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INFINITE TENSOR PRODUCTS OF VON NEUMANN ALGEBRAS, I

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§ 1. Introduction.

For the purpose of studying lattice systems of quantum statistical mechanics and representations of CCR and CAR, infinite tensor products of von Neumann algebras due to von Neumann [12] have been frequently used as shown in [1], [4], [6] and others. The problems of types of infinite tensor products of von Neumann algebras have been investigated by many authors [2], [3], [7], [9], [11]. Infinite tensor products of normal positive linear functionals have been studied by Takeda [10] and symmetric states of infinite tensor products have been recently studied by Størmer [8]. Most of these results have been treated in the cases of incomplete infinite tensor products and of factors.

When we study infinite tensor products of von Neumann algebras, we set a problem what kind of relations has a finite normal trace given in the infinite tensor product of von Neumann algebras, with a finite measure on an infinite product space of some topological spaces corresponding to given von Neumann algebras? We encounter this problem in the course of studying infinite dimensional measures such as weak distributions, cylindrical measures and integrations of functionals. In the present paper we prepare some results on infinite tensor products of operators and those of normal positive linear functionals, which are defined in complete infinite tensor products of Hilbert spaces, in order to give some informations on that problem. By utilizing the results of this paper a partial answer will be given in the subsequent paper*) of the same title. In Theorem 3.1 some conditions by which infinite tensor products of operators can be defined will be discussed, and in Theorem 3.2 the conditions that infinite tensor products of operators belong to a given infinite tensor product of von Neumann algebras or to its commutor will be obtained. In Theorem 4.1 a sufficient condition that infinite tensor products of normal positive linear functionals can be defined will be given by introducing a concept of characteristic numbers. The similar results together with the necessary condition for finite normal traces will be given in Theorem 4.2 with the aid of coupling operators. Beside this theorem will indicate a finite part of infinite tensor product of von Neumann algebras as shown in Corollary 4.2.

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§ 2. Preparatory notations and definitions.

In what follows we will have to assume that the reader is familiar with the elementary properties of von Neumann algebras which are given in [5] and those of infinite tensor products of Hilbert spaces which are given by von Neumann [12]. In this section we prepare some notations and definitions used through this paper. Some elementary facts have also been explained in the additional contexts.

von Neumann algebra: Let $\mathfrak{H}$ be a Hilbert space, x a vector in $\mathfrak{H}$ and $\mathfrak{A}$ a von Neumann algebra on $\mathfrak{H}$. $\boldsymbol{C}_{\mathfrak{H}}$ and $\mathfrak{L}(\mathfrak{H})$ stand for von Neumann algebras of all scalar operators and all operators on $\mathfrak{H}$ respectively. Denote by $E(\mathfrak{A}, x)$ the projection onto the subspace $[\mathfrak{A}, x]$ generated by $\{Ax: A\in\mathfrak{A}\}$. Let 0 and 1 denote the zero and the identity operators on $\mathfrak{H}$ respectively and A^+ the non-negative part of a self-adjoint operator A. By $\mathfrak{A}^u$ (resp. $\mathfrak{A}^+, \mathfrak{A}^p$) we mean the set of all unitary (resp. non-negative, projection) operators in $\mathfrak{A}$. We say x is cyclic (resp. separating) for $\mathfrak{A}$ if $E(\mathfrak{A}, x)=1$ (resp. $E(\mathfrak{A}', x)=1$). ω_x is a positive linear functional defined by $\omega_x(A)=(Ax, x)$ for $A\in\mathfrak{A}$. Let φ_ι be a normal positive linear functional on $\mathfrak{A}_\iota$ for $\iota\in J$, where J is a finite set, and denote the tensor product of φ_ι by $\otimes_J\varphi_\iota$. Then $\otimes_J\varphi_\iota$ is a normal positive linear functional.

Infinite tensor product of Hibert spaces: Let I be an index set. This set is used universally in this paper and is considered to be infinite if the contrary is not explicitly stated. Let's denote $J\Subset I$ if J is a finite subset of I. We often omit the index set I from some symbols such as the sum $\sum$, the product Π, the union $\cup$, the intersection $\cap$ and the tensor product $\otimes$. Let $\{\mathfrak{H}_\iota: \iota\in I\}$ be a family of non trivial Hilbert spaces, $e_\iota, x_\iota, y_\iota, z_\iota, \cdots$ elements of $\mathfrak{H}_\iota$ and employ the same symbol $\|\ \ \|$ for the norms on all $\mathfrak{H}_\iota$ for $\iota\in I$. If $0<\Pi\|x_\iota\|<+\infty$ for $x_\iota\in\mathfrak{H}_\iota$, then the set $\{x_\iota: \iota\in I\}$ is called a C_0-*sequence* and written by (x_ι). A pair of C_0-sequences (x_ι) and (y_ι) is *equivalent* if $\sum|(x_\iota, y_\iota)-1|<+\infty$, which we denote by $(x_\iota)\sim(y_\iota)$. It is already known that this relation satisfies the equivalence relation. Let Γ_0 and Γ denote respectively the set of all C_0-sequences and the set of all equivalence classes $\mathfrak{c}$ of C_0-sequences in Γ_0 classified by $\sim$.

Let $\otimes\mathfrak{H}_\iota$ denote the complete infinite tensor product of $\mathfrak{H}_\iota$ for $\iota\in I$ and $\otimes^{\mathfrak{c}}\mathfrak{H}_\iota$ the incomplete one with respect to $\mathfrak{c}\in\Gamma$. The vector $\otimes x_\iota$ which corresponds to (x_ι) is called a *tensor product vector.* If $\Pi\|x_\iota\|=0$, we define $\otimes x_\iota=0$. Zero vector which we denote by 0 is assumed to be a tensor product vector. Let $\odot\mathfrak{H}_\iota$ be the set of all finite linear combinations of tensor product vectors in $\otimes\mathfrak{H}_\iota$. Then $\odot\mathfrak{H}_\iota$ is a pre-Hilbert space being dense in $\otimes\mathfrak{H}_\iota$.

Infinite tensor product of von Neumann algebras: Let $\mathfrak{A}_\iota$ be a von Neumann algebra on $\mathfrak{H}_\iota$ for each $\iota\in I$. Denote the zero operator and the identity operator on $\mathfrak{H}_\iota$ by 0_ι and 1_ι, sometimes without suffix. Moreover $1(J)$ is the identity on $\otimes_J\mathfrak{H}_\iota$ for $J\subset I$. When $\mathfrak{A}_\iota=\boldsymbol{C}_{\mathfrak{H}_\iota}$, we write $\boldsymbol{C}_\iota$ instead of it. If an operator $A_\kappa\in\mathfrak{A}_\kappa$ is given, then there exists a unique operator $\bar{A}_\kappa\in\mathfrak{L}(\mathfrak{H})$ with $\mathfrak{H}=\otimes\mathfrak{H}_\iota$ such that for all $(x_\iota)\in\Gamma_0$

$$\bar{A}_\kappa(\otimes x_\iota)=\bar{A}_\kappa(x_\kappa\otimes(\bigotimes_{\iota\neq\kappa} x_\iota))=A_\kappa x_\kappa\otimes(\bigotimes_{\iota\neq\kappa} x_\iota).$$

We shall often denote such an operator $\bar{A}_\kappa$ by $A_\kappa\otimes(\otimes_{\iota\neq\kappa}1_\iota)$ and denote the set of all $\bar{A}_\kappa$ for $A_\kappa\in\mathfrak{A}_\kappa$ by $\bar{\mathfrak{A}}_\kappa$ or $\mathfrak{A}_\kappa\otimes(\otimes_{\iota\neq\kappa}\boldsymbol{C}_\iota)$. Then $\bar{\mathfrak{A}}_\kappa$ is a von Neumann algebra. Indeed, since the correspondence Φ_κ: $A_\kappa\rightarrow\bar{A}_\kappa$ is an isomorphism of $\mathfrak{A}_\kappa$ to $\bar{\mathfrak{A}}_\kappa$ for the structure of $*$-algebra and it carries the operator 0_κ and 1_κ into 0 and 1 (zero and identity operator on $\otimes\mathfrak{H}_\iota$), the correspondence Φ_κ is an isomorphism of von Neumann algebra $\mathfrak{A}_\kappa$ into $\mathfrak{L}(\mathfrak{H})$ such that $\Phi_\kappa(1_\kappa)=1$. Thus $\bar{\mathfrak{A}}_\kappa=\Phi_\kappa(\mathfrak{A}_\kappa)$ is a von Neumann algebra [5; p 57].

