KŌDAI MATH. SEM. REP.
21 (1969), 448—458

ISOMETRIC IMMERSIONS OF SASAKIAN MANIFOLDS IN SPHERES

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Introduction. A Sasakian manifold M^m which is isometrically immersed in a Riemannian manifold $*M^{m+1}$ of constant curvature $C\neq 1$ is of constant curvature 1, as was shown by Takahashi [7]. The case where the constant curvature of $*M^{m+1}$ is 1 is in a very different situation. In this paper we study the case. The rank of the second fundamental form is called the type number of the immersion.

THEOREM. *Let M^m be a Sasakian manifold which is isometrically immersed in a Riemannian manifold $*M^{m+1}$ of constant curvature* 1. *Then*

(i) *the type number $k\leqq 2$, and*

(ii) *M^m is of constant curvature* 1 *if and only if the scalar curvature S is equal to $m(m-1)$.*

In an η-Einstein space for any point p the Ricci curvature $R_1(X, X)$ is constant for any unit vector X at p such that $\eta(X)=0$. In [7] it was also proved that an η-Einstein Sasakian manifold M^m $(m\geqq 5)$ which is isometrically immersed in a Riemannian manifold $*M^{m+1}$ of constant curvature 1 is of constant curvature 1. We generalize this in the following form.

THEOREM. *Let M^m $(m\geqq 5)$ be a Sasakian manifold which is isometrically immersed in a Riemannian manifold $*M^{m+1}$ of constant curvature* 1. *Assume that at any point p of M^m we have a subspace F_p of the tangent space at p to M^m such that*

(i) $\dim F_p=m-2$,

(ii) $\eta(X)=0$ *for any* $X\in F_p$,

(iii) $R_1(X, X)=$*constant for any unit* $X\in F_p$.

Then M^m is of constant curvature 1.

In § 2 we study some properties of contact Riemannian manifolds which satisfy some conditions on the Ricci tensor or the Riemannian curvature tensor, for example, $R(X, Y)\cdot R=0$ or $R(X, Y)\cdot R_1=0$.

In the last section, we consider invariant submanifolds of Sasakian manifolds. We see that they are minimal. As a special case, we have invariant submanifolds M^m of a unit sphere S^{2r+1} considered as a Sasakian manifold, which are shown to be unit spheres if and only if $S=m(m-1)$.

Received June 19, 1969.

§1. Structure tensors.

Let (ϕ, ξ, η, g) be structure tensors of a contact Riemannian manifold M of m-dimension. They satisfy

(1. 1) $$\phi\xi=0, \qquad \eta(\xi)=1,$$

(1. 2) $$\phi\phi X=-X+\eta(X)\xi,$$

(1. 3) $$g(X, Y)=g(\phi X, \phi Y)+\eta(X)\eta(Y),$$

(1. 4) $$2g(X, \phi Y)=d\eta(X, Y), \qquad \eta(X)=g(\xi, X)$$

for any vector fields X and Y on M. When ξ is a Killing vector field, M is said to be a K-contact Riemannian manifold, and we have

(1. 5) $$\nabla_X\xi=-\phi X,$$

(1. 6) $$R_1(X, \xi)=(m-1)\eta(X)$$

(1. 7) $$g(R(X, \xi)Y, \xi)=g(X, Y)-\eta(X)\eta(Y),$$

(1. 7)′ $$R(X, \xi)\xi=-X+\eta(X)\xi,$$

where ∇ is the Riemannian connection, R_1 and R are the Ricci curvature tensor and the Riemannian curvature tensor, respectively (cf. [1], [8]). If we have

(1. 8) $$(\nabla_X\phi)Y=g(X, Y)\xi-\eta(Y)X,$$

then M is called a Sasakian manifold or a normal contact Riemannian manifold, and we have

(1. 9) $$R(X, \xi)Y=g(X, Y)\xi-\eta(Y)X.$$

A Sasakian manifold is a K-contact Riemannian manifold. The Ricci curvature tensor R_1 on a Sasakian manifold satisfies

(1. 10) $$R_{rk}\phi_j^r=-R_{rj}\phi_k^r$$

where indices $i, j, r, \cdots \in (1, \cdots, m)$ (cf. [5]). Operating ϕ_i^k to (1. 10) and using (1. 6), we have

(1. 11) $$R_1(\phi X, \phi Y)=R_1(X, Y)-(m-1)\eta(X)\eta(Y).$$

§2. Some conditions on the Ricci curvature tensor and the Riemannian curvature tensor.

