for the equation (8). However, the point $t=0$ is clearly not a turning point of (8) and then a solution of (8) may give us the value at the original turning point $x=0$.

If we could obtain the solution of the equation (8) in the region of t for all t such that $0\leqq|t|<\infty$ (except for neighborhoods of roots of $t^3-1=0$), this solution would play a role of the inner one. Hence, naturally the third equation, i.e., (9) is unnecessary for our turning point problem.

1.5. The problem. Under the above situation, we set the problem.

The differential equation to be considered in the present paper is

$$\varepsilon\frac{dY}{dx}=\begin{bmatrix}0 & 1\\ x^3-\varepsilon & 0\end{bmatrix}Y, \tag{10}$$

or equivalently

$$\varepsilon^2\frac{dy^2}{dx^2}-(x^3-\varepsilon)y=0, \tag{10'}$$

in the (x, ε)-region such as

$$D:\quad |x|\leqq x_0,\qquad 0<\varepsilon\leqq\varepsilon_0.$$

The problem is to get the solution of (10) in the full neighbourhood of the origin as ε tends to zero. In order to do it, we shall use the matching method.

1° In the region $M\varepsilon^{1/3}\leqq|x|\leqq x_0$, the differential equation (10) is reduced by the transformation

$$Y=\begin{bmatrix}1 & 0\\ 0 & x^{3/2}\end{bmatrix}Z \tag{11}$$

to

$$(x^{-3}\varepsilon)x^{3/2}\frac{dZ}{dx}=\left\{\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}+(x^{-3}\varepsilon)\begin{bmatrix}0 & 0\\ -1 & -\dfrac{3}{2}x^{1/2}\end{bmatrix}\right\}Z, \tag{12}$$

or by changing a notation $x^{-3}\varepsilon=\lambda$,

$$\lambda x^{3/2}\frac{dZ}{dx}=\left\{\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}+\lambda\begin{bmatrix}0 & 0\\ -1 & -\dfrac{3}{2}x^{1/2}\end{bmatrix}\right\}Z. \tag{12'}$$

2° For all t such as $|t|<\infty$ except for neighbourhoods of the roots of $t^3-1=0$, we shall consider

$$\varepsilon^{1/6}\frac{dU}{dt}=\begin{bmatrix}0 & 1\\ t^3-1 & 0\end{bmatrix}U, \tag{13}$$

where

$$t=x\varepsilon^{-1/3},\quad Y=\begin{bmatrix}1 & 0\\ 0 & \varepsilon^{1/2}\end{bmatrix}U. \tag{14}$$

The former we call the outer differential equation and the later the inner one. Since two regions overlap for all ε sufficiently small, the outer and the inner solutions can be matched if they are obtained.

Thus we shall obtain the formal outer solutions in § **2** and the formal inner solutions in § **3.** In § **4** we shall consider the topological properties of the inner region, because the formal solutions obtained in § **3** are asymptotic expansions of the actual solutions in the subsets of the inner region called canonical regions.

In §§ **5** and **6**, the existence of actual outer and inner solutions will be showed. Since the canonical regions are subsets of the inner region and the existence of actual solutions is proved in each canonical region, we must relate all the inner solutions if we wish to know the properties of the solutions in the full neighbourhood of the origin. Therefore in § 7 we shall get the relations between inner solutions, i.e., the connection matrices between inner solutions in one canonical region and another canonical region. In § 8 the inner and the outer solutions will be matched. The last section (§ 9) summarizes the results.

§ 2. Formal outer solutions.

In this section we consider the differential equation *1.5.*(12) defined outside the turning point $x=0$, and we shall obtain formal outer solutions, whose asymptoticity will be given in § 5.

2.1. Simplification. First of all, we notice that the constant coefficient of *1.5.*(12) possesses two different eigenvalues 1 and -1. Thus, if we transform it by

$$Z=QV, \qquad Q=\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \tag{1}$$

it is changed to

$$\lambda x^{3/2}\frac{dV}{dx}=\left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}+\lambda\begin{bmatrix} a & b \\ a & b \end{bmatrix}\right\}V, \tag{2}$$

where $\lambda=x^{-3}\varepsilon$, $a=-1/2-3x^{1/2}/4$ and $b=1/2-3x^{1/2}/4$.

Let us write (2) in

$$\lambda x^{3/2}\frac{dV}{dx}=AV, \qquad A(x, \lambda)=A_0(x)+\lambda A_1(x). \tag{2'}$$

Furthermore let us transform (2) by

$$V=PW, \qquad P(x, \lambda)=\sum_{r=0}^{\infty}P_r(x)\lambda^r, \qquad P_0=I \text{ (the 2-by-2 unit matrix)}. \tag{3}$$

Then we have

$$\lambda x^{3/2}\frac{dW}{dx}=BW, \qquad B(x, \lambda)=P^{-1}AP-\lambda x^{3/2}P^{-1}P'=\sum_{r=0}^{\infty}B_r(x)\lambda^r. \tag{4}$$

Equating the same powers of λ of the second equality of (4), we get

$$A_0P_0-P_0B_0=0, \qquad A_0P_1-P_1B_0=xP_0'-A_1P_0+P_0B_1,$$

$$A_0P_r+A_1P_{r-1}-\sum_{j=0}^{r}P_jB_{r-j}=x^{3/2}P_{r-1}' \qquad (r=2, 3, \cdots),$$

or, by remembering $P_0=I$,

$$(5)_0 \qquad B_0=A_0,$$

$$(5)_1 \qquad A_0P_1-P_1A_0=B_1-A_1,$$

$$(5)_r \qquad A_0P_r-P_rA_0=\left[x^{3/2}P'_{r-1}-A_1P_{r-1}+\sum_{j=1}^{r-1}P_jB_{r-j}\right]+B_r \qquad (r=2,3,\cdots).$$

We want to determine B_r and P_r such types as

$$B_r=\begin{bmatrix}B_r^1 & 0\\ 0 & B_r^2\end{bmatrix} \ (r=0,1,2,\cdots), \quad \text{and} \quad P_r=\begin{bmatrix}0 & P_r^1\\ P_r^2 & 0\end{bmatrix} \ (r=1,2,\cdots).$$

That is to say, we want to choose the transformation (3) so that the matrix of the coefficient B of (4) is diagonal. In fact, we can show at once from $(5)_1$ that $B_1=\begin{bmatrix}a & 0\\ 0 & b\end{bmatrix}$ by choosing $P_1=\begin{bmatrix}0 & -b/2\\ a/2 & 0\end{bmatrix}$. Other B_r's will be determined successively by $(5)_r$. Because, since the bracket of the right side of $(5)_r$ contains $B_0, B_1, \cdots, B_{r-1}$ and $P_0, P_1, \cdots, P_{r-1}$, the P_r and B_r would be determined in desired forms if we assume that B_j, P_j $(j=0,1,2,\cdots,r-1)$ were known already. Thus follow the equalities

$$B_r^1=bP_{r-1}^2, \qquad B_r^2=aP_{r-1}^1,$$

$$P_r^1=\frac{1}{2}\left\{P_{r-1}^{1\prime}x^{3/2}-aP_{r-1}^1+\sum_{j=1}^{r-1}P_j^1B_{r-j}^2\right\},$$

and

$$P_r^2=-\frac{1}{2}\left\{P_{r-1}^{2\prime}x^{3/2}-bP_{r-1}^2+\sum_{j=1}^{r-1}P_j^2B_{r-j}^1\right\}.$$

Therefore we could get the differential equation (4) with B the diagonal coefficient by the formal transformation (3).

2.2. Formal solutions. Let

$$B=\sum_{r=0}^{\infty}B_r(x)\lambda^r=\sum_{r=0}^{\infty}\begin{bmatrix}B_r^1(x) & 0\\ 0 & B_r^2(x)\end{bmatrix}\lambda^r=\begin{bmatrix}B^1 & 0\\ 0 & B^2\end{bmatrix}$$

and

$$P=\sum_{r=0}^{\infty}P_r(x)\lambda^r=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\sum_{r=1}^{\infty}\begin{bmatrix}0 & P_r^1(x)\\ P_r^2(x) & 0\end{bmatrix}\lambda^r=\begin{bmatrix}1 & P^1\\ P^2 & 1\end{bmatrix}.$$

From (4) it follows, by putting $W=\begin{bmatrix}W^1 & 0\\ 0 & W^2\end{bmatrix}$

$$\lambda x^{3/2}\frac{dW^k}{dx}=B^k W^k \qquad (k=1, 2).$$

Since the above differential equations are of the first order we obtain their solutions:

$$W^1=x^{-3/4}\exp\left[\frac{1}{\varepsilon}\frac{2}{5}x^{5/2}+x^{-1/2}\right]\cdot\left[1+\left(-\frac{9}{80}x^{-5/2}+\frac{1}{28}x^{-7/2}\right)\varepsilon+O(\varepsilon^2)\right],$$

$$W^2=x^{-3/4}\exp\left[\frac{1}{\varepsilon}\frac{2}{5}x^{5/2}-x^{-1/2}\right]\cdot\left[1+\left(-\frac{9}{80}x^{-5/2}-\frac{1}{28}x^{-7/2}\right)\varepsilon+O(\varepsilon^2)\right],$$

where $O(\varepsilon^2)$'s represent functions of ε of order two with coefficients bounded for x such that $M\varepsilon^{1/3}\leqq|x|\leqq x_0$.