DEFINITION 1. 1. Denote by $\otimes\mathfrak{A}_\iota$ the von Neumann algebra on $\otimes\mathfrak{H}_\iota$ generated by $\bar{A}_\iota$ satisfying $A_\iota\in\mathfrak{A}_\iota$ for all $\iota\in I$.

Let $\odot\mathfrak{A}_\iota$ be the union of $\Pi_J\bar{\mathfrak{A}}_\iota$ for all $J\Subset I$. Then $\odot\mathfrak{A}_\iota$ is a weakly dense sub-$*$-algebra of $\otimes\mathfrak{A}_\iota$.

§ 3. Infinite tensor products of operators.

LEMMA 3. 1. *Let U_ι be a partially isometric operator on $\mathfrak{H}_\iota$ for each $\iota\in I$. Then there exists uniquely a partially isometric operator U on $\otimes\mathfrak{H}_\iota$ such that $U(\otimes x_\iota)=\otimes U_\iota x_\iota$ for every $(x_\iota)\in\Gamma_0$.*

Proof. Let $\mathfrak{D}_\iota$ and $\mathfrak{R}_\iota$ be the initial and final spaces of U_ι respectively. The tensor products $\mathfrak{D}=\otimes\mathfrak{D}_\iota$ and $\mathfrak{R}=\otimes\mathfrak{R}_\iota$ are canonically identified with the subspace of $\otimes\mathfrak{H}_\iota$. Then, (I) for every $(x_\iota)\in\Gamma_0$ with $x_\iota\in\mathfrak{D}_\iota$ an element $\otimes U_\iota x_\iota$ of $\mathfrak{R}$ is defined; (II) $(\otimes U_\iota x_\iota, \otimes U_\iota y_\iota)=\Pi(x_\iota, y_\iota)$ for every (x_ι) and $(y_\iota)\in\Gamma_0$ with x_ι and $y_\iota\in\mathfrak{D}_\iota$; (III) all the finite linear combinations of $\otimes U_\iota x_\iota$ forms a dense linear subset of $\mathfrak{R}$. It follows from Theorem IV in [12; p 33] that there exists uniquely an isomorphism V of $\mathfrak{D}$ onto $\mathfrak{R}$ such that $V(\otimes x_\iota)=\otimes U_\iota x_\iota$. Define an operator U on $\otimes\mathfrak{H}_\iota$ by $U=V$ on $\mathfrak{D}$ and $U=0$ on $\mathfrak{D}^\perp$. Then U is a desired partially isometric operator on $\otimes\mathfrak{H}_\iota$ satisfying $U(\otimes x_\iota)=\otimes U_\iota x_\iota$ for $(x_\iota)\in\Gamma_0$, because it is obvious that if $\otimes x_\iota\in\mathfrak{D}^\perp$ then $\otimes U_\iota x_\iota=0$.

In the following we shall denote by $\otimes U_\iota$ the partially isometric operator U obtained in the above.

LEMMA 3. 2. *Let T_ι be a bounded operator on $\mathfrak{H}_\iota$ for each $\iota\in I$. Assume that $\Pi\|T_\iota\|<+\infty$.*

(i) *If $(x_\iota)\sim(y_\iota)$ and $(T_\iota x_\iota)$ and $(T_\iota y_\iota)\in\Gamma_0$, then $(T_\iota x_\iota)\sim(T_\iota y_\iota)$; and*

(ii) *if (x_ι) and $(T_\iota x_\iota)\in\Gamma_0$, and $T_\iota=U_\iota|T_\iota|$ the polar decomposition, then $(T_\iota x_\iota)\sim(U_\iota x_\iota)$.*

Proof. (i) Since $(T_\iota x_\iota)\in\Gamma_0$, it follows that $0<\Pi(T_\iota^*T_\iota x_\iota, x_\iota)<+\infty$ and hence

$$0<\Pi\|T_\iota^*T_\iota x_\iota\|\leqq(\Pi\|T_\iota\|)(\Pi\|T_\iota x_\iota\|),$$

that is, $(T_\iota^*T_\iota x_\iota)\in\Gamma_0$. Since $(T_\iota x_\iota)\in\Gamma_0$, it follows $0<\Pi(T_\iota^*T_\iota x_\iota, x_\iota)<+\infty$ and hence

$(T_\iota^* T_\iota x_\iota)\sim(x_\iota)$. Since $(x_\iota)\sim(y_\iota)$, it follows that $(T_\iota^* T_\iota x_\iota)\sim(y_\iota)$.

(ii)[1] Since $\Pi\| |T_\iota| x_\iota\| = \Pi\|T_\iota x_\iota\|$, it follows that $(|T_\iota| x_\iota)\in\Gamma_0$. Suppose that (x_ι) and $(T_\iota x_\iota)\in\Gamma_0$. Denote $x_\iota' = \|x_\iota\|^{-1} x_\iota$ and $T_\iota' = \|T_\iota\|^{-1} T_\iota$. Then $(|T_\iota'|^2 x_\iota', x_\iota') \leqq (|T_\iota'| x_\iota', x_\iota')$. Since $\sum|(|T_\iota'|^2 x_\iota', x_\iota')-1|<+\infty$, it follows that $\sum|(|T_\iota'| x_\iota, x_\iota)-1|<+\infty$ and hence $(|T_\iota'| x_\iota)\sim(x_\iota')$, that is, $(|T_\iota| x_\iota)\sim(x_\iota)$. Let $x_\iota = x_\iota^1 + x_\iota^2$ be the decomposition such that $x_\iota^1\in\mathfrak{N}_\iota^\perp$ and $x_\iota^2\in\mathfrak{N}_\iota$, where $\mathfrak{N}_\iota$ is the kernel of T_ι. Since $T_\iota x_\iota = T_\iota x_\iota^1$, follows that $\Pi\|T_\iota x_\iota\| \leqq (\Pi\|T_\iota\|)(\Pi\|x_\iota^1\|)$ and so $(x_\iota^1)\in\Gamma_0$. Since $(T_\iota x_\iota) = (U_\iota|T_\iota| x_\iota)\in\Gamma_0$, it follows from the above that $(T_\iota x_\iota)\sim(U_\iota x_\iota^1) = (U_\iota x_\iota)$.

Before going into the following lemma, recall that, for any $0<\varepsilon<1$ and $J\Subset I$, if $\sum_J|\alpha_\iota - 1|<\varepsilon/2$, then $|\Pi_J \alpha_\iota - 1|<\varepsilon$.

