KŌDAI MATH. SEM. REP.
**21** (1969), 496—509

# NOTES ON SUBMANIFOLDS IN A RIEMANNIAN MANIFOLD

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## §1. Introduction.

Many attempts have been done to generalize, to the case of hypersurfaces in a Riemannian manifold, a famous theorem of Liebmann [8] and Süss [14]: The only convex hypersurface with constant mean curvature is a sphere. See for example, Hsiung [1], Katsurada [2], [3], Koyanagi [7], Ōtsuki [13], Tani [15], [21] and the present author [16], [17], [21].

In these papers, the authors prove that, under certain conditions, a closed hypersurface with constant mean curvature is umbilical or pseudo-umbilical.

The present author [17] gave certain conditions under which a closed hypersurface with constant mean curvature be isometric to a sphere.

Also attempts have been recently started to generalize the theorem of Liebmann and Süss to the case of submanifolds in a Riemannian manifold. See for example, Katsurada [4], [5], [6], Kôjyô [5], Nagai [6], [9], Okumura [11], [20], Tani [21] and the present author [18], [20], [21].

Katsurada [4], [5], [6], Kôjyô [5] and Nagai [6], [9] assume the existence of a conformal Killing vector field in the ambient manifold and that this vector field is contained in the linear space spanned by the mean curvature vector of the submanifold and the tangent space to the submanifold.

The present author [18] recently weakened this assumption and obtained similar results to those of Katsurada, Kôjyô and Nagai.

The main purpose of the present paper is to generalize the methods and results in [17] to the case of general submanifolds in a Riemannian manifold admitting a scalar field $v$ such that $\nabla_j\nabla_i v=f(v)g_{ji}$ and give conditions for a submanifold to be isometric to a sphere.

Similar attempt has been already done by Nagai [9], but he assumes that the vector field $v^h=(\nabla_i v)g^{ih}$ lies in the linear space spanned by the mean curvature vector and the tangent plane of the submanifold. We study the problem under a condition which is weaker than this.

## §2. Preliminaries.

Let $M$ be an $m$-dimensional orientable Riemannian manifold of differentiability class $C^\infty$ covered by a system of coordinate neighborhoods $\{U\colon \xi^h\}$ and $g_{ji}$, $\{{}_j{}^h{}_i\}$,

Received August 2, 1969.

1) This work was supported by the National Research Council of Canada.

$\nabla_i$, $K_{kji}{}^h$ and $K_{ji}$, the metric tensor, the Christoffel symbols formed with $g_{ji}$, the operator of covariant differentiation with respect to the Christoffel symbols, the curvature tensor and the Ricci tensor respectively, where and in the sequel the indices $h, i, j, k, \cdots$ run over the range $\{1, 2, \cdots, m\}$.

Let $N$ be an $n$-dimensional compact and orientable manifold of differentiability class $C^\infty$ covered by a system of coordinate neighborhoods $\{V; \eta^a\}$ and $C^\infty$ differentiably imbedded in $M$ and let

$$\xi^h = \xi^h(\eta^a) \tag{2.1}$$

be the local parametric expression of $N$, where and in the sequel the indices $a, b, c, d, e$ run over the range $\{1, 2, \cdots, n\}$ and $1<n<m$.

If we put

$$B_b{}^h = \partial_b \xi^h, \qquad \partial_b = \partial/\partial\eta^b,$$

then, the Riemannian metric of $N$ induced from that of $M$ is given by

$$g_{cb} = g_{ji} B_c{}^j B_b{}^i \tag{2.2}$$

and the Christoffel symbols formed with $g_{cb}$ by

$$\{{}_c{}^a{}_b\} = (\partial_c B_b{}^h + \{{}_j{}^h{}_i\} B_c{}^j B_b)^i B^a{}_h, \tag{2.3}$$

where

$$B^a{}_h = B_b{}^i g^{ba} g_{ih},$$

$g^{ba}$ being the contravariant components of the metric tensor of $N$.

