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# GENERALIZATIONS OF THE CONNECTION OF TZITZÉICA

BY KENTARO YANO

Dobrescu [1] has recently studied what he calls the connection of Tzitzéica [4] on hypersurfaces in a Euclidean space.

The main purpose of the present paper is to define the connection of Tzitzéica on hypersurfaces in an affinely connected manifold along which a torse-forming [6] or a concurrent vector field [5] is given and to study the properties of the connection of Tzitzéica thus defined.

## § 1. The connection of Tzitzéica on a hypersurface in a centro-affine space.

Let $A^n$ be an $n$-dimensional centro-affine space ($n\geqq 3$), that is, an affine space in which a point O is specified. Then any point P in $A^n$ is represented by the so-called position vector $X=\overrightarrow{\mathrm{OP}}$. This means that with every point P of $A^n$, there is associated a vector $X$.

We now assume that there is given a hypersurface $V^{n-1}$ in $A^n$ and denote by

$$X=X(u^1, u^2, \cdots, u^{n-1})$$

the parametric representation of $V^{n-1}$, where $(u^a)$ $(a, b, c, \cdots=1, 2, 3, \cdots, n-1)$ are local parameters on $V^{n-1}$ such that vectors

$$X_b=\partial_b X$$

tangent to $V^{n-1}$ are linearly independent, $\partial_b$ denoting the differential operators $\partial_b=\partial/\partial u^b$.

We assume that the vector $X$ at P on $V^{n-1}$ is never tangent to $V^{n-1}$, that is, the vector $X$ is linearly independent of $X_b$.

We then have, for the vectors $\partial_c X_b$, the equations of the form

$$\partial_c X_b=\Gamma_c{}^a{}_b X_a+h_{cb}X, \tag{1.1}$$

where $\Gamma_c{}^a{}_b$, symmetric in $c$ and $b$, define an affine connection on $V^{n-1}$ called the connection of Tzitéica [1], [4] and $h_{cb}$, symmetric in $c$ and $b$, define a tensor field on $V^{n-1}$ called the second fundamental tensor.

The equations (1.1) are so-called equations of Gauss for the $V^{n-1}$, the pseudo-

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affine normal being the vector $X$ associated with the point P on $V^{n-1}$. The equations of Weingarten in this case are

$$\partial_c X = X_c. \tag{1 2}$$

Computing the integrability conditions of (1. 1) and (1. 2) regarded as a completely integrable system of partial differential equations with unknown vectors $X_b$ and $X$, we find

$$R_{dcb}{}^a + \delta_d^a h_{cb} - \delta_c^a h_{db} = 0, \tag{1. 3}$$

and

$$\nabla_d h_{cb} - \nabla_c h_{db} = 0, \tag{1. 4}$$

where

$$R_{dcb}{}^a = \partial_d \Gamma_c{}^a{}_b - \partial_c \Gamma_d{}^a{}_b + \Gamma_d{}^a{}_e \Gamma_c{}^e{}_b - \Gamma_c{}^a{}_e \Gamma_d{}^e{}_b \tag{1. 5}$$

are components of the curvature tensor of the connection of Tzitzéica and $\nabla_d$ denotes the covariant differentiation with respect to the connection of Tzitzéica.

From (1. 3) we have

$$R_{cb} + (n-2) h_{cb} = 0, \tag{1. 6}$$

where $R_{cb}$ are components of the Ricci tensor

$$R_{cb} = R_{acb}{}^a. \tag{1. 7}$$

Equation (1. 6) shows that the Ricci tensor of the connection of Tzitzéica is symmetric:

$$R_{cb} = R_{bc}, \tag{1. 8}$$

from which we see that the connection of Tzitzéica is volume-preserving.

From (1. 3), (1. 4) and (1. 6), we find, for $n \geqq 3$,

$$R_{dcb}{}^a - \frac{1}{n-2}(\delta_d^a R_{cb} - \delta_c^a R_{db}) = 0 \tag{1. 9}$$

and

$$\nabla_d R_{cb} - \nabla_c R_{db} = 0, \tag{1. 10}$$

which show that the connection of Tzitzéica is projectively flat.