LEMMA 3. 3. *Let (x_ι) and (y_ι) be elements of Γ_0. Then $(x_\iota)\sim(y_\iota)$ is necessary and sufficient that for any $0<\varepsilon<1$ there is $J\Subset I$ such that*

$$\|\bigotimes_K x_\iota - \bigotimes_K y_\iota\| < \varepsilon$$

for every $K\Subset J^c$

Proof. Necessity: For any $0<\varepsilon<1$ there exists $J\Subset I$ such that for any $K\Subset J^c$

$$\sum_K | \|x_\iota\|^2 - 1| < \frac{\varepsilon^2}{8}, \qquad \sum_K | \|y_\iota\|^2 - 1| < \frac{\varepsilon^2}{8} \quad \text{and} \quad \sum_K |(x_\iota, y_\iota) - 1| < \frac{\varepsilon^2}{8}.$$

The first two inequalities follow from the facts that (x_ι) and $(y_\iota)\in\Gamma_0$ and the last inequality from $(x_\iota)\sim(y_\iota)$. Combining these inequalities, we have

$$\|\bigotimes_K x_\iota - \bigotimes_K y_\iota\| < \varepsilon.$$

Sufficiency: Since for any $0<\varepsilon<1/4$ there exists $J\Subset I$ such that for any $K\Subset J^c$

$$\|\bigotimes_K x - \bigotimes_K y\| < \varepsilon, \qquad | \|\bigotimes_K x_\iota\|^2 - 1| < \varepsilon \quad \text{and} \quad | \|\bigotimes_K y_\iota\|^2 - 1| < \varepsilon.$$

Combining these three inequalities and combining the last two inequalities, we have

$$|1 - \Re \prod_K (x_\iota, y_\iota)| < \varepsilon + \frac{1}{2}\varepsilon^2$$

and

$$\prod_K |(x_\iota, y_\iota)| < 1 + \varepsilon$$

1) Another proof (due to Araki): Since

$$0 < \prod \|T_\iota x_\iota\| \cdot \|T_\iota\|^{-1} \leqq \prod(|T_\iota| x_\iota, x_\iota) \leqq (\prod\|T_\iota x_\iota\|)(\prod\|x_\iota\|) < +\infty$$

it follows that $(x_\iota)\sim(y_\iota)$, where $y_\iota = |T_\iota| x_\iota$. Since

$$0 < \prod(|T_\iota| x_\iota, x_\iota) = \prod(U_\iota y_\iota, U_\iota x_\iota) \leqq (\prod\|U_\iota x_\iota\|)(\prod\|y_\iota\|),$$

it follows that $(U_\iota x_\iota)\in\Gamma_0$ and $(U_\iota x_\iota)\sim(U_\iota y_\iota) = (T_\iota x_\iota)$.

respectively. Hence

$$|1-\prod_K (x_\iota, y_\iota)|<2\varepsilon+4\varepsilon^2.$$

Consequently $\sum|(x_\iota, y_\iota)-1|<+\infty$ and so $(x_\iota)\sim(y_\iota)$.

THEOREM 3.1. *Let $T_\iota\in\mathfrak{L}(\mathfrak{H}_\iota)$ with $T_\iota\neq 0$ for each $\iota\in I$ and let $T_\iota=U_\iota|T_\iota|$ be the polar decomposition.*

I. *Denoting $T^J=(\otimes_J T_\iota)\otimes(\otimes_{J^c}U_\iota)$ for $J\Subset I$.*

(i) *If $\Pi\|T_\iota\|<+\infty$, then $\{T^J: J\Subset I\}$ converges strongly to a unique $T\in\mathfrak{L}(\otimes\mathfrak{H}_\iota)$ and $\|T\|\leqq\Pi\|T_\iota\|$; and*

(ii) *if $\{T^J: J\Subset I\}$ converges strongly to some $T\in\mathfrak{L}(\otimes\mathfrak{H}_\iota)$, then $\Pi\|T_\iota x_\iota\|<+\infty$ and $T(\otimes x_\iota)=\otimes T_\iota x_\iota$ for every $(x_\iota)\in\Gamma_0$.*

II. *The following four conditions are equivalent:*

(i) *$0<\Pi\|T_\iota\|<+\infty$ and there is $(x_\iota)\in\Gamma_0$ with $(T_\iota x_\iota)\in\Gamma_0$;*

(ii) *$0<\Pi\|T_\iota\|<+\infty$ and each $|T_\iota|$ except for a countable[2] number of ι's in I has a proper value* 1;

(iii) *there exists uniquely $T\in\mathfrak{L}(\otimes\mathfrak{H}_\iota)$ such that $T\neq 0$ and $\{T^J: J\Subset I\}$ converges strongly to T; and*

(iv) *there exists uniquely $T\in\mathfrak{L}(\otimes\mathfrak{H}_\iota)$ such that $T\neq 0$, $\Pi\|T_\iota x_\iota\|<+\infty$ and $T(\otimes x_\iota)=\otimes T_\iota x_\iota$ for every $(x_\iota)\in\Gamma_0$.*

In the case II, $\|T\|=\Pi\|T_\iota\|$.

Proof. I. (i) Let $(x_{i\iota})\in\Gamma_0$ and $\alpha_i\in\boldsymbol{C}$ for $i=1,2,\cdots,n$. If $\sum_{i=1}^n\alpha_i(\otimes x)_i=0$ for $(\otimes x)_i=\otimes x_{i\iota}$, then for any $(y_\iota)\in\Gamma_0$

$$\begin{aligned}0&=\left(\sum_{i=1}^n\alpha_i(\otimes x)_i,\ \otimes T_\iota^* y_\iota\right)=\sum_{i=1}^n\alpha_i((\otimes x)_i,\ \otimes T_\iota^* y_\iota)\\&=\sum_{i=1}^n\alpha_i\Pi(x_{i\iota},\ T_\iota^* y_\iota)=\sum_{i=1}^n\alpha_i\Pi(T_\iota x_{i\iota}, y_\iota)^{3)}\\&=\sum_{i=1}^n\alpha_i(\otimes T_\iota x_{i\iota},\ \otimes y_\iota)=\left(\sum_{i=1}^n\alpha_i(\otimes T_\iota x_{i\iota}),\ \otimes y_\iota\right)\end{aligned}$$

and therefore $\sum_{i=1}^n\alpha_i(\otimes T_\iota x_{i\iota})=0$. Thus we may define an an operator T on $\odot\mathfrak{H}_\iota$ by

$$T\left(\sum_{i=1}^n\alpha_i(\otimes x)_i\right)=\sum_{i=1}^n\alpha_i(\otimes T_\iota x_{i\iota}).$$

If $\otimes T_\iota x_\iota=0$ for $(x_\iota)\in\Gamma_0$, then for any $\varepsilon>0$ there exists $J_0\Subset I$ such that for any $J\Subset I$ with $J_0\subset J$ we have $\Pi_J\|T_\iota x_\iota\|<\varepsilon$ and hence

$$\|T^J(\otimes x_\iota)-\otimes T_\iota x_\iota\|\leqq M_1\varepsilon,$$

2) "Countable" is either finite or countably infinite.

