KŌDAI MATH. SEM. REP.
20 (1968), 29—53

NORMAL CIRCLE BUNDLES OF COMPLEX HYPERSURFACES

BY KENTARO YANO AND SHIGERU ISHIHARA

Dedicated to Professor Hitoshi Hombu on his sixtieth birthday

Introduction. The main purpose of the present paper is to study the so-called normal circle bundle $\mathfrak{N}(V)$ of a complex hypersurface V, that is, a $2n$-dimensional submanifold in a $(2n+2)$-dimensional Kählerian manifold M, whose tangent space is invariant by the complex structure of M. For a complex hypersurface V of even-dimensional Euclidean space with natural Kählerian structure, we make use of the natural mappings p: $\mathfrak{N}(V)\to V$, ψ: $\mathfrak{N}(V)\to S^{2n+1}$, a $(2n+1)$-dimensional unit sphere and π: $S^{2n+1}\to CP^n$, a complex n-dimensional projective space, and introduce a mapping φ: $V\to CP^n$, which may be considered as the Gauss map of V. The study of the Gauss map of V in this sense is one of purposes of the present paper.

We first state in § 1 some of important formulas for complex hypersurfaces in a general Kählerian manifold and then specialize in § 2 these formulas for complex hypersurfaces in a Kählerian manifold of constant holomorphic sectional curvature. These formulas permit us to prove some of recent results of Ako [1], Smyth [6] and Takahashi [8].[1)]

§ 3 is devoted to the study of normal circle bundles of complex hypersurfaces in a Kählerian manifold and § 4 to the study of Gauss maps of complex hypersurfaces in even-dimensional Euclidean spaces.

We study in the last § 5 Einstein complex hypersurfaces in an even-dimensional Euclidean space with natural complex structure.

§ 1. Complex hypersurfaces in a Kählerian manifold.

Let M be a Kählerian manifold of $2n+2$ dimensions with Kählerian structure (G, F), where G is a Riemannian metric tensor and F a tensor field of type (1, 1) such that[2)]

$$F^2=-I, \quad \text{i.e.,} \quad F_B{}^A F_C{}^B=-\delta_C^A, \tag{1.1}$$

$$F_C{}^E F_B{}^D G_{ED}=G_{CB}, \tag{1.2}$$

$$\nabla F=0, \qquad \text{i.e.,} \qquad \nabla_C F_B{}^A=0, \tag{1.3}$$

Received July 6, 1967.

1) The numbers between brackets refer to References at the end of the paper.

2) The indices A, B, C, D, E run over the range $\{1, 2, \cdots, 2n+2\}$ and the so-called Einstein's convention is used with respect to this system of indices.

I being the unit tensor of type (1, 1), $F_B{}^A$ and G_{CB} the components of F and G respectively and ∇ the operator of covariant differentiation with respect to G (Cf. Yano [9]). The tensor field F is called the *complex structure* of M. If we put

(1.4) $$F_{CB}=F_C{}^D G_{DB},$$

we easily see that

(1.5) $$F_{CB}+F_{BC}=0.$$

Let there be given in M a differentiable submanifold V of class C^∞ and of codimension 2. Suppose that V is expressed in each neighborhood $\bar{U}$ of M by equations[3)]

$$x^A=x^A(u^a),$$

where (x^A) are local coordinates of M in $\bar{U}$ and (u^a) local coordinates of V in $U=\bar{U}\cap V$. We have in U $2n$ local vector fields B_b having components

$$B_b{}^A=\partial_b x^A$$

and spanning the tangent space of V at each point of U, where ∂_b denotes the operator $\partial/\partial u^b$. The submanifold V is a *complex hypersurface* when and only when the complex structure F leaves invariant the tangent space of V at each point of V. In the sequel, we shall restrict ourselves only to complex hypersurfaces. For a complex hypersurface V, FB_b is a linear combination of B_a in U, that is,

(1.6) $$FB_b=f_b{}^a B_a, \quad \text{i.e.,} \quad F_B{}^A B_b{}^B=f_b{}^a B_a{}^A,$$

where the functions $f_b{}^a$ are components of a tensor field f of type (1, 1) defined globally in V. Applying the operator F to both sides of (1.6) and taking account of (1, 1), we find

(1.7) $$f^2=-I, \quad \text{i.e.,} \quad f_b{}^a f_c{}^b=-\delta_c^a.$$

The *Riemannian metric g induced in V* has components of the form

(1.8) $$g_{cb}=G_{CB}B_c{}^C B_b{}^B$$

in each neighborhood U of V. Thus we obtain

(1.9) $$f_c{}^e f_b{}^d g_{ed}=g_{cb}$$

as a direct consequence of (1.2) and (1.8). On putting

(1.10) $$f_{cb}=f_c{}^e g_{eb},$$

we have

3) The indices a, b, c, d, e, f run over the range $\{1, 2, \cdots, 2n\}$ and the so-called Einstein's convention is also used with respect to this system of indices.

$$f_{cb}+f_{bc}=0 \tag{1.11}$$

by virtue of (1.5).

Since V is a complex hypersurface, the normal plane of V is left invariant by the complex structure F of M at each point of V. Thus there exist, in each neighborhood U of V, two local unit vector fields C and D normal to V such that

$$\begin{aligned} FC=D, \qquad & FD=-C, \quad \text{i.e.,} \\ F_B{}^A C^B=D^A, \qquad & F_B{}^A D^B=-C^A, \end{aligned} \tag{1.12}$$

where C^A and D^A denote the components of C and D respectively, C and D being necessarily perpendicular to each other.

We have, as is well known, the following equations:

$$\begin{aligned} \nabla_c B_b{}^A &= \quad h_{cb}C^A+k_{cb}D^A, \\ \nabla_c C^A &= -h_c{}^a B_a{}^A+l_c D^A, \\ \nabla_c D^A &= -k_c{}^a B_a{}^A-l_c C^A, \end{aligned} \tag{1.13}$$

which are respectively *the equations of Gauss* and *those of Weingarten* for the complex hypersurface V. The left hand sides of these equations are defined by

$$\nabla_c B_b{}^A=\partial_c B_b{}^A+\{_C{}^A{}_B\}B_c{}^C B_b{}^B-\{_c{}^a{}_b\}B_a{}^A,$$

$$\nabla_c C^A=\partial_c C^A+\{_C{}^A{}_B\}B_c{}^C C^B, \qquad \nabla_c D^A=\partial_c D^A+\{_C{}^A{}_B\}B_c{}^C D^B$$

respectively, where $\{_C{}^A{}_B\}$ and $\{_c{}^a{}_b\}$ are Christoffel symbols determined respectively by G_{CB} and g_{cb}. The functions $h_b{}^a$ and $k_b{}^a$ appearing in (1.13) are the components of the so-called *second fundamental tensors* h and k respectively, where h and k are local tensor fields of type (1, 1) defined in each neighborhood U of V with respect to the choice of the unit normal vector fields C and D. The functions h_{cb} and k_{cb} appearing in (1.13) are respectively defined by

$$h_{cb}=h_c{}^a g_{ab}, \qquad k_{cb}=k_c{}^a g_{ab}. \tag{1.14}$$

We easily see that

$$h_{cb}=h_{bc}, \qquad k_{cb}=k_{bc} \tag{1.15}$$

hold. The functions l_c appearing in (1.13) are the components of the so-called *third fundamental tensor* l, which is a local covector field defined in each neighborhood U of V with respect to the choice of the unit normal vector fields C and D.