The curvature transformation $R(X, Y)$ acts on the tensor algebra as a derivation. And the condition $R(X, Y)\cdot R=0$ was discussed by Nomizu [2] for hypersurfaces of Euclidean spaces. While the condition $R(X, Y)\cdot R_1=0$ was studied by Tanno [10] for hypersurfaces of Euclidean spaces.

PROPOSITION 2.1. *Let M be a K-contact Riemannian manifold. Then the following conditions are equivalent*:

(i) *M is an Einstein space*; $R_1=ag$,

(ii) *The Ricci curvature tensor is parallel*; $\nabla R_1=0$,

(iii) $R(X, Y)\cdot R_1=0$ *for any X and Y*,

(iv) $R(X, \xi)\cdot R_1=0$ *for any Y*.

Proof. (i)→(ii)→(iii)→(iv) is clear. Assume condition (iv), which is equivalent to

$$R_1(R(X, \xi)U, V)+R_1(U, R(X, \xi)V)=0 \tag{2.1}$$

for any vectors U and V. Put $V=\xi$. Then, using (1.6), (1.7) and (1.7)′ we have

$$(m-1)g(X, U)-R_1(X, U)=0. \tag{2.2}$$

Therefore M is an Einstein space. Q.E.D.

If the Ricci curvature tensor R_1 is of the form

$$R_1=ag+b\eta\otimes\eta, \tag{2.3}$$

where a and b are functions on M, then M is called an η-Einstein space. If $m>3$, a and b are constant on a K-contact Riemannian manifold. A Sasakian manifold M is an η-Einstein space if and only if

$$\begin{aligned}(R(X, Y)\cdot R_1)(U, V)=&\,b[\eta(U)g(V, X)+\eta(V)g(U, X)]\eta(Y)\\ &-b[\eta(U)g(V, Y)+\eta(V)g(U, Y)]\eta(X)\end{aligned} \tag{2.4}$$

holds good for some function b on M. The necessity is an immediate consequence of (1.9) and (2.3). While the sufficiency will be verified in proposition 2.2. By putting $Y=\xi$ in (2.4), we have

$$(R(X, \xi)\cdot R_1)(U, V)=b[\eta(U)g(V, X)+\eta(V)g(U, X)-2\eta(U)\eta(V)\eta(X)]. \tag{2.5}$$

PROPOSITION 2.2. *A K-contact Riemannian manifold is an η-Einstein space if and only if* (2.5) *holds for some function b on M*.

Proof. Assume that (2.3) holds. Then by (1.7) we have (2.5). Conversely assume that (2.5) holds. Then as in the proof of Proposition 2.1, we have

$$(m-1)g(X, U)-R_1(X, U)=b[g(U, X)-\eta(U)\eta(X)].$$

Therefore we have

$$R_1(X, U)=(m-1-b)g(X, U)+b\eta(X)\eta(U), \tag{2.6}$$

which shows that M is an η-Einstein space.

THEOREM 2. 3. *Let M be a K-contact Riemannian manifold. Then the following conditions are equivalent*:

(i) *M is of constant curvature* 1,

(ii) *M is locally symmetric*; $\nabla R=0$,

(iii) $R(X, Y)\cdot R=0$ *for any X and Y*,

(iv) $R(X, \xi)\cdot R=0$ *for any X*.

Proof. (i)→(ii)→(iii)→(iv) is clear. Assume condition (iv), which is equivalent to

$$R(X, \xi)(R(U, V)W)-R(R(X, \xi)U, V)W -R(U, R(X, \xi)V)W-R(U, V)(R(X, \xi)W)=0 \tag{2. 7}$$

for any U, V and W. If we put $U=W=\xi$ in (2. 7), using (1. 7), we have

$$R(X, V)\xi+R(\xi, V)X+\eta(V)X-2\eta(X)V+g(X, V)\xi=0. \tag{2. 8}$$

Consider the inner product of Y and both sides of (2. 8). Then we have

$$g(R(X, V)\xi, Y)+g(R(\xi, V)X, Y)+\eta(V)g(X, Y)-2\eta(X)g(V, Y)+\eta(Y)g(X, V)=0. \tag{2. 9}$$

Interchanging X and Y in (2. 9) and subtracting the result from (2. 9), we get

$$g(R(\xi, V)X, Y)=\eta(X)g(V, Y)-\eta(Y)g(X, V), \tag{2. 10}$$

where we have used the Bianchi's identity. (2. 10) is written as

$$R(\xi, V)X=\eta(X)V-g(X, V)\xi. \tag{2. 11}$$

If we put $U=\xi$ in (2. 7), using (2. 11) we have

$$R(X, V)W=g(X, W)V-g(W, V)X,$$

which shows that M is of constant curvature 1.