Conversely suppose that there is given, in an $(n-1)$-dimensional differentiable manifold $V^{n-1}$ ($n \geqq 3$), a symmetric affine connection $\Gamma_c{}^a{}_b$ which is volume-preserving and projectively flat. We then have equations (1. 8), (1. 9) and (1. 10), which show that the system of partial differential equations

$$\partial_c X_b = \Gamma_c{}^a{}_b X_a + \frac{1}{n-2} R_{cb} X, \qquad (1.11)$$
$$\partial_c X = X_c$$

is completely integrable. Thus we have

PROPOSITION 1.1. *A symmetric volume-preserving projectively flat affine connection is always realized as a connection of Tzitzéica on a hypersurface in a centro-affine space.*

## § 2. A characterization of projectively flat manifold.

Let $M^n$ be an $n$-dimensional differentiable manifold ($n \geqq 3$) with a symmetric affine connection $\nabla$ with components $\Gamma_j{}^h{}_i(x)$, where $(x^h)$ are local coordinate system in $M^n$ and the indices $h, i, j, \cdots$ run over the range $\{1, 2, 3, \cdots, n\}$.

The curvature tensor field $R$ of $M^n$ is given by

$$R(Z, Y)X = \nabla_Z \nabla_Y X - \nabla_Y \nabla_Z X - \nabla_{[Z,Y]} X$$

for any vector fields $Z, Y, X$ in $M^n$.

We assume that the vector $R(Z, Y)X$ is always in the three-dimensional linear space spanned by the vectors $Z$, $Y$ and $X$, that is,

$$R(Z, Y)X = \gamma(Y, X)Z + \beta(Z, X)Y + \alpha(Z, Y)X,$$

where $\gamma$, $\beta$ and $\alpha$ are all tensors of type (0, 2).

Since $R(Z, Y)X$ is skew-symmetric in $Z$ and $Y$, we see that

$$\gamma(Y, X) + \beta(Y, X) = 0,$$
$$\alpha(Z, Y) + \alpha(Y, Z) = 0$$

and consequently we have

$$R(Z, Y)X = \gamma(Y, X)Z - \gamma(Z, X)Y + \alpha(Z, Y)X, \qquad \text{(P)}$$

$\alpha(Z, Y)$ being skew-symmetric.

If (P) is satisfied, then we say that the affine connection has the property (P).

Denoting the local components of $R$, $\gamma$ and $\alpha$ by $R_{kji}{}^h$, $-\gamma_{ji}$ and $\alpha_{ji}$ respectively, we find, from (P),

$$R_{kji}{}^h + \delta_k^h \gamma_{ji} - \delta_j^h \gamma_{ki} - \alpha_{kj} \delta_i^h = 0, \qquad (2.1)$$

from which, by contraction with respect to $k$ and $h$,

$$R_{ji} + (n-1)\gamma_{ji} - \alpha_{ij} = 0, \qquad (2.2)$$

and, by contraction with respect to $i$ and $h$,

$$-R_{kj}+R_{jk}+\gamma_{jk}-\gamma_{kj}-n\alpha_{kj}=0,$$

or

$$(2.3)\qquad -R_{ij}+R_{ji}+\gamma_{ji}-\gamma_{ij}-n\alpha_{ij}=0$$

by virtue of the first Bianchi identity

$$R_{kji}{}^{h}+R_{jik}{}^{h}+R_{ikj}{}^{h}=0.$$

Forming (2. 2)$\times n-$(2. 3), we find

$$(2.4)\qquad (n-1)R_{ji}+R_{ij}+(n^2-n-1)\gamma_{ji}+\gamma_{ij}=0,$$

from which

$$(2.5)\qquad (n-1)R_{ij}+R_{ji}+(n^2-n-1)\gamma_{ij}+\gamma_{ji}=0.$$

Forming (2. 4)$\times(n^2-n-1)-$(2. 5), we find

$$[(n-1)(n^2-n-1)-1]R_{ji}+[(n^2-n-1)-(n-1)]R_{ij}+[(n^2-n-1)^2-1]\gamma_{ji}=0,$$

or

$$(2.6)\qquad nR_{ji}+R_{ij}+(n^2-1)\gamma_{ji}=0,$$

from which

$$(2.7)\qquad \gamma_{ji}=-\frac{1}{n^2-1}(nR_{ji}+R_{ij})$$

and

$$(2.8)\qquad \gamma_{ji}-\gamma_{ij}=-\frac{1}{n+1}(R_{ji}-R_{ij}).$$

Substituting (2. 8) into (2. 3), we find

$$\alpha_{ji}=-\frac{1}{n+1}(R_{ji}-R_{ij}),$$

that is,

$$(2.9)\qquad \alpha_{ji}=\gamma_{ji}-\gamma_{ij}.$$

Thus (2. 1) gives

$$(2.10)\qquad R_{kji}{}^{h}+\delta_k^h\gamma_{ji}-\delta_j^h\gamma_{ki}-(\gamma_{kj}-\gamma_{jk})\delta_i^h=0,$$

where $\gamma_{ji}$ is given by (2. 7), and consequently the manifold $M^n$ $(n\geqq 3)$ is projectively flat. Thus we have