Differentiating (1.6) covariantly along V and taking account of (1.12) and (1.13), we obtain

$$\nabla f=0, \quad \text{i.e.,} \quad \nabla_c f_b{}^a=0 \tag{1.16}$$

and

$$h_{cb}=k_{ce}f_b{}^e, \qquad k_{cb}=-h_{ce}f_b{}^e, \tag{1.17}$$

where the operator ∇ appearing in (1.16) denotes the covariant differentiation with respect to the induced metric g in V. The equations (1.17) are equivalent to the conditions

$$\begin{aligned} h&=-fk, \quad \text{i.e.,} \quad h_c{}^a=-f_e{}^a k_c{}^e, \\ k&=fh, \quad \text{i.e.,} \quad k_c{}^a=f_e{}^a h_c{}^e, \end{aligned} \tag{1.18}$$

which imply together with (1.11) and (1.15)

$$h_e{}^e=0, \qquad k_e{}^e=0. \tag{1.19}$$

The equations (1.18) imply

$$fh+hf=0, \qquad fk+kf=0 \tag{1.20}$$

by virtue of (1.11) and (1.15). We have moreover the conditions

$$h^2=k^2, \qquad hk+kh=0 \tag{1.21}$$

and

$$f_c{}^e f_b{}^d \gamma_{ed}=\gamma_{cb}, \qquad \gamma_{cb}=\gamma_{bc} \tag{1.22}$$

by virtue of (1.18) and (1.20), where we have put

$$\gamma_{cb}=h_c{}^e h_{be}. \tag{1.23}$$

If we take account of (1.7), (1.9), (1.16) and (1.19), we see that *any complex hypersurface in a Kählerian manifold is a minimal surface and is itself a Kählerian manifold with the induced Kählerian structure* (g, f) (Cf. Schouten and Yano [7]).

The set of all vectors normal to the complex hypersurface V is called the *normal bundle*, which is orientable. Thus, taking two intersecting neighborhoods U and $\bar{U}$ of V, we can choose pairs (C, D) and $(\bar{C}, \bar{D})$ of normal vector fields defined respectively in U and $\bar{U}$ such that they are related to each other by

$$\bar{C}=C\cos\theta-D\sin\theta, \qquad \bar{D}=C\sin\theta+D\cos\theta \tag{1.24}$$

in $U\cap\bar{U}$, θ being a certain function in $U\cap\bar{U}$, where (C, D) and $(\bar{C}, \bar{D})$ satisfy respectively the condition (1.12). If we denote by $\bar{h}$, $\bar{k}$ and $\bar{l}$ respectively the second and the third fundamental tensors in $\bar{U}$ with respect to $\bar{C}$ and $\bar{D}$, then we easily obtain in $U\cap\bar{U}$

$$\begin{aligned} \bar{h}=h\cos\theta-k\sin\theta, &\qquad \bar{k}=h\sin\theta+k\cos\theta, \\ \bar{l}&=l-d\theta \end{aligned} \tag{1.25}$$

by virtue of (1.21), (1.24) and the definitions of h, k and l. Therefore we get

$$\bar{h}^2=h^2, \qquad \bar{k}^2=k^2, \qquad \bar{h}\bar{k}=hk, \qquad \bar{k}\bar{h}=kh, \qquad d\bar{l}=dl$$

in $U\cap\bar{U}$. Consequently, taking account of (1.21) and (1.23), we have

PROPOSITION 1.1. *For any complex hypersurface V in a Kählerian manifold, h^2, k^2, hk, kh and $\Omega=dl$ determine global tensor fields of corresponding type in V, respectively. They satisfy the conditions*

$$h^2=k^2, \qquad hk+kh=0.$$

The local tensor field γ_{cb} defined by (1.23) *determines a global tensor field γ of type* (0, 2) *in V.*

For a complex hypersurface V in a Kählerian manifold, we have, as is well known, the structure equations

$$'K_{DCBA}B_d{}^D B_c{}^C B_b{}^B B_a{}^A=K_{dcba}-(h_{da}h_{cb}-h_{ca}h_{db})-(k_{da}k_{cb}-k_{ca}k_{db}), \tag{1.26}$$

$$'K_{DCBA}B_d{}^D B_c{}^C B_b{}^B C^A=\nabla_d h_{cb}-\nabla_c h_{db}-l_d k_{cb}+l_c k_{db}, \tag{1.27}$$

$$'K_{DCBA}B_d{}^D B_c{}^C B_b{}^B D^A=\nabla_d k_{cb}-\nabla_c k_{db}+l_d h_{cb}-l_c h_{db}, \tag{1.28}$$

$$'K_{DCBA}B_d{}^D B_c{}^C C^B D^A=\nabla_d l_c-\nabla_c l_d+h_d{}^e k_{ec}-h_c{}^e k_{ed}, \tag{1.29}$$

where $'K_{DCBA}$ and K_{dcba} are components of the curvature tensors of the enveloping manifold M and the complex hypersurface V respectively.

If we transvect the first Bianchi identity

$$'K_{DCBA}+'K_{CBDA}+'K_{BDCA}=0$$

with F^{CB}, then we find

$$'K_{DCBA}F^{CB}=-\frac{1}{2}\,'K_{CBDA}F^{CB}, \tag{1.30}$$

F^{CB} being defined by

$$F^{CB}=G^{CD}F_D{}^B,$$

where $(G^{CD})=(G_{CB})^{-1}$.