REMARK. Proposition 2. 1 for a Sasakian manifold was given by Takahashi [7]. (i)↔(ii) in Theorem 2. 3 was given by Tanno [9], generalizing a result by Okumura [3] for a Sasakian manifold. For (iii) on a Sasakian manifold, see Takahashi [6].

§ 3. Sasakian manifolds which are hypersurfaces of a Riemannian manifold of constant curvature 1.

Let M^m be a Sasakian manifold of m-dimension which is isometrically immersed in a Riemannian manifold $^*M^{m+1}$ of constant curvature 1. Then we have the Gauss and Codazzi equations:

$$-R(X, Y)=X\wedge Y+AX\wedge AY, \quad \text{or} \tag{3. 1}$$

$$-R(X, Y)Z=g(Y, Z)X-g(X, Z)Y+g(AY, Z)AX-g(AX, Z)AY, \tag{3. 1$'$}$$

$$(3.2)\qquad (\nabla_X A)Y=(\nabla_Y A)X,$$

where A is a (local) (1, 1)-tensor associated with the second fundamental form B by $B(X, Y)=g(X, AY)\zeta$ corresponding to a (local) field of unit normal vectors ζ to M. A is symmetric with respect to g. By (3. 1)′, putting $Y=\xi$, we have

$$(3.3)\qquad -R(X, \xi)Z=\eta(Z)X-g(X, Z)\xi+\eta(AZ)AX-g(AX, Z)A\xi.$$

By (1. 9) and (3. 3) we have

$$(3.4)\qquad \eta(AY)AX=g(AX, Y)A\xi.$$

When the trace of A vanishes, M or the immersion is called minimal. The rank of A is called the type number of the immersion.

LEMMA 3. 1. *Denoting by θ the trace of A we have*

$$(3.5)\qquad \theta A-AA+\theta\phi A\phi-\phi AA\phi=0.$$

Proof. Since the Ricci curvature tensor R_1 is given by

$$R_1(X, Y)=\text{trace}\ [U\rightarrow R(X, U)Y],$$

by (3. 1)′ we have

$$(3.6)\qquad R_1(X, Y)=(m-1)g(X, Y)+(\text{trace}\ A)g(AX, Y)-g(AAX, Y).$$

Replacing X and Y by ϕX and ϕY in (3. 6), we have

$$(3.7)\qquad R_1(\phi X, \phi Y)=(m-1)g(\phi X, \phi Y)+\theta g(A\phi X, \phi Y)-g(AA\phi X, \phi Y).$$

By (1. 4) ϕ is skew symmetric, and we have

$$g(A\phi X, \phi Y)=-g(\phi A\phi X, Y),$$

$$g(AA\phi X, \phi Y)=-g(\phi AA\phi X, Y).$$

By (1. 3) and (1. 11), (3. 7) is written as

$$(3.8)\qquad R_1(X, Y)=(m-1)g(X, Y)-g((\theta\phi A\phi-\phi AA\phi)X, Y).$$

Then (3. 6) and (3. 8) imply

$$(3.9)\qquad g((\theta A-AA+\theta\phi A\phi-\phi AA\phi)X, Y)=0.$$

That is, we have (3. 5). Q.E.D.