3) The convergence of $\prod(x_{i\iota}, T_\iota^* y_\iota)$ and $\prod(T_\iota x_{i\iota}, y_\iota)$ is in the sense of quasi-convergence.

where $M_1=\sup_K \Pi_K\|x_\iota\|$. If $\otimes T_\iota x_\iota \neq 0$ for $(x_\iota)\in\Gamma_0$, then $(T_\iota x_\iota)\sim(U_\iota x_\iota)$ by Lemma 3.2. It follows from Lemma 3.3 that for any $0<\varepsilon<1$ there exists $J_0\Subset I$ such that for any $K\Subset J_0^c$

$$\|\bigotimes_K U_\iota x_\iota - \bigotimes_K T_\iota x_\iota\|<\varepsilon$$

and therefore for any $J\Subset I$ with $J_0\subset J$

$$\|T^J(\otimes x_\iota)-\otimes T_\iota x_\iota\|<M_2\varepsilon$$

where $M_2=\sup_L \Pi_L\|T_\iota x_\iota\|$. Hence for any $(x_{i\iota})\in\Gamma_0$ and $\alpha_i\in \boldsymbol{C}$, $i=1,2,\cdots,n$, define

$$M=\sum_{i=1}^n|\alpha_i|(\sup_J \prod_J\|x_{i\iota}\|+\sup_J\prod_J\|T_\iota x_{i\iota}\|).$$

Then for any $0<\varepsilon<1$ there exists $J_0\Subset I$ such that for any $J\Subset I$ with $J_0\subset J$.

$$\left\|T^J\sum_{i=1}^n\alpha_i(\otimes x)_i-\sum_{i=1}^n\alpha_i(\otimes T_\iota x_{i\iota})\right\|<M\varepsilon.$$

Since $\|T^J\|=\Pi_J\|T_\iota\|$, it follows that $\|T\|\leqq\lim\|T^J\|=\Pi\|T_\iota\|$, that is, T is bounded. Thus T on $\odot\mathfrak{H}_\iota$ has a unique continuous extension to $\otimes\mathfrak{H}_\iota$, which we denote by the same letter T. Consequently $\{T^J\colon J\Subset I\}$ converges strongly to T.

(ii) Suppose that T is a strong limit of $\{T^J\colon J\Subset I\}$. If $(x_\iota)\in\Gamma_0$ then $\Pi\|T_\iota x_\iota\|=\lim\|T^J(\otimes x_\iota)\|<+\infty$. If $\otimes T_\iota x_\iota=0$, then $\lim T^J(\otimes x_\iota)=0$ similarly as in (i), and hence $T(\otimes x_\iota)=\otimes T_\iota x_\iota$. If $\otimes T_\iota x_\iota\neq 0$, then $(T_\iota x_\iota)\sim(U_\iota x_\iota)$ by Lemma 3.2 and therefore

$$T(\otimes x_\iota)=\lim_J(\bigotimes_J T_\iota x_\iota)\otimes(\bigotimes_{J^c}U_\iota x_\iota)=\otimes T_\iota x_\iota.$$

II. (i) implies (ii): Since $0<\Pi\|T_\iota\|<+\infty$ and (x_ι) and $(T_\iota x_\iota)\in\Gamma_0$, it follows that $\|T_\iota\|=\|x_\iota\|=\|T_\iota x_\iota\|=1$ except for a countable number of ι's. Let's denote such a countable set by I_0. In general, if $A\geqq 0$ and $\|A\|=\|x\|=\|Ax\|=1$, then $Ax=x$. Since $\||T_\iota|\|=\|T_\iota\|$ and $\||T_\iota|x_\iota\|=\|T_\iota x_\iota\|$, it follows that $|T_\iota|x_\iota=x_\iota$ for $\iota\in I-I_0$.

(ii) implies (iii): Since the unique existence follows from (i) of I, it suffices to show that $T\neq 0$. Let I_1 be the set of ι's such that $|T_\iota|x_\iota=x_\iota$ for some $\|x_\iota\|=1$ and $\|T_\iota\|=1$. Then $I_0=I-I_1$ is a countable subset, $I_0=\{1,2,\cdots,\iota,\cdots\}$ say, by (ii). For any $\varepsilon>0$ and $i\in I_0$ there exists $x_i\in\mathfrak{H}_i$ such that $\|x_i\|=1$ and $\|T_i\|-\varepsilon/2^i<\|T_ix_i\|$ and so $\sum_{I_0}(1-\|T_ix_i\|)<\varepsilon$, which implies $\Pi_{I_0}\|T_\iota x_\iota\|\neq 0$, if $\varepsilon/2<\inf\|T_i\|$. Hence

$$\|T(\otimes x_\iota)\|=\prod_{I_0}\|T_\iota x_\iota\|=\Pi\|T_\iota x_\iota\|\neq 0$$

and so $T\neq 0$.

(iii) implies (iv): It is clear from (ii) of I.

(iv) implies (i): Since $T(\otimes x_\iota)=\otimes T_\iota x_\iota$ and $T\neq 0$, there exists $(x_\iota)\in\Gamma_0$ such that $\|T(\otimes x_\iota)\|\neq 0$. It follows that

$$0<\Pi\|T_\iota x_\iota\|=\|T(\otimes x_\iota)\|\leqq\|T\|\Pi\|x_\iota\|$$

and hence $(T_\iota x_\iota)\in\Gamma_0$ and $0<\Pi\|T_\iota\|=\|T\|$. Q.E.D.

In the last theorem, if $\Pi\|T_\iota\|<+\infty$ and if any condition in II are not satisfied, then $T=0$. Thus T is considered to be an infinite tenson product of operators T_ι.

DEFINITION 3.1. The operator T obtained in the last theorem is denoted by $\otimes T_\iota$ symbolically.

The following corollary is an immediate consequence of the last theorem.

COROLLARY 3.1. (i) $(\otimes T_\iota)^*=\otimes T_\iota^*$;
(ii) $(\otimes T_\iota)(\otimes S_\iota)=\otimes T_\iota S_\iota$;
(iii) *if* $\Pi\alpha_\iota$ *is convergent, then* $\otimes\alpha_\iota T_\iota=(\Pi\alpha_\iota)(\otimes T_\iota)$
(iv) *if* T_ι *is invertible for all* $\iota\in I$ *and* $\Pi\|T_\iota^{-1}\|<+\infty$, *then* $(\otimes T_\iota)^{-1}=\otimes T_\iota^{-1}$.

This corollary tells us that the set of all finite linear combinations of $\otimes T_\iota$ satisfying $T_\iota\in\mathfrak{A}_\iota$ for $\iota\in I$ forms a normed $*$-algebra on $\otimes\mathfrak{H}_\iota$ which depends on the choice of $\mathfrak{A}_\iota$ for $\iota\in I$. Thus its weak closure is a von Neumann subalgebra of $\mathfrak{L}(\otimes\mathfrak{H}_\iota)$. But we have few knowledges about this algebra such as its type, its commutor, its relation to $\otimes\mathfrak{A}_\iota$, its interpretation in physics and so on.

In what follows we shall denote by $P_\mathfrak{c}$ the projection of $\otimes\mathfrak{H}_\iota$ onto the incomplete infinite tensor product $\otimes^\mathfrak{c}\mathfrak{H}_\iota$ for $\mathfrak{c}\in\Gamma$. Then it is easily vertified that $P_\mathfrak{c}\in(\otimes\mathfrak{A}_\iota)'$ by the similar methods in [12; p 54].

THEOREM 3.2. *Assume that* $\Pi\|T_\iota\|<+\infty$.
I. *If* $T_\iota\in\mathfrak{A}_\iota$, *then the following three conditions are equivalent*:
(i) $\otimes T_\iota\in\otimes\mathfrak{A}_\iota$;
(ii) *for any* $\mathfrak{c}\in\Gamma$ *and any* $(x_\iota)\in\mathfrak{c}$, $(T_\iota x_\iota)\in\mathfrak{c}$ *or* $\otimes T_\iota x_\iota=0$; *and*
(iii) $\otimes T_\iota$ *is a strong limit of* $\{T_J\colon J\Subset I\}$, *where* $T_J=(\otimes_J T_\iota)\otimes 1(J^c)$ *for* $J\Subset I$.
II. *If* $T_\iota\in\mathfrak{A}_\iota^+$, *then* $\otimes T_\iota\in(\otimes\mathfrak{A}_\iota)^+$.
III. *If* $T_\iota\in\mathfrak{A}_\iota'$, *then* $\otimes T_\iota\in(\otimes\mathfrak{A}_\iota)'$.