If we take account of (1.3) and the Ricci formula, we obtain

$$0=\nabla_D\nabla_C F_B{}^A-\nabla_C\nabla_D F_B{}^A='K_{DCE}{}^A F_B{}^E-'K_{DCB}{}^E F_E{}^A,$$

from which, transvecting with G^{CB},

$$'K_D{}^E F_E{}^A='K_{DCB}{}^A F^{CB},$$

or

$$'K_{BE}F_A{}^E = \frac{1}{2}\,'K_{DCBA}F^{CB} \tag{1.31}$$

by virtue of (1.30), the Ricci tensors $'K_{BE}$ and $'K_B{}^E$ being defined by $'K_{BE} = 'K_{CBE}{}^C$ and $'K_B{}^E = 'K_{BD}G^{DE}$ respectively.

On the other hand, we have the formula

$$F^{DC} = B_d{}^D B_c{}^C f^{dc} - C^D D^C + C^C D^D \tag{1.32}$$

along the complex hypersurface V, where $f^{dc} = g^{db}g^{ca}f_{ba}$.

Now transvecting (1.31) with $B_b{}^B B_a{}^A$ and taking account of (1.6) and (1.32), we find

$$'K_{BA}B_b{}^B B_e{}^A f_a{}^e = \frac{1}{2}\,'K_{DCBA}B_d{}^D B_c{}^C B_b{}^B B_a{}^A f^{dc} - 'K_{DCBA}D^D C^C B_b{}^B B_a{}^A. \tag{1.33}$$

Substituting (1.26) and (1.29) into (1.33), we obtain

$$'K_{be}f_a{}^e = K_{be}f_a{}^e + (\nabla_b l_a - \nabla_a l_b), \tag{1.34}$$

or, equivalently

$$\nabla_b l_a - \nabla_a l_b = ('K_{be} - K_{be})f_a{}^e \tag{1.35}$$

by virtue of

$$K_{be}f_a{}^e = \frac{1}{2}K_{dcba}f^{dc}, \tag{1.36}$$

where we have put

$$'K_{cb} = 'K_{CB}B_c{}^C B_b{}^B$$

(Smyth [6]).

§ 2. Complex hypersurfaces in a Kählerian manifold of constant holomorphic sectional curvature.

We assume in this section that the enveloping manifold M is a Kählerian manifold of constant holomorphic sectional curvature c. Then the components $'K_{DCBA}$ of the curvature tensor of M have the form

$$'K_{DCBA} = \frac{c}{4}[(G_{DA}G_{CB} - G_{CA}G_{DB}) + (F_{DA}F_{CB} - F_{CA}F_{DB}) - 2F_{DC}F_{BA}]$$

(Cf. Yano [9]). Substituting this expression of $'K_{DCBA}$ into (1.26)~(1.29), we respectively obtain

$$\begin{aligned} K_{dcba} = \frac{c}{4}[(g_{da}g_{cb} - g_{ca}g_{db}) + (f_{da}f_{cb} - f_{ca}f_{db}) - 2f_{dc}f_{ba}] \\ + (h_{da}h_{cb} - h_{ca}h_{db}) + (k_{da}k_{cb} - k_{ca}k_{db}), \end{aligned} \tag{2.1}$$

$$(2.2)\qquad \nabla_d h_{cb}-\nabla_c h_{db}-l_d k_{cb}+l_c k_{db}=0,$$

$$(2.3)\qquad \nabla_d k_{cb}-\nabla_c k_{db}+l_d h_{cb}-l_c h_{db}=0,$$

$$(2.4)\qquad \nabla_d l_c-\nabla_c l_d+h_d{}^e k_{ce}-h_c{}^e k_{de}+\frac{c}{2}f_{dc}=0.$$

Transvecting (2.1) with g^{dc} and taking account of (1.23), we obtain

$$(2.5)\qquad \begin{aligned} K_{cb}&=\frac{(n+1)c}{2}g_{cb}-2h_c{}^e h_{be}\\ &=\frac{(n+1)c}{2}g_{cb}-2\gamma_{cb}\end{aligned}$$

by virtue of (1.19) and (1.21), where K_{cb} are components of the Ricci tensor of V. Thus we have

PROPOSITION 2.1. *For a complex hypersurface V in a Kählerian manifold of non-positive constant holomorphic sectional curvatute c, the Ricci form of V satisfies the inequality $K_{cb}X^cX^b\leqq 0$ for any values of variables X^a. In this case, the equality $K_{cb}=0$ holds identically if and only if $c=0$ and V is totally geodesic (i.e. $h_b{}^a=0$, $k_b{}^a=0$).*

Taking account of (1.17), we find from (2.4)

$$(2.6)\qquad \frac{1}{2}(\nabla_d l_c-\nabla_c l_d)=f_d{}^a\left(h_a{}^e h_{ce}-\frac{c}{4}g_{ac}\right),$$

which reduces to

$$(2.7)\qquad \frac{1}{2}(\nabla_d l_c-\nabla_c l_d)=\frac{1}{2}f_d{}^e\left(\frac{nc}{2}g_{ec}-K_{ec}\right)$$

by virtue of (2.5).

If we transvect (2.2) and (2.3) with $f_e{}^c$, we respectively find

$$(2.8)\qquad \begin{aligned}&(\nabla_d k_{eb}+l_d h_{eb})-f_e{}^c(\nabla_c h_{db}-l_c k_{db})=0,\\ &(\nabla_d h_{eb}-l_d k_{eb})+f_e{}^c(\nabla_c k_{db}+l_c h_{db})=0\end{aligned}$$

by virtue of (1.17). These two equations imply

$$(\nabla_d h_{cb}-l_d k_{cb})+f_d{}^f f_c{}^e(\nabla_f h_{eb}-l_f k_{eb})=0,$$

which is equivalent to

$$(2.9)\qquad \nabla_d h_{cb}=l_d k_{cb}+f_d{}^f f_c{}^e(\nabla_f h_{eb}-l_f k_{eb}).$$

Taking account of (2.5), we see that the condition $\nabla_d K_{cb}=0$ is equivalent to $\nabla_d(h_c{}^e h_{be})=0$, or, to

$$(\nabla_d h_{cb}) h_a{}^b + h_c{}^b (\nabla_d h_{ba}) = 0. \tag{2.10}$$

If we assume that $\nabla_d K_{cb}=0$ holds, substituting (2.9) into (2.10), we find

$$f_c{}^e(\nabla_d h_{eb} - l_d k_{eb}) h_a{}^b + f_b{}^e h_c{}^b (\nabla_d h_{ea} - l_d k_{ea}) = 0,$$

from which, transvecting with $-f_f{}^c$,

$$(\nabla_d h_{fb} - l_d k_{fb}) h_a{}^b - h_f{}^e (\nabla_d h_{ea} - l_d k_{ea}) = 0$$

by virtue of (1.18) and (1.20). This equation reduces to

$$(\nabla_d h_{cb} - l_d k_{cb}) h_a{}^b = 0 \tag{2.11}$$

because of (1.21) and (2.9). If we assume conversely that the condition (2.11) is satisfied, then we get by virtue of (1.21) the condition (2.10), which is equivalent to the condition $\nabla_d K_{cb}=0$ by virtue of (2.5).