If we put

$$\nabla_c B_b{}^h = \partial_c B_b{}^h + \{{}_j{}^h{}_i\} B_c{}^j B_b{}^i - \{{}_c{}^a{}_b\} B_a{}^h, \tag{2.4}$$

then we see from (2.3) that

$$g_{ji}(\nabla_c B_b{}^j) B_a{}^i = 0, \tag{2.5}$$

which says that $\nabla_c B_b{}^h$, as vectors of $M$, are orthogonal to the submanifold $N$. The $\nabla_c B_b{}^h$ defined by (2.4) is called the *van der Waerden-Bortolotti covariant derivative* of $B_b{}^h$ along the submanifold $N$. Thus,

$$H^h = \frac{1}{n} g^{cb} \nabla_c B_b{}^h \tag{2.6}$$

is an intrinsic vector field of $M$ defined along $N$ and is orthogonal to $N$. $H^h$ is called the *mean curvature vector* of $N$.

We assume that the mean curvature vector $H^h$ of $N$ never vanishes along $N$ and take the first unit normal $C^h$ to $N$ in the direction of $H^h$ and put

$$(\nabla_c B_b{}^i) C_i = h_{cb}, \tag{2.7}$$

where $C_i$ are the covariant components of $C^h$. The $h_{cb}$ are components of the *second fundamental tensor* of $N$ with respect to the mean curvature unit normal $C_h$. We then have

$$g^{cb}\nabla_b B_b{}^h = h_a{}^a C^h, \tag{2.8}$$

where

$$h_c{}^a = h_{cb} g^{ba}.$$

If we denote by $k_1, k_2, \cdots, k_n$ the eigenvalues of $h_c{}^a$, we then have

$$\sum_{a=1}^{n} k_a = h_a{}^a$$

and

$$\sum_{b<a}^{1\cdots n} k_b k_a = \frac{1}{2}(h_b{}^b h_a{}^a - h_b{}^a h_a{}^b).$$

The scalars $H_1$ and $H_2$ defined by

$$nH_1 = \sum_{a=1}^{n} k_a = h_a{}^a \tag{2.9}$$

and

$$\binom{n}{2} H_2 = \sum_{b<a}^{1\cdots n} k_b k_a = \frac{1}{2}(h_b{}^b h_a{}^a - h_b{}^a h_a{}^b) \tag{2.10}$$

are called *the first and the second mean curvatures* of $N$ with respect to $C^h$ respectively. We note here that

$$\begin{aligned} H_1^2 - H_2 &= \frac{1}{n(n-1)}\left(h_b{}^a h_a{}^b - \frac{1}{n} h_b{}^b h_a{}^a\right) \\ &= \frac{1}{n^2(n-1)} \sum_{b<a}^{1\cdots n} (k_b - k_a)^2, \end{aligned} \tag{2.11}$$

and consequently, if

$$H_1^2 - H_2 = 0 \qquad \text{or} \qquad h_b{}^a h_a{}^b - \frac{1}{n} h_b{}^b h_a{}^a = 0,$$

then

$$k_1 = k_2 = \cdots = k_n = k,$$

that is,

$$h_{cb} = k g_{cb}.$$

A submanifold for which $k_1=k_2=\cdots=k_n$ or $h_{cb}=kg_{cb}$ is said to be *umbilical* with respect to the mean curvature unit normal $C^h$, or simply *pseudo-umbilical.*

We now take $m-n$ mutually orthogonal unit normals $C_x{}^h$ in such a way that the first normal $C_{n+1}{}^h$ coincides with the mean curvature unit normal $C^h$ and $B_b{}^h$, $C_x{}^h$ form the positive orientation of $M$, where and in the sequel the indices $x$, $y$, $z$ take the values $n+1$, $n+2$, $\cdots$, $m$. Then, since $\nabla_c B_b{}^h$ are orthogonal to the submanifold $N$, they can be expressed as

$$\nabla_c B_b{}^h=h_{cbx}C_x{}^h, \tag{2.12}$$

which are *equations of Gauss*, where

$$h_{cb,\,n+1}=h_{cb}.$$

To get the *equations of Weingarten*, we put

$$\nabla_c C_x{}^h=\partial_c C_x{}^h+\{{}_j{}^h{}_i\}B_c{}^j C_x{}^i. \tag{2.13}$$

Then we have

$$\nabla_c C_x{}^h=-h_c{}^a{}_x B_a{}^h+l_{cxy}C_y{}^h, \tag{2.14}$$

where

$$h_c{}^a{}_x=h_{cbx}g^{ba}$$

and

$$l_{cxy}=-l_{cyx}$$

is the *third fundamental tensor* with respect to the normals $C_x{}^h$ and defines the connection induced on the normal bundle.