By combining two transformations (1) and (3), we obtain the formal solutions of (2), $Z=(QP)W$, that is to say,

$$Z=\begin{bmatrix} Z^{11} & Z^{12} \\ Z^{21} & Z^{22}\end{bmatrix}=\begin{bmatrix} 1-P^2 & -1+P^1 \\ 1+P^2 & 1+P^1\end{bmatrix}\begin{bmatrix} W^1 & 0 \\ 0 & W^2\end{bmatrix}=\begin{bmatrix} (1-P^2)W^1 & -(1-P^1)W^2 \\ (1+P^2)W^1 & (1+P^1)W^2\end{bmatrix},$$

where

$$1\pm P^1=1\pm\sum_{r=1}^{\infty}P_r^1\lambda^r=1\pm\sum_{r=1}^{\infty}x^{-3r}P_r^1\varepsilon^r=1\mp\varepsilon(2x^{-3}-3x^{-5/2})/8+O(\varepsilon^2),$$

$$1\pm P^2=1\pm\sum_{r=1}^{\infty}P_r^2\lambda^r=1\mp\varepsilon(2x^{-3}+3x^{-5/2})/8+O(\varepsilon^2),$$

and $O(\varepsilon^2)$'s represent functions with the same property as already mentioned. Hence we get the fundamental matrix of the differential equation *1.5.*(12) and it consists of

$$(6)_1 \quad Z^{11}=x^{-3/4}\exp\left[\frac{1}{\varepsilon}\frac{2}{5}x^{5/2}+x^{-1/2}\right]\cdot\left[1+\left(\frac{1}{28}x^{-7/2}+\frac{1}{4}x^{-3}+\frac{21}{80}x^{-5/2}\right)\varepsilon+O(\varepsilon^2)\right],$$

$$(6)_2 \quad Z^{12}=x^{-3/4}\exp\left[-\frac{1}{\varepsilon}\frac{2}{5}x^{5/2}-x^{-1/2}\right]\cdot\left[-1+\left(\frac{1}{28}x^{-7/2}-\frac{1}{4}x^{-3}+\frac{39}{80}x^{-5/2}\right)\varepsilon+O(\varepsilon^2)\right],$$

$$(6)_3 \quad Z^{21}=x^{-3/4}\exp\left[\frac{1}{\varepsilon}\frac{2}{5}x^{5/2}+x^{-1/2}\right]\cdot\left[1+\left(\frac{1}{28}x^{-7/2}-\frac{1}{4}x^{-3}-\frac{39}{80}x^{-5/2}\right)\varepsilon+O(\varepsilon^2)\right],$$

$$(6)_4 \quad Z^{22}=x^{-3/4}\exp\left[-\frac{1}{\varepsilon}\frac{2}{5}x^{5/2}-x^{-1/2}\right]\cdot\left[1+\left(-\frac{1}{28}x^{-7/2}-\frac{1}{4}x^{-3}+\frac{21}{80}x^{-5/2}\right)\varepsilon+O(\varepsilon^2)\right],$$

where $O(\varepsilon^2)$'s have the obvious meaning. Therefore, the given equation *1.5.*(10)

has the solution $Y=\mathrm{diag}\,[1, x^{3/2}]Z$, i.e.,

$$(7)\qquad Y^{11}=Z^{11}, \quad Y^{12}=Z^{12}, \quad Y^{21}=x^{3/2}Z^{21}, \quad Y^{22}=x^{3/2}Z^{22}.$$

Summing up the above statements we have proved the following

THEOREM A. [Formal outer solution]. *The differential equation 1.5.*(12) *possesses the formal solution* (6), *then* (7) *is the formal outer solutions of the differential equation 1.5.*(10):

$$Y(x, \varepsilon)=x^{\frac{3}{4}\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}}\cdot \tilde{Y}(x, \varepsilon)\cdot \exp\left\{\left(\frac{1}{\varepsilon}\frac{2}{5}x^{5/2}+x^{-1/2}\right)\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\right\},$$

where

$$\tilde{Y}(x, \varepsilon)=\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}+O(\varepsilon)$$

and $O(\varepsilon)$ *represents a formal power series of* ε *beginning with the degree one with coefficients bounded for* x *containing in the whole outer region* $M\varepsilon^{1/3}\leqq|x|\leqq x_0$.

§ 3. Formal inner solutions.

In this section, we shall consider the differential equation *1.5.*(13) and obtain their formal solutions, which are formal inner solutions of *1.5.*(10). The technique used here is very similar to one in the previous section.

3.1. Simplification. Let us write *1.5.*(13) as

$$(1)\qquad \rho\frac{dU}{dt}=CU, \qquad C(t)=\begin{bmatrix}0 & 1\\ p(t) & 0\end{bmatrix},$$

where $\rho=\varepsilon^{1/6}$, $p(t)=t^3-1$ and U is a 2-by-2 matrix or a 2-dim. vector.

If we consider (1) outside neighbourhoods of the roots of $p(t)=0$ (as mentioned in *1.5.2°*), the matrix C has always two different eigenvalues.

Transforming (1) by

$$(2)\qquad U=RX, \qquad R(t)=\begin{bmatrix}1 & -1\\ p^{1/2} & p^{1/2}\end{bmatrix},$$

we have

$$(3)\qquad \rho\frac{dX}{dt}=DX, \qquad D(t, \rho)=D_0(t)+\rho D_1(t),$$

where

$$D_0=\begin{bmatrix} p^{1/2} & 0 \\ 0 & -p^{1/2} \end{bmatrix} \quad \text{and} \quad D_1=-\frac{1}{4}p'p^{-1}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Further, let S be a matrix of formal power series of ρ and transform (3) by

$$X=SH, \qquad S(t,\rho)=\sum_{r=0}^{\infty} S_r(t)\rho^r. \tag{4}$$

Then (3) becomes

$$\rho\frac{dH}{dt}=EH, \qquad E(t,\rho)=S^{-1}DS-\rho S^{-1}S'=\sum_{r=0}^{\infty} E_r(t)\rho^r. \tag{5}$$

We want to determine matrices S and E so that they have forms

$$S=\begin{bmatrix} 1 & S^1 \\ S^2 & 1 \end{bmatrix}=S_0+\sum_{r=1}^{\infty}\begin{bmatrix} 0 & S_r^1 \\ S_r^2 & 0 \end{bmatrix}\rho^r, \quad S_0=I$$

and

$$E=\begin{bmatrix} E^1 & 0 \\ 0 & E^2 \end{bmatrix}=\sum_{r=0}^{\infty}\begin{bmatrix} E_r^1(t) & 0 \\ 0 & E_r^2(t) \end{bmatrix}\rho^r.$$

Such choices are possible as performed in the previous section. Indeed, we get the following recurrence formulae:

$$D_0S_0-S_0E_0=0, \qquad (D_0S_1+D_1S_0)-(S_0E_1+S_1E_0)=S_0',$$

$$(D_0S_r+D_1S_{r-1})-\sum_{j=0}^{r} S_jE_{r-j}=S_{r-1}' \qquad (r=2,3,\cdots).$$

From the above formulae we see

$$E_0=D_0, \qquad E_1=-\frac{1}{4}p'p^{-1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad E_2=\frac{1}{32}p'p^{-5/2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

and

$$S_0=I, \qquad S_1=\frac{1}{8}p'p^{-3/2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad S_2=\frac{1}{16}p^{-1/2}(p'p^{-3/2})'\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

in general

$$E_r^1=-\frac{1}{4}p'p^{-1}S_{r-1}^2, \qquad E_r^2=-\frac{1}{4}p'p^{-1}S_{r-1}^1;$$

$$S_r^1=\frac{1}{2}p^{-1/2}\left\{S_{r-1}^{2\prime}+\frac{1}{4}p'p^{-1}S_{r-1}^1+\sum_{j=1}^{r-1}S_j^1E_{r-j}^2\right\},$$

and

$$S_r^2 = -\frac{1}{2}p^{-1/2}\left\{S_{r-1}^2 + \frac{1}{4}p'p^{-1}S_{r-1}^2 + \sum_{j=1}^{r-1} S_j^2 E_{r-j}^1\right\}, \qquad r=1, 2, \cdots.$$

As shown easily by short calculation, the following relations are valid:

$$S_r^1 = (-1)^r S_r^2, \qquad B_r^1 = (-1)^{r+1} E_r^2 \qquad (r=1, 2, \cdots).$$

3. 2. Formal solutions. Since the coefficient of the differential equation (5) is diagonal, we get at once its solutions and hence the solutions of (1) by considering transformations (2) and (4). The matrix

$$U=(RS)H=\begin{bmatrix} (1-S^2)H^1 & -(1-S^1)H^2 \\ p^{1/2}(1+S^2)H^1 & p^{1/2}(1+S^1)H^2 \end{bmatrix}$$

is a solution of the differential equation (1) and its elements are given by

$$(6)_1 \qquad U^{11} = (1-S^2)H^1 = p(t)^{-1/4}\exp\left[\frac{1}{\rho}\int^t p(t)^{1/2}dt\right]\cdot[1+\varphi(t)\rho+O(\rho^2)],$$

$$(6)_2 \qquad U^{12} = -(1-S^1)H^2 = p(t)^{-1/4}\exp\left[-\frac{1}{\rho}\int^t p(t)^{1/2}dt\right]\cdot[-1+\varphi(t)\rho+O(\rho^2)],$$

$$(6)_3 \qquad U^{21} = p^{1/2}(1+S^2)H^1 = p(t)^{1/4}\exp\left[\frac{1}{\rho}\int^t p(t)^{1/2}dt\right]\cdot[1-\phi(t)\rho+O(\rho^2)],$$

$$(6)_4 \qquad U^{22} = p^{1/2}(1+S^1)H^2 = p(t)^{1/4}\exp\left[-\frac{1}{\rho}\int^t p(t)^{1/2}dt\right]\cdot[1+\phi(t)\rho+O(\rho^2)],$$

where

$$\varphi(t) = \frac{1}{8}p'(t)p(t)^{-3/2} + \frac{1}{32}\int^t p(t)^2 p'(t)^{-5/2}dt$$

$$= \int^t \left\{\frac{1}{8}p''(t)p(t)^{-3/2} - \frac{5}{32}p'(t)^2 p(t)^{-5/2}\right\}dt,$$

$$\phi(t) = \int^t \left\{\frac{1}{8}p''(t)p(t)^{-3/2} - \frac{7}{32}p'(t)^2 p(t)^{-5/2}\right\}dt$$

and $O(\rho^2)$'s are to be understood to posses the similar nature to the $O(\varepsilon^2)$'s in the previous section though the regions of definition are different.