PROPOSITION 2. 1. *When a symmetric affine connection in an* $n(\geqq 3)$*-dimensional manifold has property* (P), *the manifold is projectively flat.* (See, Ogiue [2])

PROPOSITION 2. 2. *When the curvature tensor of a symmetric affine connection in an* $n(\geqq 3)$*-dimensional manifold satisfies an equation of the form* (2. 1), *the manifold is projectively flat.*

## § 3. Fundamental equations of a hypersurface in a manifold with symmetric affine connection.

We next consider a hypersurface $V^{n-1}$ in $M^n$ and let

$$x^h = x^h(u^a) \tag{3. 1}$$

be its parametric representation. The rank of

$$B_b{}^h = \partial_b x^h$$

is assumed to be $n-1$ everywhere along $V^{n-1}$.

We take a vector field $C^h$ defined along $V^{n-1}$ such that $C^h$ is linearly independent of $B_b{}^h$ and consequently the vectors $B_b{}^h$ and $C^h$ form a frame on $V^{n-1}$. We denote by $B^a{}_i$ and $C_i$ the components of covectors which form the dual coframe.

If we put

$$\Gamma_c{}^a{}_b = (\partial_c B_b{}^h + \Gamma_j{}^h{}_i B_c{}^j B_b{}^i) B^a{}_h, \tag{3. 2}$$

then $\Gamma_c{}^a{}_b$ define a symmetric affine connection induced on the hypersurface with respect to the pseudo-affine normal $C^h$.

Then the so-called van der Waerden-Bortolotti covariant derivative of $B_b{}^h$:

$$\nabla_c B_b{}^h = \partial_c B_b{}^h + \Gamma_j{}^h{}_i B_c{}^j B_b{}^i - \Gamma_c{}^a{}_b B_a{}^h \tag{3. 3}$$

is written as

$$\nabla_c B_b{}^h = h_{cb} C^h, \tag{3. 4}$$

where $h_{cb}$ is the second fundamental tensor of the hypersurface $V^{n-1}$ with respect to $C^h$. The equation (3. 4) is that of Gauss of the hypersurface $V^{n-1}$. The equation of Weingarten of the hypersurface $V^{n-1}$ is given by

$$\nabla_c C^h = -h_c{}^a B_a{}^h + l_c C^h, \tag{3. 5}$$

where $h_c{}^a$ are components of the second fundamental tensor and $l_c$ those of the third fundamental tensor of $V^{n-1}$.

From (3. 4) and (3. 5), we find

$$\begin{aligned} &R_{kji}{}^h B_d{}^k B_c{}^j B_b{}^i \\ &= [K_{dcb}{}^a - (h_d{}^a h_{cb} - h_c{}^a h_{db})] B_a{}^h + [\nabla_d h_{cb} - \nabla_c h_{db} + l_d h_{cb} - l_c h_{db}] C^h \end{aligned} \tag{3. 6}$$

and

$$R_{kji}{}^{h}B_{d}{}^{k}B_{c}{}^{j}C^{i} = -[\nabla_{d}\nabla_{c}{}^{a}-\nabla_{c}h_{d}{}^{a}-l_{d}h_{c}{}^{a}+l_{c}h_{d}{}^{a}]B_{a}{}^{h}+[\nabla_{d}l_{c}-\nabla_{c}l_{d}-h_{da}h_{c}{}^{a}+h_{ca}h_{d}{}^{a}]C^{h} \tag{3.7}$$

by virtue of the Ricci identities

$$\nabla_{d}\nabla_{c}B_{b}{}^{h}-\nabla_{c}\nabla_{d}B_{b}{}^{h}=R_{kji}{}^{h}B_{d}{}^{k}B_{c}{}^{j}B_{b}{}^{i}-K_{dcb}{}^{a}B_{a}{}^{h}$$

and

$$\nabla_{d}\nabla_{c}C^{h}-\nabla_{c}\nabla_{d}C^{h}=R_{kji}{}^{h}B_{d}{}^{k}B_{c}{}^{j}C^{i}.$$

We now assume that the hypersurface $V^{n-1}$ has the property that

$$R(Z', Y')X'=\gamma(Y', X')Z'-\gamma(Z', X')Y'+\alpha(Z', Y')X' \tag{P'}$$

is satisfied for any vector fields $Z', Y', X'$ tangent to the hypersurface. Denoting the local components of $\gamma$ and $\alpha$ on the hypersurface by $-\gamma_{cb}$ and $\alpha_{cb}$ respectively, we find

$$R_{kji}{}^{h}B_{d}{}^{k}B_{c}{}^{j}B_{b}{}^{i}+(\delta_{d}^{a}\gamma_{cb}-\delta_{c}^{a}\gamma_{db}-\alpha_{dc}\delta_{b}^{a})B_{a}{}^{h}=0. \tag{3.8}$$