In fact, a vector field $X^h$ which is defined along $N$ and is normal to $N$ is expressed as

$$X^h=X_x C_x{}^h$$

and consequently

$$\begin{aligned}\nabla_c X^h&=(\partial_c X_x)C_x{}^h+X_x(-h_c{}^a{}_x B_a{}^h+l_{cxy}C_y{}^h)\\&=-h_c{}^a{}_x X_x B_a{}^h+(\partial_c X_x+l_{cyx}X_y)C_x{}^h.\end{aligned}$$

Thus, if we put

$$'\nabla_c X_x=(\nabla_c X^i)C_{xi},$$

where $C_{xi}=C_x{}^j g_{ji}$, we get

$$'\nabla_c X_x=\partial_c X_x+l_{cyx}X_y. \tag{2.15}$$

If $\nabla_c X^h$ is tangent to $N$, that is, if $'\nabla_c X_x=0$, we say that $X^h$ is *parallel* with respect to the induced connection $'\nabla$.

Now the equations of Gauss, those of Mainardi-Codazzi and those of Ricci-Kühne for $N$ are respectively written as

$$K_{kjih}B_d{}^kB_c{}^jB_b{}^iB_a{}^h=K_{dcba}-(h_{dax}h_{cbx}-h_{cax}h_{dbx}), \tag{2. 16}$$

$$K_{kjih}B_d{}^kB_c{}^jB_b{}^iC_x{}^h=\nabla_d h_{cbx}-\nabla_c h_{dbx}+l_{dyx}h_{cby}-l_{cyx}h_{dby}, \tag{2. 17}$$

and

$$\begin{aligned}&K_{kjih}B_d{}^kB_c{}^jC_y{}^iC_x{}^h\\ &=\nabla_d l_{cyx}-\nabla_c l_{dyx}+h_d{}^a{}_y h_{cax}-h_c{}^a{}_y h_{dax}-l_{dyz}l_{czx}+l_{cyz}l_{dzx}.\end{aligned} \tag{2. 18}$$

From (2. 17), we have, by transvection with $g^{cb}$,

$$'K_{kh}B_d{}^kC_x{}^h=\nabla_d h_a{}^a{}_x-\nabla_a h_d{}^a{}_x+l_{dyx}h_a{}^a{}_y-l_{ayx}h_d{}^a{}_y, \tag{2. 19}$$

where

$$'K_{kh}=K_{kjih}B^{ji}, \qquad B^{ji}=B_c{}^jB_b{}^ig^{cb}.$$

We notice here that $'K_{ji}$ is symmetric in $j$ and $i$, because

$$'K_{ji}=K_{jtsi}B^{ts}=K_{sijt}B^{ts}=K_{istj}B^{ts}={}'K_{ij},$$

$B^{ts}$ being symmetric.

## §3. Submanifold of a Riemannian manifold admitting a scalar field $v$ such that $\nabla_j\nabla_i v=f(v)g_{ji}$.

We now assume that the Riemannian manifold $M$ admits a scalar field $v$ such that