Thus we have proved the following

THEOREM B. [Formal inner solution]. *The differential equation* (1) *possesses* *the formal solutions* (6), *which are related to the formal inner solutions of the differential equation 1.5.*(10) *by 1.5.*(14):

$$Y(x, \varepsilon)=\begin{bmatrix}1 & 0\\ 0 & \rho^3\end{bmatrix}\cdot p^{\frac{1}{4}\left[\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right]}\cdot \hat{Y}(t, \rho)\cdot \exp\left\{\frac{1}{\rho}\int^t p^{1/2}\,dt\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\right\},$$

$$t=x\varepsilon^{-1/3}, \qquad \rho=\varepsilon^{1/6},$$

where

$$\hat{Y}(t, \rho)=\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}+O(\rho)$$

and $O(\rho)$ *represents a formal power series of* ρ *beginning with the degree one with coefficients bounded for* t *such that* $|t|<\infty$ *except for the neighbourhoods of zeros of* $p(t)$.

§ 4. Canonical regions.

In order to state the asymptotic nature of the inner solutions, we have to choose appropriate integration paths of integral equations presentted in the section six. For this purpose, we shall consider some properties of the t-plane in slight detail. Also, we choose appropriate sectors for outer solutions in accordance with canonical regions for inner solutions.

4.1. Turning points and Stokes curves. We consider the following second order linear ordinary differential equation containing a positive small parameter ρ,

$$\rho^2 y''-p(t)y=0, \tag{1}$$

where $p(t)$ is a polynomial and especially $p(t)=t^3-1$.

The point t_0 is called a (secondary) turning point of (1) if it is a zero of $p(t)$, i.e., $p(t_0)=0$. The order of the turning point t_0 is the multiplicity of the zero of $p(t)$ at $t=t_0$.

A Stokes curve of (1) is a curve proceeding from a turning point $t=t_0$ along which

$$\mathrm{Re}\int_{t_0}^t p(t)^{1/2}dt=0.$$

By a short calculation, we can show that there proceed $m+2$ Stokes curves from a turning point of order m, and neighbouring curves meet at an angle $2\pi/(m+2)$.

Let

$$\xi(t_0, t)=\int_{t_0}^t p(t)^{1/2}dt.$$

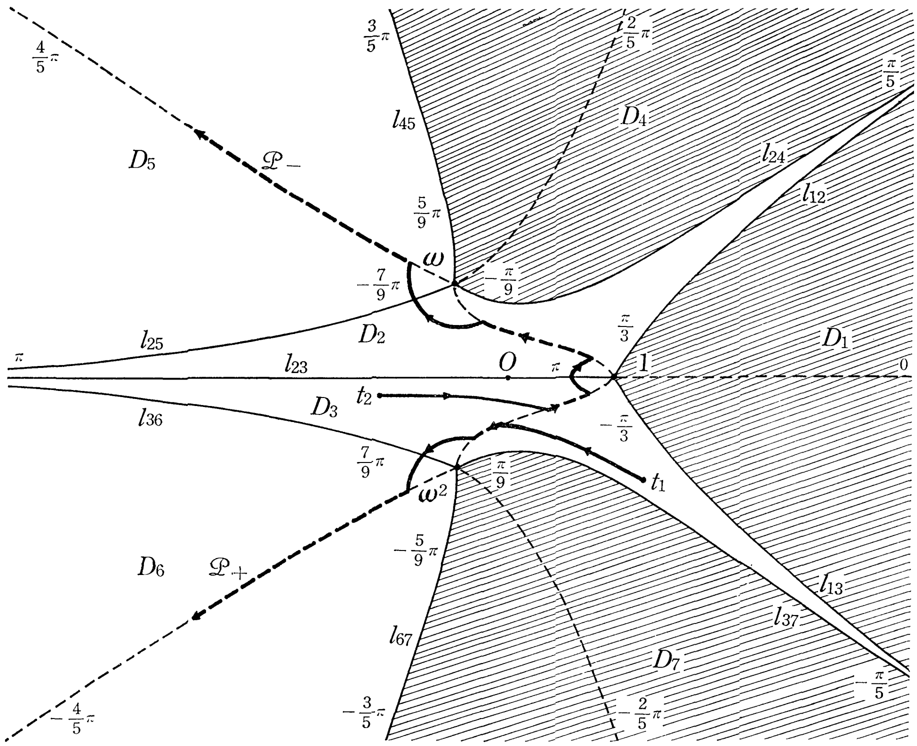

Fig. 2. Stokes curves for $p(t)=t^3-1$ are represented by solid curves.

For the case $p(t)=t^3-1$, the Stokes curves are roughly sketched in Fig. 2. Since each turning point is of order one, there proceed three Stokes curves and neighbouring curves make an angle $2\pi/3$.

In the figure, small numbers near the turning points and Stokes curves represent respectively angles of tangents of the Stokes curves at the turning points and angles of asymptotes as t tends to infinity. Dotted lines are curves proceeding from turning points along which $\operatorname{Im}\xi(t_0, t)=0$.

4.2. Canonical regions. Let D_i be an unbounded open set whose boundaries are Stokes curves l_{jk}'s. A set sum of some D_i's and l_{jk}'s would be transformed into a set in the ξ-plane with the orthogonal coordinate system $(\operatorname{Re}\xi, \operatorname{Im}\xi)$ by the transformation $\xi=\xi(t_0, t)$ for a fixed turning point t_0. If a sum of neighbouring D_i's and l_{jk}'s is mapped all over the ξ-plane simply or not doubly even in a part, then the sum is called the *canonical region* of (1). For instance, consider the case $p(t)=t^3-1$ and see Fig. 2. Let

$$\mathscr{D}_0=D_5\cup l_{25}\cup D_2\cup l_{23}\cup D_3\cup l_{36}\cup D_6.$$

Then $\mathscr{D}_0$ is a canonical region of (1) for $p(t)=t^3-1$. Here we must notice the set $\mathscr{D}_0$ is mapped onto the ξ-plane in two different ways corresponding to the choice of the branches of $p(t)^{1/2}$. Indeed, if the branch is chosen so that $\mathrm{Im}\,\xi(1,t)\geqq 0$ on l_{25}, l_{23} and l_{36}, then the point set $\mathscr{D}_0$ is transformed as shown in Fig. 3. On the other hand, if the branch is chosen such that $\mathrm{Im}\,\xi(1,t)\leqq 0$ on all of them, $\xi(\mathscr{D}_0)$ is illustrated as in Fig. 4, where $\xi(\mathscr{D}_0)=\cup_{t\in\mathscr{D}_0}\xi(1,t)$.

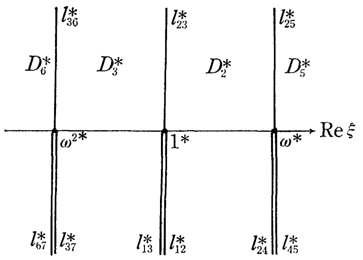

Fig. 3. Image of $\mathscr{D}_0$.

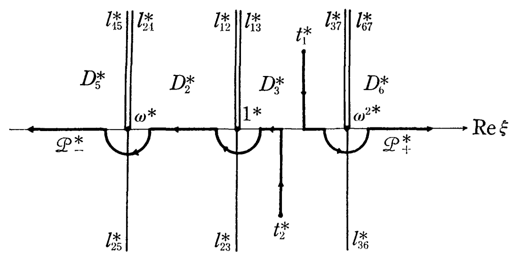

Fig. 4. Image of $\mathscr{D}_0$.[3)]

For the case $p(t)=t^3-1$, we want to choose the canonical regions of the type not in Fig. 3, but in Fig. 4, namely, of the type such that all the cuts are situated in the upper half plane. Moreover we choose several canonical regions so that neighbouring canonical regions overlap each other and all of them cover doubly all the ξ-plane. This reason will be clarified in the section seven.

All of our canonical regions $\mathscr{D}_j$ are as follows:

$$\mathscr{D}_0=D_5\cup l_{25}\cup D_2\cup l_{23}\cup D_3\cup l_{36}\cup D_6,\qquad \mathscr{D}_1=D_7\cup l_{37}\cup D_3\cap l_{13}\cup D_1,$$

$$\mathscr{D}_2=D_2\cup l_{12}\cup D_2\cup l_{24}\cup D_4,\qquad \mathscr{D}_3=D_6\cup l_{67}\cup D_7,\qquad \mathscr{D}_4=D_4\cup l_{45}\cup D_5.$$

These images in the ξ-plane are similar to $\mathscr{D}_0$. Other choices of canonical regions are possible, but the above choice appears to be most convenient.