Proof. I. (i) implies (ii): Suppose $\otimes T_\iota\in\otimes\mathfrak{A}_\iota$ and $\otimes T_\iota x_\iota\neq 0$ for $(x_\iota)\in\mathfrak{c}$. Since $P_\mathfrak{c}(\otimes T_\iota x_\iota)=P_\mathfrak{c}(\otimes T_\iota)(\otimes x_\iota)=\otimes T_\iota x_\iota$, it follows that $(T_\iota x_\iota)\in\mathfrak{c}$.

(ii) implies (iii): Applying the similar methods as the proof of (i) of I in Theorem 3.1, we can find for any $0<\varepsilon<1$ a finite subset $J_0\Subset I$ such that for any $J\Subset I$ with $J_0\subset J$

$$\|T_J(\otimes x_\iota)-\otimes T_\iota x_\iota\|<\varepsilon,$$

where $T_J=(\otimes_J T_\iota)\otimes 1(J^c)$.

(iii) implies (i): Since $T_J\in\otimes\mathfrak{A}_\iota$, it follows that the strong limit $\otimes T_\iota\in\otimes\mathfrak{A}_\iota$.

II. It is clear from I and the proof of Lemma 3.2.

III. An operator A of the form $(\otimes_J A_\iota)\otimes 1(J^c)$ for some $J\Subset I$ commutes with $\otimes T_\iota$ for $T_\iota\in\mathfrak{A}_\iota'$, because

$$A(\otimes T_\iota)(\otimes x_\iota)=A(\otimes T_\iota x_\iota)=\otimes A_\iota T_\iota x_\iota$$

where $A_\iota=1$ for $\iota\in I-J$, and so

$$A(\otimes T_\iota)(\otimes x_\iota)=\otimes T_\iota A_\iota x_\iota=(\otimes T_\iota)(\otimes A_\iota x_\iota)=(\otimes T_\iota)A(\otimes x_\iota).$$

Hence $\otimes T_\iota$ belongs to $(\odot\mathfrak{A}_\iota)'=(\otimes\mathfrak{A}_\iota)'$.

The following corollaries are easily verified.

COROLLARY 3. 2. *Let $A_\iota\in\mathfrak{A}_\iota$ and $A_\iota\neq 0$ for each $\iota\in I$. If $\sum\|A_\iota-1\|<+\infty$, then $A_J=(\otimes_J A_\iota)\otimes 1(J^c)$ converges uniformly to $\otimes A_\iota\in\otimes\mathfrak{A}_\iota$. If $U_\iota\in\mathfrak{A}_\iota^{\mathrm{u}}$ for each $\iota\in I$ and $\otimes U_\iota\in\otimes\mathfrak{A}_\iota$, then $\sum\|U_\iota-1\|<+\infty$.*

COROLLARY 3. 3. *Let $E_\iota\in\mathfrak{A}_\iota^{\mathrm{p}}$ and $E_\iota\neq 0$ and let $\mathfrak{R}_\iota$ be the projected subspace of $\mathfrak{H}_\iota$. Then*

(i) *$E=\otimes E_\iota\in(\otimes\mathfrak{A}_\iota)^{\mathrm{p}}$; and*

(ii) *the range of E coincides with $\otimes\mathfrak{R}_\iota$.*

COROLLARY 3. 4. *Let $\Pi\|T_\iota\|<+\infty$ and $T_\iota=U_\iota|T_\iota|$ be the polar decomposition of T_ι. Then $(\otimes T_\iota)=(\otimes U_\iota)(\otimes|T_\iota|)$ is the polar decomposition of $\otimes T_\iota$.*

§ 4. Infinite tensor products of normal positive linear functionals.

Let $\mathfrak{A}$ be a von Neumann algebra on $\mathfrak{H}$ and φ a normal positive linear functional on $\mathfrak{A}$. Then it is well known that φ can be written in the form $\varphi=\sum_{i=1}^\infty\omega_{x_i}$ for $x_i\in\mathfrak{H}$ $(i=1,2,\cdots)$ and $\|\varphi\|=\sum_{i=1}^\infty\|x_i\|^2$.

DEFINITION 4. 1. Let φ be a normal positive linear functional on a von Neumann algebra $\mathfrak{A}$ on $\mathfrak{H}$. γ is a *characteristic number* of φ with respect to $\mathfrak{A}$, if

$$\gamma=\sup\left\{\|x_1\|^2:\quad\varphi=\sum_{i=1}^\infty\omega_{x_i}\right\},$$

where the supremum is taken over all expansions of φ. Particularly, if $\gamma=\omega_{x_1}(1)$ and $\varphi=\sum_{i=1}^\infty\omega_{x_i}$, x_1 is called a *characteristic vector* of φ with respect to $\mathfrak{A}$. Let $\mathfrak{A}_\iota$ be a von Neumann algebra and φ_ι a normal positive linear functional on $\mathfrak{A}_\iota$ having a characteristic vector x_ι of φ_ι. The equivalence class $\mathfrak{c}\in\Gamma$ which contains (x_ι) is called a *characteristic class* of (φ_ι) whenever (x_ι) is a C_0-sequence.

THEOREM 4. 1. *Let $\mathfrak{A}_\iota$ be a von Neumann algebra and φ_ι a normal positive linear functional on $\mathfrak{A}_\iota$ whose characteristic number is γ_ι for each $\iota\in I$. If $0<\Pi\varphi_\iota(1)<+\infty$ and there is a countable subset J_0 of I such that $\sum_{J_0}(\varphi_\iota(1)-\gamma_\iota)<+\infty$ and $\varphi_\iota=\omega_{x_\iota}$ for $\iota\in I-J_0$, then there exists uniquely a normal positive linear functional φ on $\otimes\mathfrak{A}_\iota$ such that $\varphi(\Pi_K\bar{A}_\iota)=(\Pi_K\varphi_\iota(A_\iota))(\Pi_{K^c}\varphi_\iota(1))$ for $A_\iota\in\mathfrak{A}_\iota$ and every $K\Subset I$.*

Proof. Since J_0 is at most countable, we may identify J_0 with $\{i\colon i=1,2,\cdots\}$ in the following. For any $\varepsilon>0$ we have $x_i\in\mathfrak{H}_i$ for $i\in J_0$ such that $\varphi_\iota-\omega_{x_\iota}\geqq 0$ and $0\leqq\gamma_i-\|x_i\|^2<\varepsilon/2^i$. It follows that

$$\sum\|\varphi_\iota-\omega_{x_\iota}\|=\sum_{J_0}\|\varphi_\iota-\omega_{x_\iota}\|=\sum_{i=1}^{\infty}(\varphi_i(1)-\|x_i\|^2)$$

$$=\sum_{i=1}^{\infty}(\varphi_i(1)-\gamma_i)+\varepsilon<+\infty,$$

and hence $\sum|1-\|x_\iota\|^2|<+\infty$, that is, $(x_\iota)\in\Gamma_0$. Since $(x_\iota)\in\Gamma_0$ and $0<\Pi\varphi_\iota(1)<+\infty$, there is $M>0$ such that

$$\prod_K\|x_\iota\|^2\leqq M \qquad \text{and} \qquad \prod_K\|\varphi_\iota\|\leqq M$$