$$\nabla_j\nabla_i v=f(v)g_{ji}, \tag{3. 1}$$

or

$$\nabla_j v_i=f(v)g_{ji}, \tag{3. 2}$$

where $v_i=\nabla_i v$. Substituting (3. 2) into the Ricci identity

$$\nabla_k\nabla_j v_i-\nabla_j\nabla_k v_i=-K_{kji}{}^h v_h,$$

we find

$$f'(v)v_kg_{ji}-f'(v)v_jg_{ki}=-K_{kji}{}^h v_h,$$

or

$$K_{kjih}v^h=-f'(v)(v_kg_{ji}-v_jg_{ki}), \tag{3. 3}$$

where $v^h = v_i g^{ih}$, from which

$$K_{kjih} C_x{}^k B^{ji} v^h = -nf'(v) \alpha_x,$$

$$'K_{ji} v^j C_x{}^i = -nf'(v) \alpha_x, \tag{3. 4}$$

where $\alpha_x = v_k C_x{}^k$. We put

$$v^h = B_a{}^h v^a + C_x{}^h \alpha_x, \tag{3. 5}$$

where

$$v^a = v_b g^{ba}, \qquad B_b{}^i v_i = \partial_b v.$$

We also put

$$\alpha_{n+1} = \alpha.$$

From

$$\nabla_j v_i = f(v) g_{ji},$$

we find

$$B_c{}^j B_b{}^i \nabla_j v_i = f(v) B_c{}^j B_b{}^i g_{ji},$$

or

$$\nabla_c v_b = f(v) g_{cb} + h_{cbx} \alpha_x \tag{3. 6}$$

and

$$B_c{}^j C_x{}^i \nabla_j v_i = f(v) B_c{}^j C_x{}^i g_{ji},$$

or

$$\nabla_c \alpha_x = -h_c{}^a{}_x v_a + l_{cxy} \alpha_y. \tag{3. 7}$$

Now, substituting (3. 5) into (3. 4), we find

$$'K_{ji}(B_a{}^j v^a + C_y{}^j \alpha_y) C_x{}^i = -nf'(v) \alpha_x,$$

or

$$'K_{ji} B_a{}^j C_x{}^i v^a + 'K_{ji} C_y{}^j C_x{}^i \alpha_y = -nf'(v) \alpha_x,$$

or, using (2. 19),

$$v^d \nabla_d h_a{}^a{}_x - \nabla_a(h_d{}^a{}_x v^d) + h_d{}^a{}_x \nabla_a v^d + l_{dyx} v^d h_a{}^a{}_y - l_{ayx} h_d{}^a{}_y v^d + 'K_{ji} C_y{}^j C_x{}^i \alpha_y = -nf'(v) \alpha_x,$$

that is,

$$\begin{aligned} &'K_{ji} C_y{}^j C_x{}^i \alpha_y + nf'(v) \alpha_x + v^d \nabla_d h_a{}^a{}_x - \nabla_a(h_d{}^a{}_x v^d) \\ &+ f(v) h_a{}^a{}_x + h_b{}^a{}_x h_a{}^b{}_y \alpha_y + l_{dyx} h_a{}^a{}_y v^d - l_{ayx} h_d{}^a{}_y v^d = 0, \end{aligned} \tag{3. 8}$$

by virtue of (3. 6).

## §4. Integral formulas for submanifold.

From (3. 6), we have

$$g^{cb}\nabla_c v_b = nf(v) + h_a{}^a{}_x \alpha_x,$$

and consequently, integrating over $N$ and applying Green's theorem [19], we find

$$\int_N [nf(v) + h_a{}^a{}_x \alpha_x] dS = 0, \tag{4. 1}$$

$dS$ being the surface element of $N$.

Also, integrating (3. 8) over $N$ and applying Green's theorem, we find that if $C_x{}^i$ are defined globaly on $N$, then

$$\begin{aligned}\int_N [{}'K_{ji} C_y{}^j C_x{}^i \alpha_y + nf'(v)\alpha_x + + v^d \nabla_d h_a{}^a{}_x \\ + f(v) h_a{}^a{}_x + h_b{}^a{}_x h_a{}^b{}_y \alpha_y + l_{dyx} h_a{}^a{}_y v^d - l_{ayx} h_d{}^a{}_y v^d] dS = 0.\end{aligned} \tag{4. 2}$$

Now, from

$$\nabla_c B_b{}^h = h_{cbx} C_x{}^h \quad \text{and} \quad v^h = B_a{}^h v^a + C_x{}^h \alpha_x,$$

we find

$$(\nabla_c B_b{}^i) v_i = h_{cbx} \alpha_x, \tag{4. 3}$$

where

$$h_{cb,\,n+1} = h_{cb} \qquad \text{and} \qquad \alpha_{n+1} = \alpha,$$

that is,

$$(\nabla_c B_b{}^i) v_i = h_{cb}\alpha + h_{cb,\,n+2}\alpha_{n+2} + \cdots + h_{cb,\,m}\alpha_m. \tag{4. 4}$$