Many properties about the canonical regions of the entire functions are analyzed in Evgrafov-Fedoryuk [1]. In this paragraph, we have mentioned only a few properties necessary for the later studies.

4. 3. Outer regions. As shown later, the formal outer solutions obtained in the second section are asymptotic power series expansions of the actual solutions in sectors, i.e., angular regions which we call the *outer regions.* Choose the outer regions $\mathfrak{D}_j$ according to the canonical regions $\mathscr{D}_j$ as follows:

$$\mathfrak{D}_0=\{x\colon\ 3\pi/5+\theta\leqq\arg x\leqq 7\pi/5-\theta,\ \varepsilon^{1/3}M\leqq|x|\leqq x_0,\ x\notin\mathfrak{S}_0\},$$

$$\mathfrak{D}_1=\{x\colon\ -2\pi/5+\theta\leqq\arg x\leqq\pi/5-\theta,\ \varepsilon^{1/3}M\leqq|x|\leqq x_0,\ x\notin\mathfrak{S}_1\},$$

3) D_i^* designates the image of D_i by $\xi=\xi(1,t)$, i.e., $D_i^*=\xi(D_i)$, and similarly $l_{jk}^*=\xi(l_{jk})$.

$$\mathfrak{D}_2=\{x:\ -\pi/5+\theta\leqq\arg x\leqq 2\pi/5-\theta,\ \varepsilon^{1/3}M\leqq|x|\leqq x_0,\ x\notin\mathfrak{S}_2\},$$

$$\mathfrak{D}_3=\{x:\ -\pi+\theta\leqq\arg x\leqq -\pi/5-\theta,\ \varepsilon^{1/3}M\leqq|x|\leqq x_0,\ x\notin\mathfrak{S}_3\},$$

$$\mathfrak{D}_4=\{x:\ \pi/5+\theta\leqq\arg x\leqq \pi-\theta,\ \varepsilon^{1/3}M\leqq|x|\leqq x_0,\ x\notin\mathfrak{S}_4\}.$$

Here θ is a fixed small positive constant and $\mathfrak{S}_j$ is small parts of $\mathfrak{D}_j$ near the circle $|x|=\varepsilon^{1/3}M$ called shadow zones, which will be pointed out precisely in the next section.

§ 5. Existence of outer solutions.

There exist, in each outer region, actual solutions of the outer differential equation *1.5.*(12) or (12′) expansible asymptotically in the formal outer solutions obtained in the second section. Since the proof is, however, very similar to the previous paper by Nishimoto [6], we shall state only the result in the theorems and the integration paths which play an important role in their proof.

5. 1. Existence theorems.

THEOREM C. *There exists a solution $z(x,\lambda)$ of the outer differential equation which possesses the following asymptotic nature. For $0<\varepsilon\leqq\varepsilon_0'\leqq\varepsilon_0$,*

$$z(x,\lambda)=\exp\left[\int^x\left(-\frac{1}{\lambda}+b\right)x^{-3/2}dx\right]\cdot[1+O(\lambda)]\qquad \textit{as } \lambda\to 0 \textit{ uniformly for } x\in\mathfrak{D},$$

$$\frac{d}{dx}z(x,\lambda)=-(\lambda x^{3/2})^{-1}\exp\left[\int^x\left(-\frac{1}{\lambda}+b\right)x^{-3/2}dx\right]\cdot[1+O(\lambda)]\quad \textit{as } \lambda\to 0 \textit{ uniformly for } x\in\mathfrak{D},$$

where $b=1/2-(3/4)x^{1/2}$, $\lambda=x^{-3}\varepsilon$ and $\mathfrak{D}$ is any one of the outer regions.

THEOREM C′. *There exists a solution $z(x,\lambda)$ of the outer differential equation which possesses the following asymptotic nature. For $0<\varepsilon\leqq\varepsilon_0'\leqq\varepsilon_0$,*

$$z(x,\lambda)=\exp\left[\int^x\left(\frac{1}{\lambda}+a\right)x^{-3/2}dx\right]\cdot[1+O(\lambda)]\qquad \textit{as } \lambda\to 0 \textit{ uniformly for } x\in\mathfrak{D},$$

$$\frac{d}{dx}z(x,\lambda)=(\lambda x^{3/2})^{-1}\exp\left[\int^x\left(\frac{1}{\lambda}+a\right)x^{-3/2}dx\right]\cdot[1+O(\lambda)]\qquad \textit{as } \lambda\to 0 \textit{ uniformly for } x\in\mathfrak{D},$$

where $a=-1/2-(3/4)x^{1/2}$, $\lambda=x^{-3}\varepsilon$ and $\mathfrak{D}$ is any one of the outer regions.

In order to prove the above theorems we must choose the integration paths along which both of the following inequalities are valid in every outer region:

$$\mathrm{Re}\int_\tau^x x^{-3/2}\lambda^{-1}dx\leqq 0\qquad\text{for all}\quad \tau\in\mathfrak{P}_x^+,$$

and

$$\mathrm{Re}\int_{\tau}^{x} x^{-3/2}\lambda^{-1}dx \geqq 0 \qquad \text{for all} \quad \tau \in \mathfrak{P}_x^-.$$

We shall show in the next paragraph these choices are possible.

5.2. Paths of integration. We have defined five outer regions $\mathfrak{D}_j$ in *4.3*. The outer region $\mathfrak{D}$ in Theorems C and C′, can be taken any one of $\mathfrak{D}_j$'s, in which paths of integration $\mathfrak{P}_x^{\pm}$ satisfying desired conditions can be chosen.

Notice $\mathfrak{D}_2$ and $\mathfrak{D}_4$ is in position of complex conjugate of $\mathfrak{D}_1$ and $\mathfrak{D}_3$ respectively.
Let

$$\hat{\tau} = \tau^{5/2}.$$

and let for any set A

$$\hat{A} = \{\hat{\tau}: \tau \in A\}.$$

Then we see that

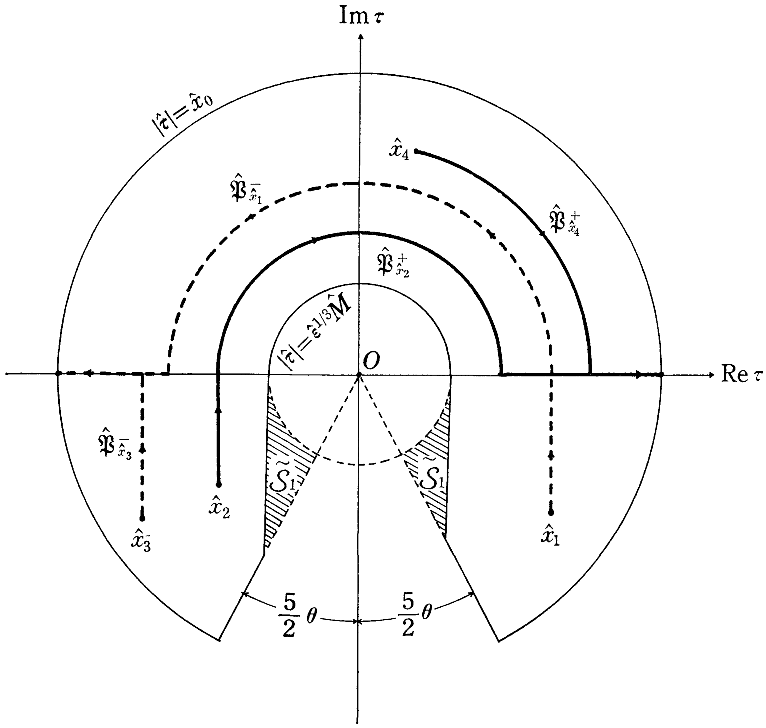

Fig. 5. $\mathcal{S}_1 = \hat{\mathfrak{D}}_0 = \hat{\mathfrak{D}}_3 = \hat{\hat{\mathfrak{D}}}_4$.

$$\hat{\mathfrak{D}}_0 = \hat{\mathfrak{D}}_3 = \hat{\bar{\mathfrak{D}}}_4,^{4)}$$

which is a sector defined by

$$\mathcal{S}_1 = \{\hat{\tau}\colon\ -\pi/2 + 5\theta/2 \leqq \arg \hat{\tau} \leqq 3\pi/2 - 5\theta/2,\ \hat{\varepsilon}^{1/3}\hat{M} \leqq |\hat{\tau}| \leqq \hat{x}_0,\ \hat{\tau} \notin \tilde{\mathcal{S}}_1\},$$

and
$$\hat{\mathfrak{D}}_1 = \hat{\bar{\mathfrak{D}}}_2,$$

which is a sector defined by

$$\mathcal{S}_2 = \{\hat{\tau}\colon\ \pi/2 - 5\theta/2 \leqq \arg \hat{\tau} \leqq \pi + 5\theta/2,\ \hat{\varepsilon}^{1/3}\hat{M} \leqq |\hat{\tau}| \leqq \hat{x}_0,\ \hat{\tau} \notin \tilde{\mathcal{S}}_2\},$$

where $\tilde{\mathcal{S}}_1$ and $\tilde{\mathcal{S}}_2$ are called shadow zones defined later.