On the other hand, taking account of (2.8), we see that the two conditions

$$(\nabla_d h_{cb} - l_d k_{cb}) h_a{}^b = 0 \qquad \text{and} \qquad (\nabla_d k_{cb} + l_d h_{cb}) k_a{}^b = 0$$

are equivalent to each other. Thus we have

LEMMA 2. 1. *For a complex hypersurface in a Kählerian manifold of constant holomorphic sectional curveture, the following three conditions* (a), (b) *and* (c) *are equivalent to each other*:

(a) $(\nabla_d h_{cb} - l_d k_{cb}) h_a{}^b = 0$, (b) $(\nabla_d k_{cb} + l_d h_{cb}) k_a{}^b = 0$, (c) $\nabla_d K_{cb} = 0$.

We now assume that the condition $\nabla_d K_{cb}=0$ is satisfied. Then, taking account of (2.5), we find

$$\nabla_d (h_c{}^e h_{be}) = 0. \tag{2.12}$$

When the complex hypersurface V is irreducible as a Riemannian manifold, (2.12) implies

$$h_c{}^e h_{be} = A g_{cb},$$

A being a constant. Thus, V is an Einstein manifold, if V is irreducible and $\nabla_d K_{cb}=0$. When V is reducible and not locally flat, taking an arbitrary coordinate neighborhood U of V, we see that there exists an irreducible factor U_1 of U in the so-called de Rham decomposition of U. Thus U is a Pythagorean product $U_1 \times U_2$, where U_1 and U_2 are two local Kählerian manifold. Let $(u^1, \cdots, u^{2r})$ and $(u^{2r+1}, \cdots, u^{2n})$ be coordinates defined in U_1 and in U_2 respectively. Then, we have at any point of U[4)]

$$g_{\alpha\lambda} = 0, \qquad f_{\alpha\lambda} = 0 \tag{2.13}$$

4) $1 \leqq \alpha, \beta, \gamma \leqq 2r$ and $2r+1 \leqq \lambda, \mu, \nu \leqq 2n$.

as consequences of (1.16) and

(2.14) $$h_\alpha{}^e h_{\lambda e}=0$$

as a consequence of (2.12). Therefore, taking account of (1.15), (1.17) and (1.21), we have

(2.15) $$h_{\alpha\lambda}=0, \qquad k_{\alpha\lambda}=0$$

by virtue of (2.14). If we put $a=\alpha$, $b=\beta$, $c=\lambda$, $d=\mu$ in (2.1), we get

(2.16) $$K_{\mu\lambda\beta\alpha}=-2f_{\mu\lambda}f_{\beta\alpha}$$

by virtue of (2.13). On the other hand, we have $K_{\mu\lambda\beta\alpha}=0$ because U is a Pythagorean product $U_1\times U_2$. This contradicts (2.16). Consequently, V is necessarily irreducible and hence an Einstein manifold when V is not locally flat. When V is locally flat, V is obviously an Einstein manifold. Thus we have

LEMMA 2.2. *A complex hypersurface of a Kählerian manifold of constant holomorphic sectional curvature is an Einstein manifold if and only if the condition* $\nabla_d K_{cb}=0$ *is satisfied.*

We assume that the complex hypersurface V is an Einstein manifold. Then we have $\nabla_d K_{cb}=0$, which implies together with Lemma 2.1 that

(2.17) $$\nabla_d h_{cb}=l_d k_{cb}, \qquad \nabla_d k_{cb}=-l_d h_{cb}.$$

If we substitute (2.17) into the equation obtained by differentiating covariantly (2.1), then we find

$$\nabla_e K_{dcba}=0.$$

Thus we have

LEMMA 2.3. *A complex hypersurface V in a Kählerian manifold of constant holomorphic sectional curvature is locally symmetric, i.e.,* $\nabla_e K_{dcba}=0$, *if V is an Einstein manifold* (Ako [1], Smyth [6]).

Combining Lemmas 2.1, 2.2 and 2.3, we have

THEOREM 2.1. *For a complex hypersurface V of a Kählerian manifold of constant holomorphic sectional curvature, the following three conditions* (a), (b) *and* (c) *are equivalent to each other:*

(a) *V is an Einstein manifold.*

(b) *The Ricci tensor of V is parallel, i.e.,* $\nabla_d K_{cb}=0$.

(c) *V is locally symmetric, i.e.,* $\nabla_e K_{dcba}=0$.

(Takahashi [8]).

Taking account of (1.17), we have from (2.4)

$$\frac{1}{2}(\nabla_c l_b - \nabla_b l_c) = f_c{}^e \left(h_e{}^f h_{bf} - \frac{c}{4} g_{eb} \right), \tag{2.17}$$

which reduces to

$$\frac{1}{2}(\nabla_c l_b - \nabla_b l_c) = \frac{1}{2} f_c{}^e \left(\frac{nc}{2} g_{eb} - K_{eb} \right) \tag{2.18}$$

by virtue of (2.5), where dim $V=2n$.

We now assume that the complex hypersurface V is an Einstein manifold with scalar curvature K, i.e., that

$$K_{cb} = \frac{K}{2n} g_{cb}.$$

Substituting this into (2.5), we get

$$h_c{}^e h_{be} = A g_{cb}, \qquad A = \frac{n(n+1)c - K}{4n} \geqq 0. \tag{2.19}$$

Thus, substituting (2.19) into (2.17), we obtain

$$\frac{1}{2}(\nabla_c l_b - \nabla_b l_c) = B f_{cb}, \qquad B = \frac{n^2 c - K}{4n}. \tag{2.20}$$

We have from (2.19)

PROPOSITION 2.2. *Let V be a complex hypersurface in a Kählerian manifold of constant holomorphic sectional curvature c. If V is an Einstein manifold, then the scalar curvature K of V satisfies the inequality $K \leqq n(n+1)c$, where* dim $V=2n$. *In this case, the equality $K=n(n+1)c$ holds when and only when V is totally geodesic.*

§ 3. Normal circle bundles.