for any $K\subset I$. Since for every $J\Subset I$

$$\bigotimes_J\varphi_\iota=\bigotimes_J\{\omega_{x_\iota}+(\varphi_\iota-\omega_{x_\iota})\}$$

$$=\bigotimes_J\omega_{x_\iota}+\sum_{\iota\in J}(\bigotimes_{\substack{\kappa\neq\iota\\ \kappa\in J}}\omega_{x_\kappa})\otimes(\varphi_\iota-\omega_{x_\iota})+\cdots+\bigotimes_J(\varphi_\iota-\omega_{x_\iota}),$$

and since for any $0<\varepsilon<1/2$ there is $J_0\Subset I$ such that $\|\varphi_\iota-\omega_{x_\iota}\|<\varepsilon$ for $\iota\in J_0{}^c$, it follows that

$$\|\bigotimes_K\varphi_\iota-\bigotimes_K\omega_{x_\iota}\|\leqq 2M\varepsilon$$

for every $K\Subset J_0{}^c$. Denote $\varphi_J=(\otimes_J\varphi_\iota)\otimes(\otimes_{J^c}\omega_{x_\iota})$ for $J\Subset I$. Then for any J and $J'\Subset I$ with $J_0\subset J$ and J'

$$\|\varphi_J-\varphi_{J'}\|\leqq M^2(\|\bigotimes_{J-J'}\varphi_\iota-\bigotimes_{J-J'}\omega_{x_\iota}\|+\|\bigotimes_{J'-J}\omega_{x_\iota}-\bigotimes_{J'-J}\varphi_\iota\|)\leqq 4M^3\varepsilon.$$

Therefore we get a Cauchy net $\{\varphi_J\colon J\Subset I\}$, whose uniform limit is a normal positive linear functional φ on $\otimes\mathfrak{A}_\iota$. If $A\in\otimes_K\mathfrak{A}_\iota$ for any $K\Subset I$, then $\varphi(A\otimes 1(K^c))=\lim\varphi_J(A\otimes 1(K^c))=(\otimes_K\varphi_\iota)(A)\Pi_{K^c}\varphi_\iota(1)$. The uniqueness follows from the coincidence of φ on a weakly dense subset $\odot\mathfrak{A}_\iota$ of $\otimes\mathfrak{A}_\iota$.

DEFINITION 4. 2. Denote by $\otimes\varphi_\iota$ the normal positive linear functional φ which is obtained in the last theorem. The equivalence class $\mathfrak{c}$ which contains the C_0-sequence (x_ι) in the last proof is called a *characteristic class* of (φ_ι) and each x_ι is called a *quasi-characteristic vector* of φ_ι.

It is not clear whether the converse of this theorem holds or not;

Let φ be a normal positive linear functional on $\otimes\mathfrak{A}_\iota$ with $\varphi(1)=1$ and φ_ι a normal positive linear functional corresponding to the restriction of φ to $\bar{\mathfrak{A}}_\iota$ by the natural isomorphism between $\mathfrak{A}_\iota$ and $\bar{\mathfrak{A}}_\iota$. If $\varphi(\Pi_K\bar{A}_\iota)=\Pi_K\varphi_\iota(A_\iota)$ for $A_\iota\in\mathfrak{A}_\iota$ and every $K\Subset I$, then there is a countable subset J_0 of I such that $\sum_{J_0}(1-\gamma_\iota)<+\infty$ and $\varphi_\iota=\omega_{x_\iota}$ for $\iota\in I-J_0$, where γ_ι is a characteristic number of φ_ι.

However if φ is a trace, then we can show in the following that the converse is valid.

Let $\mathfrak{A}$ be a von Neumann algebra on $\mathfrak{H}$ and $\mathfrak{Z}$ the center of $\mathfrak{A}$. It is well known that, since $\mathfrak{Z}$ is abelian, there exist a locally compact Hausdorff space Z, a positive Radon measure ν on Z with the carrier Z and an isometric isomorphism of a normed $*$-algebra $\mathfrak{Z}$ onto a normed $*$-algebra $L^{\infty}(Z,\nu)$. Since this isomorphism is compatible with the usual order relation, $\mathfrak{Z}^{+}$ is mapped into the set $\hat{\mathfrak{Z}}^{+}$ of non negative measurable functions on Z classified by the null set difference. Utilizing this mapping we can identify $\mathfrak{Z}^{+}$ with a subalgebra of $\hat{\mathfrak{Z}}^{+}$. Let μ be a Radon measure on Z corresponding to a normal state φ on $\mathfrak{Z}$ and $C\eta\mathfrak{Z}$ the operator corresponding to $f\in\hat{\mathfrak{Z}}^{+}$. Then we may denote $\varphi(C)=\mu(f)$. Let Φ (resp. Φ') be a canonical $\natural$-mapping of $\mathfrak{A}$ (resp. $\mathfrak{A}'$). Then there is one and only one element f in $\hat{\mathfrak{Z}}^{+}$ such that for all $x\in\mathfrak{H}$ we have $\Phi(E(\mathfrak{A}',x))=f\Phi'(E(\mathfrak{A},x))$. The operator C which corresponds to $f\in\hat{\mathfrak{Z}}^{+}$ is called a *coupling operator* of $\mathfrak{A}$. Now we extend the concept of coupling operator in more general case where $\mathfrak{A}$ is finite and $\mathfrak{A}'$ is not necessarily finite and assume that the operator admits $+\infty$ as follows. If $\mathfrak{A}$ is finite and $\mathfrak{A}'$ is not finite, we will decompose it into a finite part $\mathfrak{A}'_G$ and a properly infinite part $\mathfrak{A}'_{1-G}$ by the projection G in the center of $\mathfrak{A}$. Using the coupling operator C_G of $\mathfrak{A}_G$, we define the coupling operator C of $\mathfrak{A}$ such that C is C_G on $G\mathfrak{H}$ and $+\infty$ on $(1-G)\mathfrak{H}$.

LEMMA 4.1. *Let $\mathfrak{A}$ be a finite von Neumann algebra with the coupling operator C on $\mathfrak{H}$. If φ is a finite normal trace on $\mathfrak{A}$, then there exists a characteristic vector x of φ such that*

$$\varphi(1)-\|x\|^2=\varphi(1)-\varphi(E(\mathfrak{A}',x))=\varphi((1-C)^{+}).$$

Proof. Let $C=\int\lambda dE_\lambda$ be the spectral resolution of C and define $G=\int_{\lambda<1}dE_\lambda$. Then we have

$$\inf_{e\in\mathfrak{H}}\{\varphi(1)-\varphi(E(\mathfrak{A}',e))\}$$

$$=\inf_{e\in\mathfrak{H}}\varphi((1-G)(1-E(\mathfrak{A}',e)))+\inf_{e\in\mathfrak{H}}\varphi(G(1-E(\mathfrak{A}',e)))$$

and since in the range of $1-G$ we have $C\geqq 1$ so that there exists a separating vector y for $\mathfrak{A}_{1-G}$ in the intersection of the carrier of φ and the range of $1-G$ such that the restriction of φ to $\mathfrak{A}_{1-G}$ is ω_y and $y=0$ if the intersection is $\{0\}$. Hence the first term of the right side is 0 and therefore

$$\inf_{e\in\mathfrak{H}}\{\varphi(1)-\varphi(E(\mathfrak{A}',e))\}=\varphi(G)-\sup_{e\in\mathfrak{H}}\varphi(G\Phi(E(\mathfrak{A}',e))),$$

where Φ is the canonical $\natural$-mapping of $\mathfrak{A}$. Since $C<1$ in the intersection of the carrier of φ and the range of G, we have a cyclic vector z for $\mathfrak{A}_G$ in it such that the restriction of φ to $\mathfrak{A}_{E\circ G}$ is ω_z where $E=E(\mathfrak{A}_G,z)$. Particularly $z=0$ if the intersection is $\{0\}$. That is $\varphi(G\Phi(E(\mathfrak{A}',z)))=\varphi(GC)$. Thus we have

$$\inf_{e\in\mathfrak{H}}\{\varphi(1)-\varphi(E(\mathfrak{A}',e))\}=\varphi(G)-\varphi(GC)=\varphi((1-C)^{+}).$$