In the sequel, we assume that

$$(\nabla_c B_b{}^i) v_i = \alpha h_{cb}, \tag{4. 5}$$

that is,

$$h_{cb,\,n+2}\alpha_{n+2} + \cdots + h_{cb,\,m}\alpha_m = 0. \tag{4. 6}$$

(See also Yano [18].)

Since $v^h$ is written as

$$v^h = B_a{}^h v^a + C^h \alpha + C_{n+2}{}^h \alpha_{n+2} + \cdots + C_m{}^h \alpha_m,$$

the projection of $v^h$ on the normal plane orthogonal to $C^h$ is given by

$$C_{n+2}{}^h\alpha_{n+2}+\cdots+C_m{}^h\alpha_m.$$

For the covariant derivative of this, we have

$$\begin{aligned}&\nabla_c(C_{n+2}{}^h\alpha_{n+2}+\cdots+C_m{}^h\alpha_m)\\ &=-(h_c{}^a{}_{n+2}\alpha_{n+2}+\cdots+h_c{}^a{}_m\alpha_m)B_a{}^h\\ &\quad+(l_{c,n+2,x}\alpha_{n+2}+\cdots+l_{c,m,x}\alpha_m)C_x{}^h\\ &\quad+C_{n+2}{}^h\nabla_c\alpha_{n+2}+\cdots+C_m{}^h\nabla_c\alpha_m.\end{aligned}$$

Thus we see that the assumption (4. 5) or (4. 6) is equivalent to the fact that

$$\nabla_c(C_{n+2}{}^h\alpha_{n+2}+\cdots+C_m{}^h\alpha_m)$$

is normal to the submanifold.

As Katsurada, Kôjyô and Nagai [5], [6], [9] assumed, if $v^h$ has the form

(4. 7) $$v^h=B_a{}^hv^a+\alpha C^h,$$

that is, if $v^h$ is in the space spanned by $B_a{}^h$ and $C^h$, then

(4. 8) $$\alpha_{n+2}=\cdots=\alpha_m=0$$

and (4. 6) is satisfied.

If

(4. 9) $$\nabla_cB_b{}^h=h_{cb}C^h,$$

that is, if $\nabla_cB_b{}^h$ are in the direction of mean curvature vector, then

(4. 10) $$h_{cb,n+2}=\cdots=h_{cb,m}=0$$

and (4. 6) is satisfied.

Now, if (4. 5) or (4. 6) is satisfied, we have

(4. 11) $$h_{cbx}\alpha_x=\alpha h_{cb},$$

and consequently

(4. 12) $$h_a{}^a{}_x\alpha_x=\alpha h_a{}^a.$$

Thus we have from (4. 1)

(4. 13) $$\int_N[nf(v)+\alpha h_a{}^a]dS=0.$$

Putting $x=n+1$ in (4. 2), we have

$$\int_N ['K_{ji}{}''v^j C^i + n\alpha f'(v) + v^d \nabla_d h_a{}^a$$

(4. 14)

$$+ f(v) h_a{}^a + \alpha h_b{}^a h_a{}^b + l_{dy} h_a{}^a{}_y v^d - l_{ay} h_d{}^a{}_y v^d] dS = 0,$$

where

$$''v^j = C_x{}^j \alpha_x, \qquad l_{dy} = l_{dy,\,n+1}.$$

We moreover assume that the mean curvature vector

$$H^h = \frac{1}{n} g^{cb} \nabla_c B_b{}^h = \frac{1}{n} h_a{}^a C^h$$

is parallel with respect to the connection $'\nabla$ induced in the normal bundle, that is,

$$\nabla_c H^h = \frac{1}{n} (\nabla_c h_a{}^a) C^h + \frac{1}{n} h_a{}^a (-h_c{}^b B_b{}^h - l_{cy} C_y{}^h)$$

$$= -\frac{1}{n} h_a{}^a h_c{}^b B_b{}^h + \frac{1}{n} (\nabla_c h_a{}^a) C^h - \frac{1}{n} h_a{}^a l_{cy} C_y{}^h$$

is tangent to the submanifold. Since $l_{c,\,n+1} = 0$, we see that this assumption is equivalent to