We shall first recall the definition of almost contact structure for the later use. Let, in a differentiable manifold of odd dimension $2n+1$, there be given a tensor field $\bar{f}$, a vector field $\bar{\xi}$ and a covector field $\bar{\eta}$ such that[5)]

$$\bar{f}_j{}^h \bar{f}_i{}^j = -\delta_i^h + \bar{\xi}^h \bar{\eta}_i, \qquad \bar{f}_i{}^h \bar{\xi}^i = 0,$$
$$\bar{f}_i{}^h \bar{\eta}_h = 0, \qquad \bar{\eta}_i \bar{\xi}^i = 1,$$

where $\bar{f}_i{}^h$, $\bar{\xi}^h$ and $\bar{\eta}_i$ are respectively the components of $\bar{f}$, $\bar{\xi}$ and $\bar{\eta}$. The set $(\bar{f}, \bar{\xi}, \bar{\eta})$ is called an *almost contact structure*. When the tensor field $\bar{S}$ of type (1, 2) having components of the form

$$\bar{S}_{ji}{}^h = \bar{f}_j{}^k \partial_k \bar{f}_i{}^h - \bar{f}_i{}^k \partial_k \bar{f}_j{}^h - (\partial_j \bar{f}_i{}^k - \partial_i \bar{f}_j{}^k) \bar{f}_k{}^h + (\partial_j \bar{\eta}_i - \partial_i \bar{\eta}_j) \bar{\xi}^h,$$

5) The indices h, i, j, k run over the range $\{1, 2, \cdots, 2n+1\}$ and the so-called Einstein's convention is also used with respect to this system of indices.

constructed from $\bar{f}$, $\bar{\xi}$ and $\bar{\eta}$, vanishes identically, the almost contact structure $(\bar{f}, \bar{\xi}, \bar{\eta})$ is said to be *normal* (Cf. Sasaki [3], Sasaki and Hatakeyama [4]). Let there be given moreover a Riemannian metric $\bar{g}$ with components $\bar{g}_{ji}$ such that

$$\bar{f}_j{}^k \bar{f}_i{}^h \bar{g}_{kh} = \bar{g}_{ji} - \bar{\eta}_j \bar{\eta}_i, \qquad \bar{\eta}_i = \bar{g}_{ih} \bar{\xi}^h.$$

Then the set $(\bar{f}, \bar{\xi}, \bar{\eta}, \bar{g})$ is called an *almost contact metric structure.* If we put

$$\bar{f}_{ji} = \bar{f}_j{}^h \bar{g}_{hi},$$

we have

$$\bar{f}_{ji} + \bar{f}_{ij} = 0.$$

When the condition

$$\bar{f}_{ji} = \frac{1}{2}(\partial_j \bar{\eta}_i - \partial_i \bar{\eta}_j) \tag{3.1}$$

is satisfied, the almost contact metric structure $(\bar{f}, \bar{\xi}, \bar{\eta}, \bar{g})$ is called a *contact structure* (Cf. Hatakeyama [2], Sasaki [3], Sasaki and Hatakeyama [4]).

In the next step, we shall recall the definition of fibred spaces in the sense of [10] and [11] and some of their properties. Let V and $\tilde{V}$ be differentiable manifolds of dimensions m and $m+1$ respectively, and, suppose that there exists a differentiable mapping p: $\tilde{V} \to V$, which is onto and of the maximum rank everywhere. We assume moreover that there are given in $\tilde{V}$ a vector field $\tilde{\xi}$ and a covector field $\tilde{\eta}$ such that $\tilde{\eta}(\tilde{\xi})=1$ and $\tilde{\xi}$ is tangent to the fibre everywhere in $\tilde{V}$, where, for each point P of V, the inverse image $F_P = p^{-1}(P)$ of P is called the fibre over P and assumed to be connected. Then we call the set $(\tilde{V}, V, p; \tilde{\xi}, \tilde{\eta})$ a fibred space (Cf. [10]). If there is given moreover a Riemannian metric $\tilde{g}$ in $\tilde{V}$ such that $\mathcal{L}_{\tilde{\xi}} \tilde{g} = 0$, $\mathcal{L}_{\tilde{\xi}}$ being the operator of Lie derivation with respect to $\tilde{\xi}$, $\tilde{\xi}$ is a unit vector field with respect to $\tilde{g}$ and $\tilde{\eta}(\tilde{X}) = \tilde{g}(\tilde{\xi}, \tilde{X})$ for any vector field $\tilde{X}$ in $\tilde{V}$, then we call the set $(\tilde{V}, V, p; \tilde{\xi}, \tilde{\eta}, \tilde{g})$ a *fibred space with invariant metric* $\tilde{g}$ (Cf. [11]). We can easily prove the following Lemma 3.1 by virtue of the discussions developed in [10] and [11]:

LEMMA 3.1. *We suppose that, for a fibred space* $(\tilde{V}, V, p; \tilde{\xi}, \tilde{\eta}, \tilde{g})$ *with invariant metric* $\tilde{g}$, *the base space* V *admits a Kählerian structure* (g, f), *where* g *is the projection of* $\tilde{g}$ *in the sence of* [10] *and* [11]. *Then the set* $(\tilde{f}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ *is an almost contact metric structure in* $\tilde{V}$, *where* $\tilde{f}$ *is the horizontal lift of* f *in the sense of* [10] *and* [11]. *If* $\tilde{f}$ *and* $\tilde{\eta}$ *satisfy the condition corresponding to* (3.1), *then the almost contact metric structure* $(\tilde{f}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ *is a contact structure which is normal.*

Let V be a complex hypersurface in a Kählerian manifold M of $2n+2$ dimensions. The set of all unit normal vectors of V is called the *normal circle bundle* of V and is denoted by $\mathfrak{N}(V)$. The $\mathfrak{N}(V)$ has a circle bundle structure p: $\mathfrak{N}(V) \to V$, i.e., p: $\mathfrak{N}(V) \to V$ is a principal fibre bundle whose structure group is a compact Lie group S^1 of 1 dimension. For any coordinate neighborhood U of V endowed with local coordinates (u^a), the open set $p^{-1}(U)$ is a product $U\times S^1$, that is, any element N of $p^{-1}(U)$ is expressed as $N=C\cos\theta+D\sin\theta$, C and D being normal unit vector fields in U satisfying (1.12). Thus N has components of the form

$$N^A=C^A\cos\theta+D^A\sin\theta,$$

where C^A and D^A are the components of C and D respectively, C^A and D^A being functions depending on u^a. Therefore (u^a,θ) are regarded as local coordinates in each coordinate neighborhood $\overline{W}(\subset p^{-1}(V))$ of $\mathfrak{N}(V)$. If we put $\bar{q}^a=u^a$, $\bar{q}^{2n+1}=\theta$, then we have local coordinates $(\bar{q}^h)$ in $\overline{W}$. Taking another coordinate neighborhood $\overline{W}'$ of $\mathfrak{N}(V)$, we have in $\overline{W}\cap\overline{W}'$ the transformation of coordinates

$$'u^a={}'u^a(u^b),\qquad '\theta=\theta+\alpha(u^a),$$

$\alpha(u^a)$ being a certain function, where $('\bar{q}^h)=('u^a,{}'\theta)$ are coordinates defined in $\overline{W}'$ just as above. The Jacobian matrix of the above transformation of coordinates is given by

$$\left(\frac{\partial'\bar{q}^h}{\partial\bar{q}^i}\right)=\begin{pmatrix}\dfrac{\partial'u^a}{\partial u^b} & 0\\[2ex] \dfrac{\partial'\theta}{\partial u^b} & 1\end{pmatrix}. \tag{3.2}$$

