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# ON PRIME ENTIRE FUNCTIONS, II

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§1. An entire function $F(z)=f\circ g(z)$ is said to be prime if every factorization of the above form implies that one of the functions $f(z)$ or $g(z)$ is linear. In our previous paper [2] we proved the primeness of several functions. In this paper we shall prove the primeness of

$$\prod_{n=1}^{\infty}\left(1-\frac{z}{n^{\alpha}}\right), \qquad 2>\alpha>1.$$

In order to prove it we quote several known results.

LEMMA 1. (Edrei [1]). *Let $f(z)$ be an entire function. Assume that there exists an unbounded sequence $\{h_\nu\}_{\nu=1}^{\infty}$ such that all the roots of the equations $f(z)=h_\nu$, $\nu=1, 2, \cdots$ be real. Then $f(z)$ is a polynomial of degree at most two.*

LEMMA 2. (Valiron [4], Titchmarsh [3]). *If $f(z)$ is an entire function of order $\rho$, $0<\rho<1$, with real negative zeros, $f(0)=1$, and $n(t)\sim\lambda t^{\rho}$ $(\lambda>0)$ as $t\to\infty$, then*

$$\log f(re^{i\theta})\sim e^{i\rho\theta}\pi\lambda\frac{r^{\rho}}{\sin\pi\rho}, \qquad r\to\infty$$

*for each fixed $\theta$ in $-\pi<\theta<\pi$. Here $n(t)$ indicates the number of zeros of $f(z)$ in $|z|<t$. Further if $\varepsilon>0$ we have*

$$\log|f(-x)|<(\pi\lambda\cot\pi\rho+\varepsilon)x^{\rho}, \qquad x>x_0(\varepsilon),$$

*and if $\eta>0$ we have*

$$\log|f(-x)|>(\pi\lambda\cot\pi\rho-\varepsilon)x^{\rho}$$

*for $0<x<X$ except in a set of measure $\eta X$, provided that $X$ is sufficiently large. In particular*

$$\log|f(-x)|\sim\pi\lambda(\cot\pi\rho)x^{\rho}$$

*in a set of density* 1.

LEMMA 3. (Valiron [4], Titchmarsh [3]). *Let $f(z)$ be an entire function of order less than one and with only negative real zeros. If $f(0)=1$ and if*

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$$\log f(r) \sim \pi \frac{\lambda r^{\rho}}{\sin \pi\rho}, \qquad r\to\infty, \qquad \lambda>0,$$

*then*

$$n(r)\sim \lambda r^{\rho}, \qquad r\to\infty.$$

§2. We shall prove the following theorem.

THEOREM. *Let $F(z)$ be an entire function of order $\rho$, $1/2<\rho<1$ and with only negative real zeros. Assume that $n(r)\sim\lambda r^{\rho}$, $\lambda>0$. Further assume that there are two indices $j$ and $k$ such that $a_j$, $a_k$ are zeros of $F(z)$ whose multiplicities $p_j$, $p_k$ satisfy $(p_j, p_k)=1$. Then $F(z)$ is prime.*

*Proof.* Suppose, firstly, that $F(z)=f\circ g(z)$ with transcendental $f(w)$. Then by Edrei's theorem $g(z)$ must be a polynomial of degree at most two. Since all the zeros of $F(z)$ are real negative, $g(z)$ must be linear. This case may be put aside.