By (3. 6) the scalar curvature $S=\text{trace}\ R_1$ is given by

$$(3.10)\qquad S=m(m-1)+\theta^2-\text{trace}\ AA.$$

LEMMA 3. 2. *In a Sasakian manifold we have*

$$\phi R(X, Y)\phi=-R(X, Y)-X\wedge Y+\phi X\wedge\phi Y. \tag{3.11}$$

Proof. Assume that X, Y and Z are (local) vector fields such that $(\nabla X)_p=(\nabla Y)_p=(\nabla Z)_p=0$ for a fixed point p of M. By the Ricci identity for ϕ:

$$-(R(X, Y)\cdot\phi)Z=(\nabla_X\nabla_Y\phi)Z-(\nabla_Y\nabla_X\phi)Z,$$

we have at the point p

$$\begin{aligned}-R(X, Y)(\phi Z)+\phi(R(X, Y)Z)&=\nabla_X((\nabla_Y\phi)Z)-\nabla_Y((\nabla_X\phi)Z)\\&=\nabla_X(g(Y, Z)\xi-\eta(Z)Y)-\nabla_Y(g(X, Z)\xi-\eta(Z)X)\\&=g(Y, Z)\nabla_X\xi-(\nabla_X\eta)(Z)Y-g(X, Z)\nabla_Y\xi+(\nabla_Y\eta)(Z)X\\&=-g(Y, Z)\phi X+g(\phi X, Z)Y+g(X, Z)\phi Y-g(\phi Y, Z)X.\end{aligned}$$

Therefore operating ϕ we have

$$\begin{aligned}&-\phi R(X, Y)(\phi Z)-R(X, Y)Z+\eta(R(X, Y)Z)\xi\\&=g(Y, Z)(X-\eta(X)\xi)+g(\phi X, Z)\phi Y\\&\quad-g(X, Z)(Y-\eta(Y)\xi)-g(\phi Y, Z)\phi X.\end{aligned}$$

Since from (1.9) we have

$$\eta(R(X, Y)Z)=g(X, Z)\eta(Y)-g(Y, Z)\eta(X),$$

we have

$$-\phi R(X, Y)\phi Z-R(X, Y)Z=(X\wedge Y)Z-(\phi X\wedge\phi Y)Z. \tag{3.12}$$

LEMMA 3.3. *The (local) tensor A satisfies*

$$\phi AX\wedge\phi AY=AX\wedge AY, \tag{3.13}$$

$$\phi A\phi A+\theta A=AA. \tag{3.14}$$

Proof. By (3.1) we have

$$\begin{aligned}-\phi R(X, Y)\phi Z&=\phi(X\wedge Y+AX\wedge AY)\phi Z\\&=\phi[g(Y, \phi Z)X-g(X, \phi Z)Y+g(AY, \phi Z)AX-g(AX, \phi Z)AY]\\&=-g(\phi Y, Z)\phi X+g(\phi X, Z)\phi Y-g(\phi AY, Z)\phi AX+g(\phi AX, Z)\phi AY\\&=-(\phi X\wedge\phi Y)Z-(\phi AX\wedge\phi AY)Z.\end{aligned} \tag{3.15}$$

Then (3.13) follows from (3.1), (3.11) and (3.15). Put

$$E(X, Y, Z)=g(\phi AY, Z)\phi AX-g(\phi AX, Z)\phi AY-g(AY, Z)AX+g(AX, Z)AY.$$

Then by (3. 13) we have $E(X, Y, Z)=0$. Taking the trace $[X\to E(X, Y, Z)]$, we have

$$-g(\phi A\phi AY, Z)-\theta g(AY, Z)+g(AAY, Z)=0,$$

since trace $\phi A=0$. Consequently we have (3. 14).

If the rank of $A\leqq 1$ everywhere on M, then M is of constant curvature 1 by (3. 1). Assume that the rank of $A\geqq 2$ at some point and hence on some open set. Then we have two orthonormal vector fields X and Y (locally) such that $AX=\lambda X$ and $AY=\mu Y$ for non-zero λ and μ on the open set. By (3. 4), then, we have $\lambda\mu\eta(Y)X=\lambda g(X, Y)A\xi=0$. That is, we have $\eta(Y)=0$. Next, we put $X=Y$ in (3. 4), to get $\lambda A\xi=0$. Consequently, $A\xi=0$ on the open set. From now on in this section we assume that ξ is an eigenvector of A corresponding to eigenvalue 0. Let

$$\lambda,\ \mu,\ \cdots,\ \nu,\ 0 \qquad (\lambda\geqq\mu\geqq\cdots\geqq\nu)$$

be eigenvalues of A. They are continuous. Let p be an arbitrary point of M. We define subspaces $D_\lambda(\lambda, \mu, \cdots)$ of the tangent space T_pM by