If we introduce in $\overline{W}$ a local vector field $\bar{\xi}$ having components

$$(\bar{\xi}^h)=\begin{pmatrix}\bar{\xi}^a\\ \bar{\xi}^{2n+1}\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix} \tag{3.3}$$

with respect to $(\bar{q}^h)$, then $\bar{\xi}$ determines a global vector field, denoted also by $\bar{\xi}$, in $\mathfrak{N}(V)$ because of (3.2). The vector field thus defined is non-zero and tangent to the fibre everywhere. If we introduce in $\overline{W}$ a local covector field $\bar{\eta}$ having components

$$(\bar{\eta}_i)=(\bar{\eta}_b,\bar{\eta}_{2n+1})=(l_b,1),\quad \text{i.e.,}\quad \bar{\eta}=l_b du^b+d\theta \tag{3.4}$$

with respect to $(\bar{q}^h)$, then $\bar{\eta}$ determines a global covector field, denoted also by $\bar{\eta}$, in $\mathfrak{N}(V)$ because of (3.2), where l_b are the components of the third fundamental tensor l appearing in (1.13). Thus we easily have

$$\bar{\eta}(\bar{\xi})=\bar{\eta}_i\bar{\xi}^i=1 \tag{3.5}$$

and $\mathcal{L}_{\bar{\xi}}\bar{\eta}=0$, where $\mathcal{L}_{\bar{\xi}}$ denotes the Lie derivation with respect to $\bar{\xi}$. Therefore the set $(\bar{\xi},\bar{\eta})$ defines in $\mathfrak{N}(V)$ a structure of a *fibred space* in the sense of [10] with respect to the projection p: $\mathfrak{N}(V)\to V$.

We now define in $\overline{W}$ $2n$ local vector fields $\bar{e}_b$ and $2n$ local covector fields $\bar{e}^a$ having respectively components of the form

$$(3.6)\qquad (\bar{e}^h{}_b)=\begin{pmatrix}\delta_b^a\\ -l_b\end{pmatrix},\qquad (\bar{e}_i{}^a)=(\delta_b^a,\ 0)$$

with respect to coordinates $(\bar{q}^h)$. Then $\{\bar{e}^a, \bar{\eta}\}$ is the coframe dual to the frame $\{\bar{e}_b, \bar{\xi}\}$. We have now

$$(3.7)\qquad dp(\bar{e}_b)=B_b,\qquad dp(\bar{\xi})=0$$

directly from the definition of the projection p: $\mathfrak{N}(V)\rightarrow V$, where $\{B_b\}$ is the natural frame of coordinates (u^a) defined in each neighborhood U of V.[6]

Let T and S be two tensor fields of type (1, 1) and of type (0, 2) in V respectively. Taking account of (3.6), we see that the horizontal lifts T^L of T and S^L of S have in $\overline{W}$ respectively the components of the form

$$(3.8)\qquad (T_i{}^h)=(T_b{}^a\bar{e}_i{}^b\bar{e}^h{}_a)=\begin{pmatrix}T_b{}^a & 0\\ -T_b{}^e l_e & 0\end{pmatrix},$$

$$(3.9)\qquad (S_{ji})=(S_{cb}\bar{e}_j{}^c\bar{e}_i{}^b)=\begin{pmatrix}S_{cb} & 0\\ 0 & 0\end{pmatrix}$$

with respect to $(\bar{q}^h)$, where $T_b{}^a$ and S_{cb} are respectively the components of T and S with respect to (u^a) defined in U (Cf. [10], [11]).

If we put

$$(3.10)\qquad \bar{f}=f^L,\qquad \bar{g}=g^L+\bar{\eta}\otimes\bar{\eta},$$

(f, g) being the Kählerian structure induced in V, then we see by means of (3.6), (3.8) and (3.9) that $\bar{f}$ and $\bar{g}$ have respectively in $\overline{W}$ components of the form

$$(3.11)\qquad (\bar{f}_i{}^h)=\begin{pmatrix}f_b{}^a & 0\\ -f_b{}^e l_e & 0\end{pmatrix},\qquad (\bar{g}_{ji})=\begin{pmatrix}g_{cb}+l_c l_b & l_c\\ l_b & 1\end{pmatrix}$$

with respect to $(\bar{q}^h)$. If we put

$$\bar{f}_{ji}=\bar{f}_j{}^k\bar{g}_{ki},$$

we obtain

$$(3.12)\qquad (\bar{f}_{ji})=\begin{pmatrix}f_{cb} & 0\\ 0 & 0\end{pmatrix}$$

by virtue of (3.11). We have now

6) For a differentiable mapping ψ, we denote by $d\psi$ the differential mapping of ψ.

$$f(dp(\bar{e}_b))=f_b{}^a B_a=dp(\bar{f}\bar{e}_b),$$

i.e.,

$$f\circ dp=dp\circ\bar{f} \tag{3.13}$$

as a consequence of (3.6), (3.7) and (3.11).

We have the following equations:

$$\begin{aligned}&\bar{f}_j{}^h\bar{f}_i{}^j=-\delta_i^h+\bar{\xi}^h\bar{\eta}_i, \qquad &&\bar{f}_j{}^h\bar{\xi}^j=0,\\ &\bar{f}_j{}^h\bar{\eta}_h=0, &&\bar{\eta}_i\bar{\xi}^i=1\end{aligned} \tag{3.14}$$

as consequences of (3.3), (3.4) and (3.11). That is to say, $(\bar{f}, \bar{\xi}, \bar{\eta})$ is an almost contact structure in $\mathfrak{N}(V)$. Moreover, we obtain from (3.3), (3.4) and (3.11) the equations

$$\bar{f}_j{}^k\bar{f}_i{}^h\bar{g}_{kh}=\bar{g}_{ji}-\bar{\eta}_j\bar{\eta}_i, \qquad \bar{\eta}_j=\bar{g}_{ji}\bar{\xi}^i, \tag{3.15}$$

which show that $(\bar{f}, \bar{\xi}, \bar{\eta}, \bar{g})$ is an almost contact metric structure in $\mathfrak{N}(V)$. Summing up, we have