(4. 15) $$h_a{}^a = \text{const.} \neq 0, \qquad l_{cy} = 0.$$

In this case we have from (4. 14)

(4. 16) $$\int_N ['K_{ji}{}''v^j C^i + n\alpha f'(v) + f(v) h_a{}^a + \alpha h_b{}^a h_a{}^b] dS = 0.$$

Thus forming (4. 16)$-$(4. 13)$\times(1/n) h_a{}^a$, we find

(4. 17) $$\int_N \left[ 'K_{ji}{}''v^j C^i + n\alpha f'(v) + \alpha \left( h_b{}^a h_a{}^b - \frac{1}{n} h_b{}^b h_a{}^a \right) \right] dS = 0.$$

On the other hand, putting $x = n+1$ in (3. 4), we have

(4. 18) $$'K_{ji} v^j C^i + n\alpha f'(v) = 0,$$

from which

(4. 19) $$'K_{ji}{}''v^j C^i + n\alpha f'(v) = -'K_{ji}{}'v^j C^i,$$

where

$$'v^j = B_c{}^j v^c.$$

Thus, from (4. 17), we have

(4. 20) $$\int_N \left[ 'K_{ji}{}'v^j C^i - \alpha \left( h_b{}^a h_a{}^b - \frac{1}{n} h_b{}^b h_a{}^a \right) \right] dS = 0.$$

## § 5. Theorems.

If

$$'K_{ji}''v^jC^i+n\alpha f'(v)=0 \tag{5. 1}$$

or

$$'K_{ji}'v^jC^i=0, \tag{5. 2}$$

then we have, from (4. 17) or (4. 20),

$$\int_N \alpha\left(h_b{}^a h_a{}^b-\frac{1}{n}h_b{}^b h_a{}^a\right)dS=0. \tag{5. 3}$$

Since

$$h_b{}^a h_a{}^b-\frac{1}{n}h_b{}^b h_a{}^a=\frac{1}{n}\sum_{b<a}^{1\cdots n}(k_b-k_a)^2\geqq 0,$$

if $\alpha=C^iv_i$ has fixed sign on $S$, then we have from (5. 3)

$$h_b{}^a h_a{}^b-\frac{1}{n}h_b{}^b h_a{}^a=0,$$

that is

$$k_1=k_2=\cdots=k_n,$$

and the submanifold $N$ is pseudo-umbilical. Thus we have

THEOREM 5. 1. *Let $M$ be an $m$-dimensional orientable differentiable Riemannian manifold which admits a non-constant scalar field $v$ such that $\nabla_j\nabla_i v=f(v)g_{ji}$, $f(v)$ being a differentiable function of $v$, and $N$ an $n$-dimensional closed orientable submanifold differentiably imbedded in $M$ such that*

(i) $(\nabla_c B_b{}^i)v_i=\alpha h_{cb}$ *or* $h_{cb,n+2}\alpha_{n+2}+\cdots+h_{cb,m}\alpha_m=0$,

(ii) *the mean curvature vector $H^h\neq 0$ is parallel with respect to the connection induced on the normal bundle,*

(iii) $'K_{ji}''v^jC^i+n\alpha f'(v)=0$ *or* $'K_{ji}'v^jC^i=0$,

(iv) $\alpha=C^i\nabla_i v$ *has fixed sign on $S$.*

*Then $N$ is pseudo-umbilical.*

This generalizes Theorem 1 in [17].