$$D_\lambda=\{X\in T_pM; \qquad AX=\lambda X\}.$$

LEMMA 3. 4. *If $\lambda\neq 0$, then we have*

$$\phi D_\lambda=D_{\theta-\lambda}.$$

Proof. Let $X\in D_\lambda$. By (3. 14) we have

$$\lambda\phi A\phi X+\theta\lambda X=\lambda^2 X.$$

Since $\lambda\neq 0$, we have $\phi A\phi X=(\lambda-\theta)X$. Operating ϕ we have $\phi\phi A\phi X=(\lambda-\theta)\phi X$. Since $\eta(A\phi X)=g(A\phi X, \xi)=g(\phi X, A\xi)=0$, we have $\phi\phi A\phi X=-A\phi X$. Namely, $A\phi X=(\theta-\lambda)\phi X$. Q.E.D.

By Lemma 3. 4, A has the following components with respect to a suitable basis;

$$(3.16)\qquad A=\begin{pmatrix} \lambda E_r & & & & & & & & 0 \\ & (\theta-\lambda)E_r & & & & & & & \vdots \\ & & \mu E_s & & & & & & \vdots \\ & & & (\theta-\mu)E_s & & & & & \vdots \\ & & & & \ddots & & & & \vdots \\ & & & & & \nu E_t & & & \vdots \\ & & & & & & (\theta-\nu)E_t & & \\ 0 & \cdots & & & & & \cdots & & OE_v \end{pmatrix}$$

including the cases ($\mu=\theta$, $2\nu=\theta$, etc.), where E_r is the $r\times r$ identity matrix, etc. Trace of A is then

$$\theta=r\theta+s\theta+\cdots+t\theta,$$

which can be true only when (i) $\theta=0$, or (ii) $r=1$, $s=0$, $\cdots$, $t=0$, and $\theta\neq 0$.

LEMMA 3.5. *The rank of* $A\leqq 2$. *If the rank of* $A=2$ *at a point* p, *then non-zero eigenvalues are* λ *and* $-\lambda$, *or* λ *and* $\theta-\lambda$ *at* p.

Proof. Assume that the rank of $A\geqq 3$ at some point p. This is the case (i), since the rank of $A\leqq 2$ in the case (ii). Therefore we have $\theta=0$. Suppose that $D_\lambda\neq 0$ and $D_\mu\neq 0$ for non-zero λ, μ such that $\lambda\neq\mu$ and $\lambda\neq-\mu$. Then we have non-zero vectors $X\in D_\lambda$ and $Y\in D_\mu$, and (3.13) implies

$$\lambda\mu\phi X\wedge\phi Y=\lambda\mu X\wedge Y, \tag{3.19}$$

where $\phi X\in D_{-\lambda}$ and $\phi Y\in D_{-\mu}$. Operating (3.19) to Y we have $g(Y, Y)X=0$, contradicting $X\neq 0$ and $Y\neq 0$. Therefore non-zero eigenvalues are λ and $-\lambda$. Since the rank of $A\geqq 3$ by assumption, we have dim $D_\lambda\geqq 2$. Suppose that X and Y are orthonormal vectors in D_λ. Then (3.13) also implies $\phi X\wedge\phi Y=X\wedge Y$. Similarly we have $g(Y, Y)X=0$, which is a contradiction. Thus the rank of $A\leqq 2$.

THEOREM 3.6. *Let* M^m *be a Sasakian manifold which is isometrically immersed in a Riemannian manifold* $^*M^{m+1}$ *of constant curvature* 1. *If the scalar curvature* S *of* M^m *is equal to* $m(m-1)$, *then* M^m *is of constant curvature* 1.

Proof. By (3.10) we have $\theta^2=\text{trace}\, AA$. Assume that the rank of $A=2$ somewhere. Then by Lemma 3.5 trace $AA=\lambda^2+(\theta-\lambda)^2$ (including the case $\theta=0$). And hence we get $\lambda(\theta-\lambda)=0$, which is a contradiction.

THEOREM 3.7. *Let* M^m $(m\geqq 5)$ *be a Sasakian manifold which is isometrically immersed in a Riemannian manifold* $^*M^{m+1}$ *of constant curvature* 1. *Assume that for any point* p *of* M^m *we have a subspace* F_p *of the tangent space at* p *to* M^m *such that*

(i) dim $F_p=m-2$,

(ii) F_p *is orthogonal to* ξ,

(iii) $R_1(X, X)$ *is constant for any unit vector* $X\in F_p$.