PROPOSITION 3.1. *Let V be a complex hypersurface in a Kählerian manifold. Then the normal circle bundle $\mathfrak{N}(V)$ of V admits an almost contact metric structure $(\bar{f}, \bar{\xi}, \bar{\eta}, \bar{g})$, where $\bar{f}, \bar{\xi}, \bar{\eta}$ and $\bar{g}$ are defined by* (3.3), (3.4) *and* (3.11) *respectively.*

Let γ be the tensor field of type $(0,2)$ defined by (1.23) in the complex hypersurface V and γ^L the horizontal lift of γ. If we put

$$\bar{\gamma}=\gamma^L+\bar{\eta}\otimes\bar{\eta},$$

we see by means of (3.9) that $\bar{\gamma}$ has components of the form

$$(\bar{\gamma}_{ji})=\begin{pmatrix}\gamma_{cb}+l_c l_b & l_b\\ l_c & 1\end{pmatrix} \tag{3.16}$$

with respect to coordinates $(\bar{q}^h)$ defined in each neighborhood $\overline{W}$ of $\mathfrak{N}(V)$. Thus we have

$$\bar{f}_j{}^k\bar{f}_i{}^h\bar{\gamma}_{kh}=\bar{\gamma}_{ji}-\bar{\eta}_j\bar{\eta}_i, \qquad \bar{\eta}_j=\bar{\gamma}_{ji}\bar{\xi}^i \tag{3.17}$$

because of (3.3), (3.4), (3.11) and (3.16). On putting

$$\bar{\omega}_{ji}=\bar{f}_j{}^k\bar{\gamma}_{ki}, \tag{3.18}$$

we obtain

$$(\bar{\omega}_{ji})=\begin{pmatrix}f_c{}^e\gamma_{eb} & 0\\ 0 & 0\end{pmatrix}. \tag{3.19}$$

Therefore we have

PROPOSITION 3.2. *Let V be a complex hypersurface in a Kählerian manifold. Then the normal circle bundle $\mathfrak{N}(V)$ of V admits an almost contact metric structure $(\bar{f}, \bar{\xi}, \bar{\eta}, \bar{\gamma})$ if $\bar{\gamma}$ is non-singular everywhere in V, where $\bar{f}, \bar{\xi}, \bar{\eta}$ and $\bar{\gamma}$ are respectively defined by* (3.3), (3.4), (3.11) *and* (3.16).

Taking account of (3.4), we find $d\bar{\eta}=(1/2)(\partial_d l_c-\partial_c l_d)du^d\wedge du^c$. That is, $\bar{\Omega}=d\bar{\eta}$ has components

$$(3.20)\qquad (\bar{\Omega}_{ji})=\begin{pmatrix} \frac{1}{2}(\nabla_d l_c-\nabla_c l_d) & 0 \\ 0 & 0 \end{pmatrix}$$

with respect to $(\bar{q}^h)$. If we assume that the enveloping manifold M is locally flat, then we obtain

$$\bar{\omega}_{ji}=\bar{\Omega}_{ji}$$

by means of (3.19), (3.20) and (2.6) with vanishing c. Therefore we have

PROPOSITION 3.3. *Let V be a complex hypersurface in a locally flat Kählerian manifold. Then the normal circle bundle $\mathfrak{N}(V)$ of V admits a contact structure $(\bar{f}, \bar{\xi}, \bar{\eta}, \bar{\gamma})$, $\bar{f}, \bar{\xi}, \bar{\eta}$ and $\bar{\gamma}$ being respectively defined by* (3.3), (3.4), (3.11) *and* (3.16), *if $\bar{\gamma}$ is non-singular everywhere in V.*

We suppose now that the enveloping manifold M is an Einstein manifold with curvature scalar $'K$. If, moreover, the complex hypersurface V is an Einstein manifold with curvature scalar K, then we have from (1.34)

$$(3.21)\qquad \nabla_d l_c-\nabla_c l_d=Bf_{dc},\qquad B=\frac{1}{4}\left(\frac{'K}{n+1}-\frac{K}{n}\right),$$

where dim $V=2n$. Therefore we find

$$(3.22)\qquad (\bar{\Omega}_{ji})=\begin{pmatrix} Bf_{cb} & 0 \\ 0 & 0 \end{pmatrix},\qquad B=\frac{1}{4}\left(\frac{'K}{n+1}-\frac{K}{n}\right)$$

because of (3.19). When $'K/(n+1)\neq K/n$, we put

$$(3.23)\qquad '\bar{f}=\varepsilon\bar{f},\qquad '\bar{\xi}=|B|^{-1/2}\bar{\xi},\qquad '\bar{\eta}=B^{1/2}\bar{\eta},\qquad '\bar{g}=|B|\bar{g},$$

ε being defined by $\varepsilon=\operatorname{sgn} B$. We then see that $('\bar{f}, '\bar{\xi}, '\bar{\eta}, '\bar{g})$ is a contact structure. On the other hand, if we denote by $'f$ and $'g$ the projections of $'\bar{f}$ and $'\bar{g}$ in the sense of [10] and [11] respectively, then $('g, 'f)$ defines a Kählerian structure in V. Therefore, according to Lemma 3.1, $('\bar{f}, '\bar{\xi}, '\bar{\eta}, '\bar{g})$ is a normal contact structure in $\mathfrak{N}(V)$, because the projection Ω of $\bar{\Omega}$ has components of the form

$$\Omega_{cb}={'f_c}^a\,'g_{ba}$$

by virtue of (3.22), $'f_c{}^a$ and $'g_{ba}$ being the components of $'f$ and $'g$ respectively. Conversely, we assume that, for the complex hypersurface V of a Kähler-Einstein manifold M with curvature scalar $'K$, the almost contact metric structure $('\bar{f}, '\bar{\xi}, '\bar{\eta}, '\bar{g})$ defined in $\mathcal{N}(V)$ by (3.23) with a certain non-zero constant B is normal. In such a case, we say that $(\bar{f}, \bar{\xi}, \bar{\eta}, \bar{g})$ *is essentially a normal contact structure*. If this is the case, we obtain $\nabla_c l_b - \nabla_b l_c = Bf_{cb}$. On the other hand, substituting $'K_{cb} = ('K/2(n+1))g_{cb}$ into (1.34), we get

$$\nabla_c l_b - \nabla_b l_c = \frac{'K}{2(n+1)} f_{cb} - K_{ce} f_b{}^e, \tag{3.24}$$

which implies together with $\nabla_c l_b - \nabla_b l_c = Bf_{cb}$ that the complex hypersurface V is an Einstein manifold. Therefore we have

PROPOSITION 3.4. *Let V be a complex hypersurface in a Kähler-Einstein manifold M with curvature scalar $'K$. Then the almost contact metric structure $(\bar{f}, \bar{\xi}, \bar{\eta}, \bar{g})$ defined in $\mathcal{N}(V)$ by* (3.3), (3.4) *and* (3.11) *is essentially a normal contact structure, if and only if the complex hypersurface V is an Einstein manifold with curvature scalar K such that $K \neq (n/(n+1))\ 'K$, where* dim $V = 2n$.