The path $\hat{\mathfrak{P}}^+_{\hat{x}}$ starts from the point $\hat{x}$, runs to the real axis along the segment parallel to the imaginary axis, runs, if necessary, along a part of the circle $|\hat{\tau}| = |\hat{x}|$ with a center at the origin as far as the real axis and then runs onto the circle $|\hat{\tau}| = \hat{x}_0$ along the real axis in the positive derection.

The path $\hat{\mathfrak{P}}^-_{\hat{x}}$ is very similar to $\hat{\mathfrak{P}}^+_{\hat{x}}$ but slightly different. That is, after it runs, if necessary, on a part of the circle $|\hat{\tau}| = |\hat{x}|$ it runs either on the negative

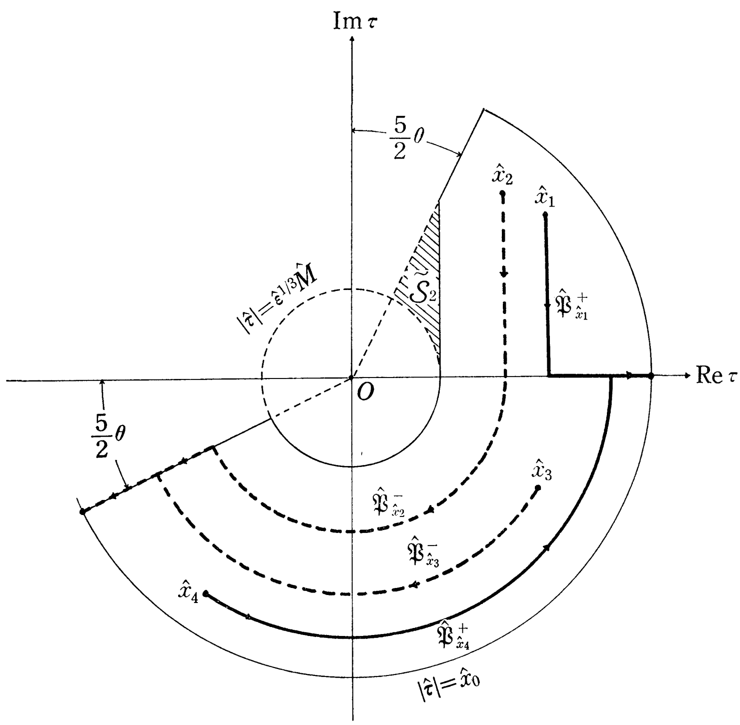

Fig. 6. $\mathcal{S}_2 = \hat{\mathfrak{D}}_1 = \hat{\bar{\mathfrak{D}}}_2$.

4) The bar besignates the complex conjugate.

real axis to the border of $\mathcal{S}_1$ for the case $\mathcal{S}_1$, or on the segment with $\arg\hat{\tau}=\pi+5\theta/2$ as far as the border of $\mathcal{S}_2$ for the case $\mathcal{S}_2$. The figures explain explicitly these aspects. However we notice that small parts of $\mathcal{S}_1$ and $\mathcal{S}_2$ near the circle $|\hat{\tau}|=\hat{\varepsilon}^{1/3}\hat{M}$ must be deleted, because if the starting point $\hat{x}$ of the paths belongs to one of them the required condition

$$\mathrm{Re}\,(\hat{x}-\hat{\tau})\leqq 0 \quad \text{or} \quad \mathrm{Re}\,(\hat{x}-\hat{\tau})\geqq 0$$

is impossible. These deleted parts $\tilde{\mathcal{S}}_1$ and $\tilde{\mathcal{S}}_2$ are called shadow zones which shrink to the origin as ε tends to zero, therefore deleting shadow zones does not affect our theory. The shadow zones $\mathfrak{S}_j$'s in the x-plane are inverse images of $\tilde{\mathcal{S}}_1$ or $\tilde{\mathcal{S}}_2$.

The paths $\mathfrak{P}_x^{\pm}$ are defined the inverse images of $\hat{\mathfrak{P}}_{\hat{x}}^{\pm}$.

§ 6. Existence of inner solutions.

This section is devoted to showing existence of actual inner solutions asymptotically expansible in the formal power series solutions.

6. 1. Existence theorem. We shall show the following:

THEOREM D. *The differential equation 1. 5.* (13) *possesses solutions such that*

$$u(t,\rho)=p(t)^{-1/4}\exp\left[-\frac{1}{\rho}\int^t p(t)^{1/2}dt\right]\cdot[1+O(\rho)], \tag{1}$$

$$\frac{d}{dt}u(t,\rho)=-\frac{1}{\rho}p(t)^{1/4}\exp\left[-\frac{1}{\rho}\int^t p(t)^{1/2}dt\right]\cdot[1+O(\rho)], \tag{1'}$$

uniformly for $t\in\mathcal{D}^\delta$ as $\rho=\varepsilon^{1/6}$ tends to zero. Here $\mathcal{D}^\delta$ is a region obtained from a canonical region by deleting δ-neighbourhoods of the boundary, δ is an arbitrarily fixed positive and small constant.

Proof. The equation *1. 5.* (13) is changed to

$$\frac{d\Phi}{dt}=\left\{\frac{p^{1/2}}{\rho}\begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix}+\begin{bmatrix}\alpha & -\beta\\ \beta & -\alpha\end{bmatrix}\rho\right\}\Phi \tag{2}$$

by the transformations

$$U=p^{-1/4}\exp\left[-\frac{1}{\rho}\int^t p(t)^{1/2}dt\right]\cdot R\cdot\hat{S}\cdot\Phi,$$

where

$$R=\begin{bmatrix}1 & -1\\ p^{1/2} & p^{1/2}\end{bmatrix},\qquad \hat{S}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\beta\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\rho,$$

$$\alpha=\frac{1}{32}\frac{p'^2}{p^{5/2}},\qquad \beta=\frac{1}{8}\left(\frac{p'}{p^{3/2}}\right)'=\frac{p''}{8p^{3/2}}-\frac{3}{16}\frac{p'^2}{p^{5/2}}\qquad \text{and}\quad p(t)=t^3-1.$$

Let

$$\Phi=\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix}.$$

Then, (2) is equivalent to the integral equations

$$(3)\qquad \begin{cases}\varphi_1(t)=\rho\displaystyle\int_{\mathscr{P}+}\{\alpha(\zeta)\varphi_1(\zeta)-\beta(\zeta)\varphi_2(\zeta)\}\exp\left[\frac{2}{\rho}\int_\zeta^t p(t)^{1/2}dt\right]d\zeta,\\ \varphi_2(t)=-1+\displaystyle\int_{\mathscr{P}+}\{\beta(\zeta)\varphi_1(\zeta)-\alpha(\zeta)\varphi_2(\zeta)\}d\zeta.\end{cases}$$

If the integration path $\mathscr{P}_+$ is chosen appropriately, the above integral equations could be solved. Along the path $\mathscr{P}_+$, we want to have the property

$$\mathrm{Re}\int_\zeta^t p(t)^{1/2}dt\leqq 0.$$

Indeed, this choice is possible if we remember how to determine the canonical regions. One way of possible choices is shown in Figures 2 and 3.

If the integration path $\mathscr{P}_+$ with the desired property can be chosen, we can obtain solutions of (3) as follows.

Let

$$(3')\qquad \Phi=\Phi_0+\mathscr{J}\Phi,\qquad \Phi_0=\begin{bmatrix}0\\ -1\end{bmatrix},$$

where $\mathscr{J}$ represents the integral operator defined by (3). Then (3′) is the same as (3). Furthermore, let

$$\Phi^{(0)}=\Phi_0,\qquad \Phi^{(n)}=\mathscr{J}^n\Phi_0\quad\text{and}\quad \Phi^{(n)}=\begin{bmatrix}\varphi_1^{(n)}\\ \varphi_2^{(n)}\end{bmatrix}.$$

This means that

$$\begin{cases}\varphi_1^{(0)}=0\\ \varphi_2^{(0)}=-1\end{cases},$$

$$\begin{cases}\varphi_1^{(n+1)}=\rho\displaystyle\int_{\mathscr{P}_+}(\alpha\varphi_1^{(n)}-\beta\varphi_2^{(n)})\exp\left[\frac{2}{\rho}\int^t p^{1/2}dt\right]d\zeta\\ \varphi_2^{(n+1)}=\rho\displaystyle\int_{\mathscr{P}_+}(\beta\varphi_1^{(n)}-\alpha\varphi_2^{(n)})d\zeta\end{cases}\qquad (n=0,1,2,\cdots).$$

We get an estimate

$$(4)\qquad |\varphi_j^{(n)}|\leqq|2\psi\rho|^n\qquad (n=0,1,2,\cdots;j=1,2),$$

where

$$\psi=\psi(t)=\int_{\mathscr{P}_+}(|\alpha|+|\beta|)|d\zeta|.$$

Notice ψ is bounded for t in the canonical region except for the neighbourhoods of the secondary turning points, and so put

$$\psi_0=\sup_{t\in\mathscr{P}_+}|\psi(t)|.$$

Now, we shall prove (4). Clearly it is true for $n=0$: $|\varphi_j^{(0)}|\leqq 1$ $(j=1,2)$. Remembering how to choose the path, we see, by induction, that

$$|\varphi_1^{(n+1)}|\leqq\rho\int_{\mathscr{P}_+}|\alpha\varphi_1^{(n)}-\beta\varphi_2^{(n)}||d\zeta|$$

$$\leqq\rho\int_{\mathscr{P}_+}(|\alpha|+|\beta|)(|\varphi_1^{(n)}|+|\varphi_2^{(n)}|)|d\zeta|\leqq|2\psi\rho|^{n+1}.$$

Similar calculation is valid for $\varphi_2^{(n+1)}$.