If $v^h$ has the form

$$v^h=B_a{}^hv^a+\alpha C^h,$$

that is, if

$$\alpha_{n+1}=\cdots=\alpha_m=0,$$

then

$$''v^h=\alpha C^h,$$

and consequently (4. 17) gives

$$(5.4)\qquad \int_N \alpha\left[('K_{ji}+nf'(v)g_{ji})C^jC^i+\left(h_b{}^a h_a{}^b-\frac{1}{n}h_b{}^b h_a{}^a\right)\right]dS=0,$$

from which we have

THEOREM 5. 2. *Let $M$ be a Riemannian manifold as in Theorem* 5. 1 *and $N$ an $n$-dimensional closed orientable submanifold differentiably imbedded in $M$ such that*

(i) $\alpha_{n+2}=\cdots=\alpha_m=0$,

(ii) *the mean curvature vector $H^h\neq 0$ is parallel with respect to the connection induced on the normal bundle,*

(iii) $('K_{ji}+nf(v)g_{ji})C^jC^i\geqq 0$,

(iv) *$\alpha=C^i\nabla_i v$ has fixed sign on $N$.*

*Then $N$ is pseudo-umbilical.*

This theorem has been obtained by Nagai [9]. But it seems to the author that the condition $h_a{}^a$=const. in his paper should be replaced by the condition (ii) above, because the assumption $h_a{}^a$=const. only gives no condition on the third fundamental tensor.

If

$$\nabla_c B_b{}^h=h_{cb}C^h,$$

that is, if

$$h_{cb,\,n+2}=\cdots=h_{cb,\,m}=0,$$

then we have, from (4. 14),

$$(5.5)\qquad \int_N ['K_{ji}{}''v^jC^i+n\alpha f'(v)+v^d\nabla_d h_a{}^a+f(v)h_a{}^a+\alpha h_b{}^a h_a{}^b]dS=0,$$

by virtue of

$$l_{d,\,n+1}=0.$$

Thus, if $h_a{}^a$=const., then we have (4. 16) and consequently (4. 20). Thus

THEOREM 5. 3. *Let $M$ be a Riemannian manifold as in Theorem* 5. 1 *and $N$ an $n$-dimensional closed orientable submanifold differentiably imbedded in $M$ such* *that*

(i) $h_{cb,n+2}=\cdots=h_{cb,m}=0$,

(ii) *the mean curvature vector* $H^h\neq 0$ *is parallel with respect to the connection induced on the normal bundle,*

(iii) $'K_{ji}{}'v^jC^i=0$,

(iv) $\alpha=C^i\nabla_i v$ *has fixed sign on* $N$.

*Then* $N$ *is pseudo-umbilical.*

Now, if the submanifold $N$ satisfies the conditions in Theorem 5. 2 or Theorem 5. 3, then we have, from (3. 6),

$$\nabla_c\nabla_b v=f(v)g_{cb}+\alpha h_{cb}$$

and, from (3. 7)

$$\nabla_c\alpha=-h_c{}^a v_a.$$

But, $N$ being pseudo-umbilical, we have

$$h_{cb}=\lambda g_{cb},$$

$\lambda$ being a constant different from zero. Thus we have

$$\nabla_c\nabla_b v=[f(v)+\lambda\alpha]g_{cb} \tag{5. 6}$$

and

$$\nabla_c\alpha=-\lambda\nabla_c v, \tag{5. 7}$$

from which

$$\alpha=-\lambda v+c,$$

$c$ being a constant. Thus (5. 6) becomes

$$\nabla_c\nabla_b v=[f(v)-\lambda^2 v+\lambda c]g_{cb}, \tag{5. 8}$$

and consequently, if

$$f(v)=kv,$$

$k$ being a constant, we have

$$\nabla_c\nabla_b v=[-(\lambda^2-k)v+\lambda c]g_{cb}, \tag{5. 9}$$

and if

$$f(v)=k,$$

we have

$$\nabla_c \nabla_b v = [-\lambda^2 v + k + \lambda c] g_{cb}. \tag{5.10}$$

Thus, if $v \neq$ constant on $N$, then as was shown in [16], we have, by a theorem of Obata [10],

THEOREM 5.4. *If $f(v)=kv$, or $f(v)=k$ in Theorem 5.2 or Theorem 5.3, and $v \neq$ constant on $N$, then the submanifold $N$ is isometric to a sphere.*

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