Define $x=y+z$. Then

$$\sup_{e\in\mathfrak{H}}\varphi(E(\mathfrak{A}', e))=\varphi(1)-\varphi((1-C)^{+})=\varphi(E(\mathfrak{A}', x))$$

and

$$\omega_x(1)=\omega_y(1-G)+\omega_z(G)=\varphi(1-G)+\varphi(CG)=\varphi(1)-\varphi((1-C)^{+}).$$

If $\|x\|^2<\gamma$ where γ is a characteristic number of φ, then there is $x'\in\mathfrak{H}$ such that $\|x'\|>\|x\|$ and $\varphi-\omega_{x'}\geqq 0$. Then

$$\varphi(E(\mathfrak{A}', x'))\geqq\|x'\|^2>\|x\|^2=\sup_{e\in\mathfrak{H}}\varphi(E(\mathfrak{A}', e)),$$

which is a contradiction. Thus x is a characteristic vector, for $\varphi-\omega_x\geqq 0$.

THEOREM 4.2. *Let $\mathfrak{A}_\iota$ be a finite von Neumann algebra with the coupling operator C_ι for every $\iota\in I$.*

(i) *Let φ_ι be a normal trace on $\mathfrak{A}_\iota$ for each $\iota\in I$ such that $0<\Pi\varphi_\iota(1)<+\infty$. If $\sum\varphi_\iota((1-C_\iota)^{+})<+\infty$, then there is one and only one normal trace φ on $\otimes\mathfrak{A}_\iota$ such that $\varphi(\Pi_K\bar{A}_\iota)=(\Pi_K\varphi_\iota(A_\iota))(\Pi_{K^c}\varphi_\iota(1))$ for $A_\iota\in\mathfrak{A}_\iota$ and every $K\Subset I$.*

(ii) *Let φ be a normal trace on $\otimes\mathfrak{A}_\iota$ with $\varphi(1)=1$ and φ_ι a normal trace corresponding to the restriction $\varphi|\bar{\mathfrak{A}}_\iota$ of φ to $\bar{\mathfrak{A}}_\iota$ by the natural isomorphism between $\mathfrak{A}_\iota$ and $\bar{\mathfrak{A}}_\iota$. If $\varphi(\Pi_K\bar{A}_\iota)=\Pi_K\varphi_\iota(A_\iota)$ for $A_\iota\in\mathfrak{A}_\iota$ and every $K\Subset I$, then $\sum\varphi_\iota((1-C_\iota)^{+})<+\infty$.*

Proof. (i) Let γ_ι be a characteristic number of φ_ι, then by Lemma 4.1 there exists a characteristic vector x_ι such that $\gamma_\iota=\|x_\iota\|^2$ and $\varphi_\iota(1)-\gamma_\iota=\varphi_\iota((1-C_\iota)^{+})$. Hence by Theorem 4.1 the desired normal positive linear functional $\varphi=\otimes\varphi_\iota$ is obtained. It suffices to show that φ is a trace. If $A\in\otimes\mathfrak{A}_\iota$ and $B\in\otimes\mathfrak{A}_\iota$, then there exist Cauchy nets A_J and B_J which converges weakly to A and B respectively as J tends to I, where $A_J=A(J)\otimes 1(J^c)$ and $B_J=B(J)\otimes 1(J^c)$ for some $A(J)$ and $B(J)$ in $\otimes_J\mathfrak{A}_\iota$. Hence by a fixed $J'\Subset I$, $A_JB_{J'}$ converges weakly to $AB_{J'}$, and therefore

$$\varphi(AB_{J'})=\lim_J\varphi(A_JB_{J'})=\lim_J\varphi(B_{J'}A_J)=\varphi(B_{J'}A).$$

It follows that

$$\varphi(AB)=\lim_{J'}\varphi(AB_{J'})=\lim_{J'}\varphi(B_{J'}A)=\varphi(BA).$$

Thus φ is a normal trace.

(ii) Since $E((\otimes\mathfrak{A}_\iota)', \otimes z_\iota)\leqq\otimes E(\mathfrak{A}_\iota', z_\iota)$ for every $(z_\iota)\in\Gamma_0$ and φ is normal, it follows that there is $(y_\iota)\in\Gamma_0$ satisfying $\varphi(\otimes E(\mathfrak{A}_\iota', y_\iota))>0$. Thus

$$0<\varphi(\otimes E(\mathfrak{A}_\iota', y_\iota))\leqq\prod_K\varphi_\iota(E(\mathfrak{A}_\iota', y_\iota))$$

for every $K\Subset I$ and hence $0<\Pi\varphi_\iota(E(\mathfrak{A}_\iota', y_\iota))\leqq\varphi(1)$, that is, $\sum(1-\varphi_\iota(E(\mathfrak{A}_\iota', y_\iota)))<+\infty$. Define

$$\psi_\iota(A)=\varphi_\iota(AE(\mathfrak{A}_\iota', y_\iota))$$

for $A\in\mathfrak{A}_\iota$. Then $\psi_\iota=\omega_{x_\iota}$ for some $x_\iota\in\mathfrak{H}_\iota$ and $\varphi_\iota-\omega_{x_\iota}\geqq 0$. Since $E(\mathfrak{A}_\iota', y_\iota)=E(\mathfrak{A}_\iota', x_\iota)$, it follows that $\omega_{x_\iota}(1)=\varphi_\iota(E(\mathfrak{A}_\iota', y_\iota))$. Hence by Lemma 4.1,

$$\varphi_\iota((1-C_\iota)^+)\leqq\varphi_\iota(1)-\varphi_\iota(E(\mathfrak{A}_\iota', y_\iota)),$$

which implies $\sum\varphi_\iota((1-C_\iota)^+)<+\infty$. Q.E.D.

The relation between infinite tensor products of operators and that of normal positive linear functionals is given in the following corollary.

COROLLARY 4.1. (i) $(\otimes\varphi_\iota)(\otimes A_\iota)=\Pi\varphi_\iota(A_\iota)$ *for* $\otimes A_\iota\in\otimes\mathfrak{A}_\iota$; *and*

(ii) *if* $(\otimes\varphi_\iota)(\otimes A_\iota)>0$ *then for any* $\varepsilon>0$ *there exists* $J_0\Subset I$ *such that for any* $J\Subset I$ *with* $J_0\subset J$

$$|(\underset{J^c}{\otimes}\varphi_\iota)(A(J^c))-1|<\varepsilon,$$

where $A(K)=\otimes_K A_\iota$ *for* $K\subset I$.

The expression of the central carrier of $P_{\mathfrak{c}}$ which is given in the following Lemma is suggested by Araki.

LEMMA 4.2. *Let's denote*

$$P(\mathfrak{c})=\lim_{J\Subset I}1(J)\otimes E(\mathfrak{A}(J^c)', x(J^c))$$

where $x(K)=\otimes_K x_\iota$ *for* $(x_\iota)\in\mathfrak{c}$ *and* $\mathfrak{A}(K)=\otimes_K\mathfrak{A}_\iota$ *for* $K\subset I$. *Then* $P(\mathfrak{c})$ *is the central carrier of* $P_{\mathfrak{c}}$.