Then M^m *is of constant curvature* 1.

Proof. Assume that the rank of $A=2$ on some open set U containing a point p. For a unit vector $X\in D_\lambda$ we have $\phi X\in D_{\theta-\lambda}$ (including the case $\theta=0$). Since $\eta(X)=0$ and dim $F_p=m-2$ we have some real numbers c and d such that $c^2+d^2=1$ and

$$cX+d\phi X\in F_p.$$

Then by (3.6) we have

$$R_1(cX+d\phi X, cX+d\phi X)=(m-1)+\theta(c^2\lambda+d^2(\theta-\lambda))-(c^2\lambda^2+d^2(\theta-\lambda)^2)$$
$$=(m-1)+\theta\lambda-\lambda^2.$$

On the other hand, since $m \geqq 5$, we have $Z \in D_0$ such that $\eta(Z)=0$ and $\phi Z \in D_0$. Similarly there are some real numbers $*c$ and $*d$ such that $*c^2 + *d^2 = 1$ and

$$*cZ + *d\phi Z \in F_p.$$

In this case we have

$$R_1(*cZ + *d\phi Z, *cZ + *d\phi Z) = m-1.$$

Now by condition (iii) we have $\lambda\theta - \lambda^2 = 0$, which is a contradiction.

Takahashi [7] proved that: A Sasakian manifold M^m (with pseudo-Riemannian metric) which is (properly and) isometrically immersed in a (pseudo-) Riemannian manifold $*M^{m+1}$ of constant curvature $C \neq 1$ is of constant curvature 1.

Therefore we have

THEOREM 3. 8. *A Sasakian manifold M^m with scalar curvature $m(m-1)$ which is isometrically immersed in a Riemannian manifold $*M^{m+1}$ of constant curvature C is of constant curvature* 1.

REMARK. One may apply the same arguments to get Lemmas and Theorems above for properly and isometrically immersed Sasakian manifolds M^m with indefinite metrics in a pseudo-Riemannian manifold $*M^{m+1}$ of constant curvature 1.

REMARK. If M^m is an η-Einstein space, then there exists a field of subspaces F_p, $p \in M$, satisfying the conditions (i), (ii) and (iii) of Theorem 3. 7. Therefore it is a generalization of Takahashi's result [7] on an η-Einstein Sasakian manifold.

§ 4. Invariant submanifolds of Sasakian manifolds.

A submanifold $M = M^{2n+1}$ of a Sasakian manifold $*M^{2r+1}$ with structure tensors $(*\phi, *\xi, *\eta, *g)$ is called invariant if

(i) $*\xi$ is tangent to M everywhere on M,

(ii) $*\phi X$ is tangent to M for any tangent vector X to M.

An invariant submanifold M has the induced structure tensors (ϕ, ξ, η, g) the restrictions of $*\phi, *\xi, *\eta, *g$ to M. For the Riemannian connection $*\nabla$ for $*g$, the Riemannian connection ∇ for g is given by

$$\nabla_X Y = (*\nabla_{*X} *Y)^{\mathrm{T}}, \tag{4. 1}$$

where $*X$ and $*Y$ are any local extensions of vector fields X and Y on M to those on $*M$, and Z^{T} means the tangent part of Z to M. Similarly Z^{N} means the normal part of Z in $*M$. We show that M is a Sasakian manifold: Let X and Y be vector fields on M. Then

$$\begin{aligned}(\nabla_X \phi) Y &= \nabla_X(\phi Y) - \phi(\nabla_X Y)\\ &= (*\nabla_{*X} *(\phi Y))^{\mathrm{T}} - *\phi(*\nabla_{*X} *Y)^{\mathrm{T}}\end{aligned}$$

$$=((^*\nabla_{*X}{}^*\phi)^*Y)^{\mathrm{T}}$$

$$=(^*g(^*X,{}^*Y)^*\xi-{}^*\eta(^*Y)^*X)^{\mathrm{T}} \qquad \text{(by (1. 8))}$$

$$=g(X,Y)\xi-\eta(Y)X.$$

This shows that the structure is Sasakian. Next we show

PROPOSITION 4. 1. *An invariant submanifold of a Sasakian manifold is minimal.*

Proof. We adopt the Simons' method [4]. Let $(e_1, \cdots, e_n, \phi e_1, \cdots, \phi e_n, \xi)$ be a ϕ-basis. Then for $X=e_\alpha$, we have