Suppose, next, that $F(z)=f\circ g(z)$ with a polynomial $f(w)$. In this case we have

$$F(z)=Ag_1(z)^{l_1}\cdots g_p(z)^{l_p}, \qquad g_j(z)=g(z)-w_j.$$

We put

$$F(z)=B\prod_{t=1}^{\infty}\left(1+\frac{z}{a_t}\right)^{p_t}, \qquad a_t>0.$$

In the above factorization any zero of $F(z)$ cannot be divided into two or more different factors. Then we may put

$$g_j(z)=c_j\prod_{s=1}^{\infty}\left(1+\frac{z}{b_{sj}}\right)^{q_{sj}}, \qquad b_{sj}>0,$$

$$b_{sj}\neq b_{s'i} \quad \text{for} \quad s\neq s' \quad \text{or} \quad s=s', \quad j\neq i.$$

Evidently $B=A\Pi_{j=1}^{p}c_j^{l_j}$. Firstly we have

$$\frac{|F(\mathrm{r})|}{|B|}=\frac{F(r)}{B}=\max_{|z|=r}\frac{|F(z)|}{B}\sim\frac{|A|}{|B|}\prod_{j=1}^{p}\left(\frac{g_j(r)}{c_j}\right)^{l_j}\prod_{j=1}^{p}|c_j|^{l_j}$$

as $r\to\infty$. Further

$$|g_j(r)|\sim|g_k(r)|, \qquad r\to\infty,$$

$$\frac{g_j(r)}{c_j}=\max_{|z|=r}\frac{|g_j(z)|}{c_j}=\frac{|g_j(r)|}{|c_j|}.$$

Hence

$$\frac{F(r)}{B}\sim\prod_{j=1}^{p}\left(\frac{g_j(r)}{c_j}\right)^{l_j}\sim\prod_{j=1}^{p}\frac{1}{|c_j|^{l_j}}|g_s(r)|^{\Sigma_{j=1}^{p}l}$$

as $r\to\infty$. Put

$$\alpha=\sum_{j=1}^{p}l_j.$$

By Lemma 2 we have

$$\log \frac{F(r)}{B} \sim \frac{\pi\lambda}{\sin \pi\rho} r^{\rho}, \quad r \to \infty.$$

Hence

$$\log \left( \frac{g_t(r)}{c_t} \right)^{\alpha} + \log \frac{c_t^{\alpha}}{\prod_{j=1}^{p} |c_j|^{l_j}} \sim \frac{\pi\lambda}{\sin \pi\rho} r^{\rho}$$

as $r \to \infty$. Thus for each $t$, $1 \leqq t \leqq p$

$$\log \frac{g_t(r)}{c_t} \sim \frac{\pi\lambda}{\alpha \sin \pi\rho} r^{\rho}, \quad r \to \infty.$$

Then by Lemma 3

$$n(r, g_t(z)) \sim \frac{\lambda}{\alpha} r^{\rho}, \quad r \to \infty.$$

Again by Lemma 2

$$\log \left| \frac{g_t(-x)}{c_t} \right| \sim \pi \frac{\lambda}{\alpha} x^{\rho} \cot \pi\rho, \qquad x \to \infty$$

in a set $E_t$ of density 1. Since $E_1 \cap E_2$ is of density 1,

$$\log |g(-x) - w_1| \sim \pi \frac{\lambda}{\alpha} x^{\rho} \cot \pi\rho,$$

$$\log |g(-x) - w_2| \sim \pi \frac{\lambda}{\alpha} x^{\rho} \cot \pi\rho$$

as $x \to \infty$ in $E_1 \cap E_2$. Since $1/2 < \rho < 1$, we have

$$g(-x) \to w_1, \qquad g(-x) \to w_2$$

as $x \to \infty$ in $E_1 \cap E_2$. This is clearly a contradiction. Therefore $F(z) = A(g(z) - w_1)^{l_1}$. By the existence of two zeros whose multiplicities are coprime $l_1$ must reduce to 1. Hence we have

$$F(z) = A(g(z) - w_1),$$

which is the desired result.

It should be mentioned a remark here. Our theorem does not remain true if the order is not greater than a half. The function $\cos \sqrt{z}$ is of order 1/2, which satisfies

$$\cos \sqrt{z} = 2 \cos^2 \frac{\sqrt{z}}{2} - 1.$$

When the order $\rho$ is less than 1/2, we can construct a counter example freely, for example $g(z)(g(z)-1)$.

## References

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