Proof. Since $E(\mathfrak{A}(J^c)', x(J^c))$ is a projection in $\mathfrak{A}(J^c)$, it follows that $P(\mathfrak{c})$ is a projection in $\otimes\mathfrak{A}_\iota$. Since $P(\mathfrak{c})$ commutes with every element of $\odot\mathfrak{A}_\iota$, it follows that $P(\mathfrak{c})$ is an element of $(\otimes\mathfrak{A}_\iota)'$. Thus $P(\mathfrak{c})$ is a central projection of $\otimes\mathfrak{A}_\iota$ and it majorates $P_{\mathfrak{c}}$. This is because the set of all $\otimes y_\iota$ such that $(y_\iota)\in\mathfrak{c}$ and $\{\iota\in I: y_\iota\neq x_\iota\}$ is finite, is total in $\otimes^{\mathfrak{c}}\mathfrak{H}_\iota$ and moreover for such $\otimes y_\iota$ we have

$$P(\mathfrak{c})(\otimes y_\iota)=\lim_J(1(J)\otimes E(\mathfrak{A}(J^c)', x(J^c)))(\otimes y_\iota)=\otimes y_\iota.$$

On the other hand, denote by P the central carrier of $P_{\mathfrak{c}}$. Since $E(\mathfrak{A}(I)', y(I))\leqq P$ for every $(y_\iota)\in\mathfrak{c}$, it follows that $1(J)\otimes E(\mathfrak{A}(J^c)', x(J^c))\leqq P$, which implies $P(\mathfrak{c})\leqq P$ and hence $P=P(\mathfrak{c})$.

COROLLARY 4.2. *Let* $\mathfrak{A}_\iota$ *be a von Neumann algebra and* φ_ι *a normal positive linear functional on* $\mathfrak{A}_\iota$ *for each* $\iota\in I$. *Let* G_ι *and* G *be the carrier projections of* φ_ι *and* $\otimes\varphi_\iota$ *respectively. Let* $\mathfrak{c}$ *be a characteristic class of* (φ_ι). *Then* $G=(\otimes G_\iota)P(\mathfrak{c})$.

Proof. Let $(x_\iota)\in\mathfrak{c}$ and x_ι a quasi-characteristic vector of φ_ι for each $\iota\in I$. For any $\varepsilon>0$ there is $J_0 \Subset I$ such that $\|\varphi_J - \otimes\varphi_\iota\| < \varepsilon$ for all $J \Subset I$ with $J_0 \subset J$, where $\varphi_J = (\otimes_J \varphi_\iota) \otimes \omega_{x(J^c)}$ and $x(J^c) = \otimes_{J^c} x_\iota$. Let $G_J = (\otimes_J G_\iota) \otimes E(\mathfrak{A}(J^c)', x(J^c))$. Then G_J is the carrier projection of φ_J satisfying $G_J \leqq G$ and $\{G_J : J \Subset I\}$ is a monotone increasing Cauchy net, because $\varphi_J \leqq \varphi_{J'}$ if $J \subset J' \Subset I$. Put the limit $G_0 = \lim G_J$. Then $(\otimes\varphi_\iota)(G_0) = 1$ and so $G = G_0$. On the other hand, since $\varphi_J \leqq \otimes\varphi_\iota$ and $G \leqq \otimes G_\iota$, it follows that $G_J \leqq \otimes G_\iota$ and hence

$$G_J = (\otimes G_\iota)(1(J) \otimes E(\mathfrak{A}(J^c)', x(J^c))).$$

Consequently $G_0 = (\otimes G_\iota) P(\mathfrak{c})$.

COROLLARY 4. 3. Let $(x_\iota) \in \mathfrak{c}$. *Then*
(i) $E(\mathfrak{A}(I), x(I)) = (\otimes E(\mathfrak{A}_\iota, x_\iota)) P_\mathfrak{c}$; *and*
(ii) $E(\mathfrak{A}(I)', x(I)) = (\otimes E(\mathfrak{A}_\iota', x_\iota)) P(\mathfrak{c})$.

Proof. (i) Let $\mathfrak{K}_\iota$ be the range of $E_\iota = E(\mathfrak{A}_\iota', x_\iota)$ for $(x_\iota) \in \mathfrak{c}$. By Corollary 3. 3, the range of $\otimes E_\iota$ is $\otimes \mathfrak{K}_\iota$. Since $\otimes \mathfrak{K}_\iota$ is generated by the tensor product vectors $\otimes y_\iota$ with $y_\iota \in \mathfrak{K}_\iota$ for all $\iota \in I$ and $(y_\iota) \in \Gamma_0$, if $(y_\iota) \in \mathfrak{c}$ then $(\otimes E_\iota) P_\mathfrak{c} (\otimes y_\iota) = \otimes y_\iota = P_\mathfrak{c} (\otimes E_\iota)(\otimes y_\iota)$ and if $(y_\iota) \notin \mathfrak{c}$ then $(\otimes E_\iota) P_\mathfrak{c} (\otimes y_\iota) = 0 = P_\mathfrak{c} (\otimes E_\iota)(\otimes y_\iota)$. Since the orthogonal complement of $\otimes \mathfrak{K}_\iota$ in $\otimes \mathfrak{H}_\iota$ is generated by the tensor product vectors $\otimes y_\iota$ with $y_\iota \in \mathfrak{K}_\iota^\perp$ for some $\iota \in I$ and $(y_\iota) \in \Gamma_0$, $(\otimes E_\iota) P_\mathfrak{c} (\otimes y_\iota) = 0 = P_\mathfrak{c} (\otimes E_\iota)(\otimes y_\iota)$. Consequently $(\otimes E_\iota) P_\mathfrak{c} = P_\mathfrak{c} (\otimes E_\iota)$ and the range of $(\otimes E_\iota) P_\mathfrak{c}$ is generated by $\{\otimes y_\iota : y_\iota \in \mathfrak{K}_\iota$ and $(y_\iota) \in \mathfrak{c}\}$. If $(y_\iota) \sim (x_\iota)$ and $y_\iota \in [\mathfrak{A}_\iota, x_\iota]$, we have $\otimes y_\iota \in [\otimes \mathfrak{A}_\iota, \otimes x_\iota]$ and hence $(\otimes E_\iota) P_\mathfrak{c} \leqq E(\mathfrak{A}(I), x(I))$. Since $\odot \mathfrak{A}_\iota$ is dense in $\otimes \mathfrak{A}_\iota$, the converse inequality follows.

(ii) Define $\varphi_\iota = \omega_{x_\iota}$. Then φ_ι is a normal positive linear functional on $\mathfrak{A}_\iota$. Since the carrier of φ_ι is $E(\mathfrak{A}_\iota', x_\iota)$, the carrier of $\otimes\varphi_\iota$ is $(\otimes E(\mathfrak{A}_\iota', x_\iota)) P(\mathfrak{c})$ by Corollary 4. 2. On the other hand, since $\otimes\varphi_\iota = \omega_{\otimes x_\iota}$, its carrier is $E((\otimes \mathfrak{A}_\iota)', \otimes x_\iota)$. The desired equality follows.

COROLLARY 4. 4. *Let* $(x_\iota) \in \mathfrak{c}$ *and* $(e_\iota) \in \Gamma_0$. *If* $\omega_{\otimes e_\iota}(E(\mathfrak{A}(I)', x(I))) > 0$, *then* $P(\mathfrak{c})(\otimes e_\iota) = \otimes e_\iota$.

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