Also, by taking ρ so small that the series $\sum_{n=0}^{\infty}|2\psi_0\rho|^n$ converges, we obtain the following estimates:

$$|\varphi_j|\leqq 2\ (j=1,2),\qquad |\varphi_1|\leqq 4\psi_0\rho,\qquad |\varphi_2+1|\leqq 4\psi_0\rho.$$

By noticing the above preparatory estimates and the relation

$$U=\begin{bmatrix}U_1\\U_2\end{bmatrix}=p^{-1/4}\exp\left[-\frac{1}{\rho}\int^t p^{1/2}dt\right]\cdot R\hat{S}\cdot\Phi,\qquad U_1'=\frac{1}{\rho}U_2\quad\text{and}\quad u=U_1,$$

we can show (1) and (1′), namely, we get

$$\left|u-p^{-1/4}\exp\left[-\frac{1}{\rho}\int^t p^{1/2}dt\right]\right|\leqq\left|p^{-1/4}\exp\left[-\frac{1}{\rho}\int^t p^{1/2}dt\right]\right|\cdot 4\rho(|\beta|+2\psi_0),$$

$$\left|u'+\frac{1}{\rho}p^{1/4}\exp\left[-\frac{1}{\rho}\int^t p^{1/2}dt\right]\right|\leqq\left|\frac{1}{\rho}p^{1/4}\exp\left[-\frac{1}{\rho}\int^t p^{1/2}dt\right]\right|\cdot 4\rho(|\beta|+2\psi_0).$$

These two inequalities are equivalent to the relations (1) and (1′) respectively. Therefore we have proved the theorem. *Q.E.D.*

Since we can choose the path $\mathscr{P}_-$ along which the inequality

$$\operatorname{Re}\int_\zeta^t p(t)^{1/2}dt\geqq 0$$

is valid in the canonical region, we obtain by the same discussion the following

THEOREM D′. *There exist solutions of the differential equation 1.5.*(13) *pos-*

sessing the following asymptotic nature

$$v(t,\rho)=p(t)^{-1/4}\exp\left[\frac{1}{\rho}\int^{t}p(t)^{1/2}dt\right]\cdot[1+O(\rho)],$$

$$\frac{d}{dt}v(t,\rho)=\frac{1}{\rho}p(t)^{1/4}\exp\left[\frac{1}{\rho}\int^{t}p(t)^{1/2}dt\right]\cdot[1+O(\rho)],$$

uniformly in $t\in\mathfrak{D}^{\delta}$ *as* ρ *tends to zero.*

6.2. Another asymptotic nature. Theorems D and D' state the asymptotic properties of the solutions with respect to the parameter, but if we notice the last estimates in the preceeding paragraph then we can see that the solutions possess asymptotic properties with respect to the independent variable. That is to say:

If $t\to\infty$ *so that* $\mathrm{Re}\int^{t}p(t)^{1/2}dt\to+\infty$,

$$u(t,\rho)\sim p(t)^{-1/4}\exp\left[-\frac{1}{\rho}\int^{t}p(t)^{1/2}dt\right],$$

$$\frac{d}{dt}u(t,\rho)\sim-\frac{1}{\rho}p(t)^{1/4}\exp\left[-\frac{1}{\rho}\int^{t}p(t)^{1/2}dt\right],$$

uniformly in ρ $(0<\rho\leqq\rho_0\leqq\varepsilon_0^{1/6})$.

Similarly, if $t\to\infty$ *so that* $\mathrm{Re}\int^{t}p(t)^{1/2}dt\to-\infty$,

$$v(t,\rho)\sim p(t)^{-1/4}\exp\left[\frac{1}{\rho}\int^{t}p(t)^{1/2}dt\right],$$

$$\frac{d}{dt}v(t,\rho)\sim\frac{1}{\rho}p(t)^{1/4}\exp\left[\frac{1}{\rho}\int^{t}p(t)^{1/2}dt\right],$$

uniformly in ρ $(0<\rho\leqq\rho_0\leqq\varepsilon_0^{1/6})$.

Clearly, two solutions u and v are independent and possess two asymptotic natures in any one of canonical regions. Relation between independent solutions in one canonical region and the other will be obtained in the next section.

§7. Connection between inner solutions.

By applying the theory of Evgrafov-Fedoryuk [1], we can get relations between solutions in one canonical region and in the others. Although the proof is simple, we shall state only the result.

7.1. Connection matrices (1). In the preceeding section, we have obtained the fundamental solutions u and v of the inner equation such that

$$u \sim cp^{-1/4} \exp\left[\frac{1}{\rho}\xi(t)\right],$$

$$v \sim cp^{-1/4} \exp\left[-\frac{1}{\rho}\xi(t)\right],$$

as ρ tends to zero uniformly for $t \in \mathcal{D}^\delta$, where $\xi(t)=\int_1^t p^{1/2}dt$, $p(t)=t^3-1$ and $\mathcal{D}^\delta$ is given in Theorem D or D′. The complex constant c is to be chosen such that $|c|=1$, $\lim_{t\to 1, t\in l} \arg [cp^{-1/4}]=0$, where l is a Stokes curve proceeding from the secondary turning point $t=1$. We call the fundamental matrix determined in this way the *elementary fundamental matrix defined by* $(l, 1, \mathcal{D})$ according to Evgrafov-Fedoryuk [1].

Let

$$\begin{bmatrix} U_k^{11} & U_k^{12} \\ U_k^{21} & U_k^{22} \end{bmatrix}$$

be the elementary fundamental matrix defined by $(l_k, 1, \mathcal{D}_k)$, and let Ω_{jk} be the matrix of transition from the elementary fundamental matrix defined by $(l_j, 1, \mathcal{D}_j)$ to the elementary fundamental matrix defined by $(l_k, 1, \mathcal{D}_k)$, that is,

$$\begin{bmatrix} U_k^{11} & U_k^{12} \\ U_k^{21} & U_k^{22} \end{bmatrix} = \begin{bmatrix} U_j^{11} & U_j^{12} \\ U_j^{21} & V_j^{22} \end{bmatrix} \Omega_{jk},$$

or

$$\begin{bmatrix} u_k \\ v_k \end{bmatrix} = \Omega_{jk}^{T} \begin{bmatrix} u_j \\ v_j \end{bmatrix}.^{5)}$$

Here l_0, l_1 and l_2 denote Stokes curves l_{23}, l_{13} and l_{12} respectively.

With the above preliminaries we can get the relation between three elementary fundamental matrices around the secondary turning point $t=1$. These relations are stated in the following:

THEOREM E. [Connection matrices around $t=1$]. *The connection matrices Ω_{jk}'s have the forms*

$$\Omega_{12}=e^{\pi i/6}\begin{bmatrix} -\dfrac{i}{\omega_{01}} & 1 \\ \omega_{12} & 0 \end{bmatrix}, \qquad \Omega_{20}=e^{\pi i/6}\begin{bmatrix} -\dfrac{i}{\omega_{12}} & 1 \\ \omega_{20} & 0 \end{bmatrix},$$

$$\Omega_{01}=e^{\pi i/6}\begin{bmatrix} -\dfrac{i}{\omega_{20}} & 1 \\ \omega_{01} & 0 \end{bmatrix},$$

5) A^T denotes the transposed matrix of A.

where $\omega_{jk}=1+O(\rho)$, *as* $\rho\to 0$, *and* $\omega_{12}\omega_{20}\omega_{01}=1$.

The proof is given in the Evgrafov-Fedoryuk [1], and so omitted here. The above theorem tells us all the relation between the solutions around the secondary turning point $t=1$. Moreover it enables us to know relations between the solutions around other secondary turning points, because the structures of regions near the secondary turning points are very similar from our choice of canonical regions (cf. *4.2.*). Thus the connection matrices around the secondary turning points $t=\omega$ and $t=\omega^2$ possess all the same forms as the case in the theorem.

7.2. Connection matrices (2). In addition to the above theorem, if we got connection matrices between elementary fundamental matrices around one secondary turning point and the others, we could get all the relations between any pair of two elementary fundamental matrices of inner solutions.

Indeed, after a simple calculation, we have connection matrices between two elementary fundamental matrices definee by $(l_{23}, 1, \mathscr{D}_0)$ and $(l_{36}, \omega^2, \mathscr{D}_0)$, $(l_{23}, 1, \mathscr{D}_0)$ and $(l_{25}, \omega, \mathscr{D}_0)$ respectively.