$$B(\phi X, \phi X)=(^*\nabla_{*(\phi X)}{}^*(\phi X))^{\mathrm{N}}$$
$$=((^*\nabla_{*(\phi X)}{}^*\phi)^*X)^{\mathrm{N}}+(^*\phi^*\nabla_{*(\phi X)}{}^*X)^{\mathrm{N}}$$
$$=(^*g(^*\phi^*X,{}^*X)^*\xi-{}^*\eta(^*X)^*\phi^*X)^{\mathrm{N}}+(^*\phi^*\nabla_{*(\phi X)}{}^*X)^{\mathrm{N}}$$
$$=(^*\phi^*\nabla_{*(\phi X)}{}^*X)^{\mathrm{N}}={}^*\phi(^*\nabla_{*(\phi X)}{}^*X)^{\mathrm{N}}$$
$$={}^*\phi B(^*\phi^*X,{}^*X)={}^*\phi B(^*X,{}^*\phi X)$$
$$={}^*\phi((^*\nabla_{*X}{}^*\phi)^*X+{}^*\phi^*\nabla_{*X}{}^*X)^{\mathrm{N}}$$
$$={}^*\phi(^*g(^*X,{}^*X)^*\xi-{}^*\eta(^*X)^*X+{}^*\phi^*\nabla_{*X}{}^*X)^{\mathrm{N}}$$
$$={}^*\phi^*\phi(^*\nabla_{*X}{}^*X)^{\mathrm{N}}=-B(X,X).$$

On the other hand, we have

$$B(\xi,\xi)=(^*\nabla_{*\xi}{}^*\xi)^{\mathrm{N}}$$
$$=(-{}^*\phi^*\xi)^{\mathrm{N}}=0.$$

Therefore the mean curvature K:

$$K=\sum[B(e_\alpha, e_\alpha)+B(\phi e_\alpha, \phi e_\alpha)]+B(\xi,\xi)$$

vanishes. Q.E.D.

A Euclidean sphere S^{2r+1} has the standard Sasakian structure of constant curvature 1, and we denote this space by $S^{2r+1}[1]$.

THEOREM 4. 2. *A compact and invariant submanifold M^m with scalar curvature S:*

$$S>m(m-1)-(2r+1-m)m/(4r-2m+1)$$

of a Sasakian manifold $S^{2r+1}[1]$ is an m-dimensional unit sphere.

In particular if $S=m(m-1)$, then M^m is a unit sphere.

Proof. This follows easily from Simons' result [4] that: Let M^m be a compact minimal variety immersed in a unit sphere S^n. If

$$S/m(m-1)>1-(n-m)/(2n-2m-1)(m-1),$$

then M^m is a unit sphere (Corollary 5. 3. 3, in [4]).

REFERENCES

[1] HATAKEYAMA, Y., Y. OGAWA, AND S. TANNO, Some properties of manifolds with contact metric structure. Tôhoku Math. J. **15** (1963), 42–48.

[2] NOMIZU, K., On hypersurfaces satisfying a certain condition on the curvature tensor. Tôhoku Math. J. **20** (1968), 46–59.

[3] OKUMURA, M., Some remarks on space with a certain contact structure. Tôhoku Math. J. **14** (1962), 135–145.

[4] SIMONS, J., Minimal varieties in Riemannian manifolds. Ann. of Math. **88** (1968), 62–105.

[5] TACHIBANA, S., On harmonic tensors in compact Sasakian spaces. Tôhoku Math. J. **17** (1965), 271–284.

[6] TAKAHASHI, T., Sasakian manifolds with pseudo-Riemannian metric. Tôhoku Math. J. **21** (1969), 271–290.

[7] TAKAHASHI, T., Sasakian hypersurfaces in a space of constant curvature. To appear in Tôhoku Math. J.

[8] TANNO, S., Some transformations on manifolds with almost contact and contact metric structures, II. Tôhoku Math. J. **15** (1963), 322–331.

[9] TANNO, S., Locally symmetric K-contact Riemannian manifolds. Proc. Japan. Acad. **43** (1967), 581–583.

[10] TANNO, S., Hypersurfaces satisfying a certain condition on the Ricci tensor. Tôhoku Math. J. **21** (1969), 297–303.

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