Substituting (3. 8) into (3. 6), we find

$$[K_{dcb}{}^{a}+\delta_{d}^{a}\gamma_{cb}-\delta_{c}^{a}\gamma_{db}-\alpha_{dc}\delta_{b}^{a}-(h_{d}{}^{a}h_{cb}-h_{c}{}^{a}h_{db})]B_{a}{}^{h}+[\nabla_{d}h_{cb}-\nabla_{c}h_{db}+l_{d}h_{cb}-l_{c}h_{db}]C^{h}=0,$$

from which

$$K_{dcb}{}^{a}+\delta_{d}^{a}\gamma_{cb}-\delta_{c}^{a}\gamma_{db}-\alpha_{dc}\delta_{b}^{a}-(h_{d}{}^{a}h_{cb}-h_{c}{}^{a}h_{db})=0 \tag{3.9}$$

and

$$\nabla_{d}h_{cb}-\nabla_{c}h_{db}+l_{d}h_{cb}-l_{c}h_{db}=0. \tag{3.10}$$

## § 4. Generalizations of the connection of Tzitzéica.

Suppose that a vector field $C^{h}(x)$ is given in $M^{n}$ and satisfies

$$\nabla_{i}C^{h}=\alpha\delta_{i}^{h}+\beta_{i}C^{h}, \tag{4.1}$$

where $\alpha$ is a scalar field and $\beta_{i}$ a covector field in $M^{n}$.

We then say that the vector field $C^{h}$ is torse-forming because if we develop the vetor field $C^{h}$ along a curve in the manifold $M^{n}$, we obtain a field of vectors along the curve whose prolongations are tangent to another curve. (See, [5], [6]).

When a vector field $C^{h}(u)$ is given along the hypersurface $V^{n-1}$ and satisfies

$$\nabla_{c}C^{h}=\alpha B_{c}{}^{h}+\beta_{c}C^{h}, \tag{4.2}$$

where $\alpha$ is a scalar field and $\beta_c$ a covector field of the hypersurface, we say that the vector field $C^h(u)$ is torse-forming along the hypersurface.

In this Section, we assume that there is given a torse-forming vector field $C^h$ along $V^{n-1}$ which is not tangent to the hypersurface and take this vector field as the pseudo-affine normal to the hypersurface $V^{n-1}$.

We induce an affine connection on the hypersurface with respect to this torse-forming vector field $C^h$ and call the connection of this kind the connection of Tzitzéica on the hypersurface.

When the vector field $C^h$ satisfies

$$\nabla_c C^h = \alpha B_c{}^h \tag{4.3}$$

with a non-zero constant $\alpha$, we say that the vector field $C^h$ is concurrent along $V^{n-1}$ because when we develop the vector field along a curve on the hypersurface, we obtain a field of vectors along the curve whose prolongations pass through a fixed points. (See [3], [6]). In a centro-affine space with a fixed point O, the vector field $X=\overrightarrow{\mathrm{OP}}$ attached to a point P is concurrent on any hypersurface in the sense above.

When the pseudo-affine normal $C^h$ is torse-forming, comparing (3. 5) and (4. 2), we find

$$h_c{}^a = -\alpha\delta_c^a \tag{4.4}$$

and

$$l_c = \beta_c. \tag{4.5}$$

Thus if moreover the hypersurface has the property (P′), we have, from (3. 9) and (3. 10),

$$K_{dcb}{}^a + \delta_d^a(\gamma_{cb} + \alpha h_{cb}) - \delta_c^a(\gamma_{db} + \alpha h_{db}) - \alpha_{dc}\delta_b^a = 0 \tag{4.6}$$

and

$$\nabla_d h_{cb} - \nabla_c h_{db} + \beta_d h_{cb} - \beta_c h_{db} = 0. \tag{4.7}$$

Thus from Proposition 2. 2 and (4. 6), we see that, if $n-1\geqq 3$, then the connection of Tzitzéica is projectively flat. Thus we have

THEOREM 4. 1. *The connection of Tzitzéica induced on a hypersurface $V^{n-1}$ with the property* (P′) *in $M^n$ ($n\geqq 4$) with respect to a torse-forming pseudo-affine normal is projectively flat.*

As a corollary to Theorem 4. 1, we have

THEOREM 4. 2. *The connection of Tzitzéica induced on a hypersurface $V^{n-1}$ with the property* (P′) *in $M^n$ ($n\geqq 4$) with respect to a concurrent pseudo-affine normal is projectively flat.*

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Department of Mathematics,
Tokyo Institute of Technology.