THEOREM F. *The connection matrix from the elementary fundamental matrix defined by* $(l_{23}, 1, \mathscr{D}_0)$ *to the elementary fundamental matrix defined by* $(l_{36}, \omega^2, \mathscr{D}_0)$ *is given by*

$$e^{-23\pi i/36}\begin{bmatrix} e^{-\tau_1/\rho} & 0 \\ 0 & e^{\tau_1/\rho} \end{bmatrix},$$

where

$$\tau_1=\int_1^{\omega^2} p^{1/2}dt \quad \textit{and} \quad \lim_{t\in l_{23},\, t\to 1} \arg p^{-1/4} - \lim_{t\in l_{36},\, t\to\omega^2} \arg p^{-1/4} = -\frac{3}{4}\pi-\left(-\frac{\pi}{9}\right)=-\frac{23}{36}\pi.$$

The connection matrix from the elementary fundamental matrix defined by $(l_{25}, \omega, \mathscr{D}_0)$ *to the elementary fundamental matrix defined by* $(l_{23}, 1, \mathscr{D}_0)$ *is given by*

$$e^{\pi i/9}\begin{bmatrix} e^{-\tau_2/\rho} & 0 \\ 0 & e^{\tau_2/\rho} \end{bmatrix},$$

where

$$\tau_2=\int_\omega^1 p^{1/2}dt \quad \textit{and} \quad \lim_{t\in l_{25},\, t\to\omega} \arg p^{-1/4} - \lim_{t\in l_{23},\, t\to 1} \arg p^{-1/4} = -\frac{23}{36}\pi-\left(-\frac{3}{4}\pi\right)=\frac{\pi}{9}.$$

§ 8. Matching matrix.

In this section, we will calculate the matching matrix which is a connecting relation matrix between the inner solution and the outer one.

8. 1. Matching matrix (1). We shall match the inner solution defined by $(l_0, 1, \mathcal{D}_0)$ and the outer one in the sector: $3\pi/5<\arg x<7\pi/5$ or $3\pi/5<\arg t<7\pi/5$, i.e., the sector of the outer region $\mathfrak{D}_0$ and the corresponding canonical region $\mathcal{D}_0$. In this sector, the fundamental matrix of the outer solutions possesses the following asymptotic expansion

$$(1)\qquad G_o\cdot F_o\cdot E_o=\begin{bmatrix} x^{-3/4} & 0 \\ 0 & x^{3/4}\end{bmatrix}\cdot\begin{bmatrix} z_{11} & -z_{12} \\ z_{21} & z_{22}\end{bmatrix}\cdot\exp\left\{\left(\frac{1}{\varepsilon}\frac{2}{5}x^{5/2}+x^{-1/2}\right)\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\right\},$$

where $z_{jk}=1+O(\varepsilon)$ as ε tends to zero for x belonging to $\mathfrak{D}_0$ and a branch of $x^{3/4}$ is arbitrarily chosen. The fundamental matrix of inner solutions defined by $(l_0, 1, \mathcal{D}_0)$ possesses the asymptotic expansion

$$(2)\qquad G_i\cdot F_i\cdot E_i=\begin{bmatrix} cp^{-1/4} & 0 \\ 0 & c\rho^3 p^{1/4}\end{bmatrix}\cdot\begin{bmatrix} u_{11} & -u_{12} \\ u_{21} & u_{22}\end{bmatrix}\cdot\exp\left\{\left(\frac{1}{\rho}\int_1^t p^{1/2}dt\right)\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\right\},$$

where $u_{jk}=1+O(\rho)$ as ρ tends to zero for t belonging to $\mathcal{D}_0$ and c is chosen such that $|c|=1$ and $\arg(cp^{-1/4})=0$ as t tends 1 along the Stokes curve l_0, i.e., $c=e^{3\pi i/4}$.

Then the matching matrix $\mathfrak{M}_1$ between the above two solutions is related by the equation

$$(3)\qquad G_o\cdot F_o\cdot E_o\cdot\mathfrak{M}_1=G_iF_iE_i,$$

or

$$(3')\qquad E_o\mathfrak{M}_1E_i^{-1}=F_o^{-1}G_o^{-1}G_iF_i.$$

In view of (1), (2) and (3) or (3′), we can calculate $\mathfrak{M}_1$. Indeed, we get:

THEOREM G. [Matching matrix]. *The matching matrix* $\mathfrak{M}_1$ *defined by* (3) *is of a form*

$$\mathfrak{M}_1\sim e^{3\pi i/4}\varepsilon^{1/4}\begin{bmatrix} e^{c_1} & 0 \\ 0 & e^{-c_1}\end{bmatrix},$$

as ε tends to zero, and c_1 is a constant given in the proof.

Proof. First, we notice that the two variables x and t are connected by the relations $x=t\rho^2$ and $\rho=\varepsilon^{1/6}$ and that $x=\eta\rho$ belongs to the outer region $\mathfrak{D}_0$ for ρ small and $t=\eta\rho^{-1}$ belongs to the inner region or to the canonical region $\mathcal{D}_0$ for ρ small, where η is an additional complex parameter whose magnitude may be taken constant, say, unit. Since, after a simple computation,

$$g=\frac{1}{\varepsilon}\frac{2}{5}x^{5/2}+x^{-1/2}=\frac{2}{5}\eta^{5/2}\rho^{-7/2}+\eta^{-1/2}\rho^{-1/2},$$

$$\frac{1}{\rho}\int_1^t p^{1/2}dt = g + O(\eta^{-7/2}\rho^{5/2}) + c_1,$$

where the O-symbol denotes a fractional power series of ρ and η^{-1} beginning with order $\eta^{-7/2}\rho^{5/2}$, we get

$$\exp\left[\pm g \mp \frac{1}{\rho}\int_1^t p^{1/2}\,dt\right] = e^{\mp c_1}\{1 + O(\eta^{-7/2}\rho^{5/2})\}$$

and

$$\exp\left[\pm g \mp \frac{1}{\rho}\int_1^t p^{1/2}\,dt\right] = e^{\pm c_1}e^{\pm 2g}\{1 + O(\eta^{-7/2}\rho^{5/2})\}.$$

Thus we get, denoting the (j,k)-element of $\mathfrak{M}_1$ by $\mathfrak{m}_{jk}$,

$$E_o\mathfrak{M}_1E_i^{-1} = \begin{bmatrix} e^{-c_1}\mathfrak{m}_{11} & e^{c_1}e^{2g}\mathfrak{m}_{12} \\ e^{-c_1}e^{-2g}\mathfrak{m}_{21} & e^{c_1}\mathfrak{m}_{22}\end{bmatrix}\cdot K \quad \text{and} \quad K = \begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix} + O(\eta^{-7/2}\rho^{5/2}),$$

where the O-symbol denotes a matrix whose components consist of fractional power series of ρ and η^{-1} beginning with an order $\eta^{-7/2}\rho^{5/2}$ and we remark that $O(\eta^{-7/2}\rho^{5/2}) = O(\rho^{5/2})$ for η whose magnitude is constant.

Then the right side of (3′) is

$$F_o^{-1}G_o^{-1}G_iF_i = c\begin{bmatrix} z_{11} & -z_{12} \\ z_{21} & z_{22}\end{bmatrix}^{-1}\begin{bmatrix} x^{3/4} & 0 \\ 0 & x^{-3/4}\end{bmatrix}\begin{bmatrix} p^{-1/4} & 0 \\ 0 & \rho^3p^{1/4}\end{bmatrix}\begin{bmatrix} u_{11} & -u_{12} \\ u_{21} & u_{22}\end{bmatrix},$$

in which $c = \exp(3\pi i/4)$ and the first matrix F_o and the last one F_i are in the form

$$\begin{bmatrix} 1 & -1 \\ 1 & 1\end{bmatrix} + O(\rho)$$

and $x^{3/4}p^{-1/4}$ and $\rho^3x^{-3/4}p^{1/4}$ is $\rho^{3/2}\{1 + O(\eta^{-3}\rho^3)\}$, hence we have

$$F_o^{-1}G_o^{-1}G_iF_i = c\rho^{3/2}\begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix} + O(\rho^{5/2})$$

by inserting $x = \eta\rho$ and $t = \eta\rho^{-1}$.

Remarking that K^{-1} exists for ρ small and it is of the form $I + O(\rho^{5/2})$, we get

$$(4) \qquad \begin{bmatrix} e^{-c_1}\mathfrak{m}_{11} & e^{c_1}e^{2g}\mathfrak{m}_{12} \\ e^{-c_1}e^{-2g}\mathfrak{m}_{21} & e^{c_1}\mathfrak{m}_{22}\end{bmatrix} = c\rho^{3/2}\begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix} + O(\rho^{5/2}).$$

From the above equality (4), we can determine $\mathfrak{m}_{jk}$'s. Indeed, since $e^{-c_1}\mathfrak{m}_{11}$

$=c\rho^{3/2}+O(\rho^{5/2})$, $e^{c_1}\mathfrak{m}_{22}=c\rho^{3/2}+O(\rho^{5/2})$, $e^{-c_1}\mathfrak{m}_{11}$ and $e^{c_1}\mathfrak{m}_{22}$ are clearly asymptotically equal to $c\varepsilon^{1/4}$ as ρ tends to zero for any fixed η. Hence $\mathfrak{m}_{11}\sim ce^{c_1}\varepsilon^{1/4}$ and $\mathfrak{m}_{22}\sim ce^{-c_1}\varepsilon^{1/4}$ as ε tends to zero. As for $\mathfrak{m}_{12}$ and $\mathfrak{m}_{21}$, we can determine them as follows. Since the matching matrix is not dependent on the independent variable x or t, we can choose as x or t arbitrary values in the relations

$$e^{2g}\mathfrak{m}_{12}=O(\rho^{5/2}), \tag{5}$$

and

$$e^{-2g}\mathfrak{m}_{21}=O(\rho^{5/2}). \tag{6}$$

In the equation (5), we choose t on the path $\mathcal{P}_+$, namely, $t=|t|e^{6\pi i/5}$ for $|t|$ large or $\eta=e^{6\pi i/5}$ for ρ small. Then, $\mathrm{Re}\,[e^{2g}]$ becomes large for ρ small, but the right side of (5) becomes small for ρ small. Hence the $\mathfrak{m}_{12}$ must be asymptotically zero as ρ tends to zero. On the other hand, in the equation (6), we choose t on the path $\mathcal{P}_-$, namely, $t=|t|e^{4\pi i/5}$ for $|t|$ large or $\eta=e^{4\pi i/5}$ for ρ small. Then, $\mathrm{Re}\,[e^{-2g}]$ becomes large for ρ small but the right side of (6) becomes small for ρ small. This induces that the $\mathfrak{m}_{21}$ also must be asymptotically zero as ρ tends to zero. Therefore, we have completed the theorem. *Q.E.D.*

8.2. Matching matrix (2). In the preceding paragraph, we obtained the matching matrix of the outer solution in $\mathfrak{D}_0$ and the inner solution defined by $(l_0, 1, \mathcal{D}_0)$. The matching matrix of the outer solution in $\mathfrak{D}_0$ and the inner solution defined by $(l_{36}, \omega^2\, \mathcal{D}_0)$ or $(l_{25}, \omega, \mathcal{D}_0)$ are calculated by the same method in Theorem G.

Let $\mathfrak{M}_{\omega^2}$ (or $\mathfrak{M}_\omega$) be the matching matrix of the fundamental matrix of outer solutions in the sector $\mathfrak{D}_0$ and the fundamental matrix of inner solutions defined by $(l_{36}, \omega^2, \mathcal{D}_0)$ (or $(l_{25}, \omega, \mathcal{D}_0)$). Then, noticing that

$$\frac{1}{\rho}\int_{\omega^2}^{t} p^{1/2}dt=\frac{1}{\rho}\int_{1}^{t} p^{1/2}\,dt+\frac{1}{\rho}\int_{\omega^2}^{1} p^{1/2}dt=g+O(\rho^{5/2})+c_1-\frac{1}{\rho}\tau_1,$$

and

$$\frac{1}{\rho}\int_{\omega}^{t} p^{1/2}dt=\frac{1}{\rho}\int_{1}^{t} p^{1/2}dt+\frac{1}{\rho}\int_{\omega}^{1} p^{1/2}dt=g+O(\rho^{5/2})+c_1+\frac{1}{\rho}\tau_2,$$

where τ_1 and τ_2 are constants defined by in Theorem F, and replacing c_1 in the proof of Theorem G by $c_1-(1/\rho)\tau_1$ or $c_1+(1/\rho)\tau_2$, we get the asymptotic expansions of $\mathfrak{M}_{\omega^2}$ and $\mathfrak{M}_\omega$.

THEOREM H. *The matching matrices $\mathfrak{M}_{\omega^2}$ and $\mathfrak{M}_\omega$ are given by*

$$\mathfrak{M}_{\omega^2}\sim e^{\pi i/9}\varepsilon^{1/4}\begin{bmatrix} e^{c_1} & 0 \\ 0 & e^{-c_1}\end{bmatrix}\begin{bmatrix} e^{-\tau_1/\rho} & 0 \\ 0 & e^{\tau_1/\rho}\end{bmatrix},$$

$$\mathfrak{M}_{\omega} \sim e^{23\pi i/36} \varepsilon^{1/4} \begin{bmatrix} e^{c_1} & 0 \\ 0 & e^{-c_1} \end{bmatrix} \begin{bmatrix} e^{\tau_2/\rho} & 0 \\ 0 & e^{-\tau_2/\rho} \end{bmatrix},$$

as ε tends to zero. Here, c_1, τ_j $(j=1, 2)$ are constants given in Theorem G and Theorem F respectively.

8. 3. Matching matrix (3). In the preceding paragraphs, we obtained the matching matrices in the sector according to $\mathfrak{D}_0$ or $\mathscr{D}_0$. Since the outer and inner solutions, however, possess the same asymptotic expansions in the other sectors too, and there exist paths along which the real part of g tends to $\pm\infty$ as $\rho \to 0$, the matching matrices in the other sectors must be of the same asymptotic form. Thus we have completed the relation between the outer solution and the inner one in each sector according to every canonical region.

§ 9. Example.

9. 1. Summary. We shall summarize the results of the present paper. We have considered the second order linear ordinary differential equation containing a small positive parameter (10) in § **1**. The equation contains a turning point at the origin and satisfies the (simplest) two segment condition. We have analyzed it to get solutions in two different regions which are called the outer and the inner ones and overlap each other. Therefore, we can calculate the matching matrix between two solutions. In fact, one of the matching matrices is given in Theorem G in § **8**. In the outer region, the formal outer solution was easily obtained in § **2** because the outer region does not contain the turning point, and its asymptotic nature was shown in § **5**.

The inner solution was obtained in the whole t-plane containing the turning point but except for the neighbourhoods of the secondary turning points (§ **3**) and its asymptotic nature was shown in § **6**. However, the asymptoticity is valid only in the sector with a central angle $4\pi/5$ which is not maximal and the maximal angle is $6\pi/5$ (cf. Nakano [5].) The region in which the asymptoticity of the inner solutions are valid is called the canonical region by Evgrafov-Fedoryuk [1] (§ **4**). The canonical region contains the routes or paths along which the exponential parts of the two independent inner solutions grow as t tends to infinity. This growth is essential for the proof of the asymptoticity of the solutions (§ **6**) and for the calculation of the matching matrix (§ 8).

Relations between inner solutions are given in § **7**. These relations are presented in the form of matrices in Theorems E and F.

Thus, if we want to know the relation between two outer solutions in one outer region and in the other, we have only to calculate the matching matrix from the first outer solution to the inner one in the first sector, calculate the connection matrix from the inner solution to an inner solution and calculate, if necessary, connection matrices from this inner solution to other inner solutions, the last one of which must belong to the second sector, in which we have only to calculate a matching matrix of the last inner solution and the second outer solution (cf. *9.2*).

The value at the origin, i.e., at the turning point, can be obtained from the inner solution.

Hence, we can get all values of the solution of the given equation in the region of definition except for the neighbourhoods of the secondary turning points.

9.2. Example. We shall illustrate the relation between two outer solutions by a simple example.

Let Y_o^- be an outer solution in the sector $\mathfrak{D}_0$. Then we can know values of the solution of the given differential equation in the sector $\mathfrak{D}_1$ from Y_o^-.

Let Y_i^- be an inner solution in the canonical region $\mathscr{D}_0$ which corresponds to the outer region $\mathfrak{D}_0$. By Theorem G, Y_o^- and Y_i^- are related by the equality

$$Y_i^- = Y_o^- \mathfrak{M}^-, \tag{1}$$

where $\mathfrak{M}^-$ is a matching matrix which matches Y_i^- and Y_o^- and its asymptoticity property is given by

$$\mathfrak{M}^- \sim e^{3\pi i/4}\varepsilon^{1/4}\begin{bmatrix} e^{c_1} & 0 \\ 0 & e^{-c_1} \end{bmatrix}, \tag{2}$$

as ε tends to zero.

The solution Y_i^- may be considered as the elementary fundamental solution defined by $(l_0, 1, \mathscr{D}_0)$. Then, Y_i^- can be continued by the connection matrix Ω_{01} onto the inner region $\mathscr{D}_1$. If Y_i^+ is the inner solution, continued from Y_i^-, in the canonical region $\mathscr{D}_1$, Y_i^- and Y_i^+ are related by the equality

$$Y_i^+ = Y_i^- \Omega_{01}, \tag{3}$$

$$\Omega_{01} \sim e^{\pi i/6}\begin{bmatrix} -i & 1 \\ 1 & 0 \end{bmatrix}, \tag{4}$$

as ε tends to zero.

The inner solution Y_i^+ enables us to know values in the outer region $\mathfrak{D}_1$. Indeed, since Y_i^+ is the inner solution defined by $(l_1, 1, \mathscr{D}_1)$ and a constant corresponding to c in (2) (see the proof of Theorem G) is $e^{2\pi i/3}$, if $\mathfrak{M}^+$ is a matching matrix in the sector corresponding to $\mathscr{D}_1$ or $\mathfrak{D}_1$, i.e.,

$$\mathfrak{M}^+ \sim e^{2\pi i/3}\varepsilon^{1/4}\begin{bmatrix} e^{c_1} & 0 \\ 0 & e^{-c_1} \end{bmatrix}, \tag{5}$$

as ε tends to zero, the matrix Y_o^+ defined by

$$Y_o^+ = Y_i^+(\mathfrak{M}^+)^{-1} \tag{6}$$

becomes an outer solution in $\mathfrak{D}_1$. Thus, Y_o^- and Y_o^+ are connected by the relation

$$Y_o^+ = Y_o^- \mathfrak{M}^- \Omega_{01}(\mathfrak{M}^+)^{-1},$$

$$M^- \Omega_{01}(\mathfrak{M}^+)^{-1} \sim e^{\pi i/4}\begin{bmatrix} -i & e^{2c_1} \\ e^{-2c_1} & 0 \end{bmatrix},$$

as ε tends to zero. The above equality is followed from (1), (3) and (6), and the last asymptotic relation is true from (2), (4) and (5).

Since $\mathfrak{D}_0$ contains the negative real axis and the outer solution Y_o^- gives us values on it and since $\mathfrak{D}_1$ contains the positive real axis, we can know values on the positive real axis from the values on the negative real axis. The branch of $x^{3/4}$ is to be chosen same both on the positive and negative real axis.

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