Taking account of (2.5), we have the following Proposition 3.5 as a corollary to Proposition 3.4.

PROPOSITION 3.5. *Let V be a complex hypersurface in a Kählerian manifold of constant holomorphic sectional curvature c. Then the set $(\bar{f}, \bar{\xi}, \bar{\eta}, \bar{\gamma})$ defined in $\mathcal{N}(V)$ by* (3.3), (3.4), (3.11) *and* (3.16) *is essentially a normal contact structure, if and only if the complex hypersurface V is an Einstein manifold with curvature scalar K such that $K \neq n(n+1)c$, where* dim $V = 2n$.

The 1-form $\bar{\eta}$ defined in $\mathcal{N}(V)$ by (3.4) is a connection form in the principal fibre bundle $\mathcal{N}(V)$ and the curvature form of $\bar{\eta}$ is by definition the two form Ω in V such that $*p(\Omega) = d\bar{\eta}$, where $*p$ is the mapping dual to dp. The curvature form Ω has in each neighborhood U of V components of the form $\Omega = (1/2)(\nabla_c l_b - \nabla_b l_c)\, du^c \wedge du^b$ by means of (3.20). The cohomology class $[\Omega]$ determined by Ω is the characteristic class of the circle bundle $\mathcal{N}(V)$. As is well known, $[\Omega]$ is an integral cocycle when V is compact. If, for a compact V, the cohomology class $[\Phi]$ determined by the fundamental form $\Phi = (1/2)f_{cb}\, du^c \wedge du^b$ is an integral cocycle, V is called a *Hodge manifold*. Then, taking account of (3.21), we have

$$\Omega = B\Phi, \qquad B = \frac{1}{4}\left(\frac{'K}{n+1} - \frac{K}{n}\right) \tag{3.25}$$

and hence

PROPOSITION 3.6. *Let V be a compact complex hypersurface in a Kähler-Einstein manifold M with curvature scale $'K$. Then V is a Hodge manifold if V is an Einstein manifold with curvature scalar K such that $K \neq (n/(n+1))\ 'K$, where* dim $V = 2n$.

When the characteristic class $[\Omega]$ is zero, i.e., when we have $\Omega = d\phi$, ϕ being a global 1-form in V, the structure group of $\mathfrak{N}(V)$ is reducible to a discrete group. In such a case, $\mathfrak{N}(V)$ is said to be *locally trivial*. On the other hand, $\mathfrak{N}(V)$ is said to be *locally parallelizable*, when $\Omega=0$. A field N of unit normal vectors to V, global or local, is said to be *parallel in* $\mathfrak{N}(V)$ if, for any tangent vector field X of V, $\nabla_X N$ is perpendicular to the normal plane at each point. It is easily verified that $\mathfrak{N}(V)$ is locally parallelizable if and only if there exists in each neighborhood U of V a field of unit normal vectors which is parallel in $\mathfrak{N}(V)$.

Let V be a compact complex hypersurface of constant curvature scalar K in a Kähler-Einstein manifold M with curvature scalar $'K$. Then, substituting $'K_{cb}=('K/2(n+1))g_{cb}$ in (1.34), we obtain

$$\nabla_c l_b-\nabla_b l_c=\frac{'K}{2(n+1)}f_{cb}-K_{ce}f_b{}^e. \tag{3.26}$$

On the other hand, the tensor field $K_{ce}f_b{}^e$ is harmonic in the Kählerian manifold V if and only if V is of constant curvature scalar (Cf. Yano [9]). Therefore, in our case, Ω is a hormonic form in V by virtue of (3.26). Thus we have $\Omega=d\phi$, ϕ being a global 1-form in V, if and only if $\Omega=0$ (Cf. Yano [9]). Consequently, according to (3.26), we have

PROPOSITION 3.7. *For a compact complex hypersurface V of constant curvature scalar in a Kähler-Einstein manifold M, the following three conditions* (a), (b) *and* (c) *are equivalent to each other*:

(a) *$\mathfrak{N}(V)$ is locally trivial.*
(b) *$\mathfrak{N}(V)$ is locally parallelizable.*
(c) *V is an Einstein manifold with curvature scalar K such that $K=(n/(n+1))\ 'K$, where $'K$ is the curvature scalar of M and* dim $V=2n$.

When V admits a global field of unit normal vectors, $\mathfrak{N}(V)$ is locally trivial. Thus, as a corollary to Proposition 3.7, we have

PROPOSITION 3.8. *Let V be a compact complex hypersurface of constant curvature scalar in a Kähler-Einstein manifold M with curvature scalar $'K$. Then, if V admits a global field of unit normal vectors, V is necessarily an Einstein manifold with curvature scalar K such that $K=(n/(n+1))\ 'K$, where* dim $V=2n$. *If this is the case, V admits a global field of unit normal vectors which is parallel in $\mathfrak{N}(V)$.*

Let V be a complex hypersurface, which is not necessarily compact, in an Kähler-Einstein manifold M. Then, if $\mathfrak{N}(V)$ is locally parallelizable, we find

$$K_{cb}=\frac{'K}{2(n+1)}g_{cb}$$

by virtue of (3.26) and conversely. Thus we have

PROPOSITION 3.9. *Let V be a complex hypersurface, which is not necessarily compact, in a Kähler-Einstein manifold M. Then $\mathfrak{N}(V)$ is locally parallelizable if* *and only if V is an Einstein manifold with curvature scalar $K=(n/(n+1))\ 'K$, $'K$ being the curvature scalar of M, where* dim $V=2n$.

We now assume that the enveloping manifold M is of constant holomorphic sectional curvature c. Then, according to (2.7), $\mathfrak{N}(V)$ is locally parallelizable if and only if

$$K_{cb}=\frac{n}{2}cg_{cb},$$

which implies

$$K=n^2c, \tag{3.27}$$

where K denotes the scalar curvature of V. On the other hand, in the present case, the scalar curvature $'K$ of M is given by

$$K'=(n+1)(n+2)c. \tag{3.28}$$

Thus, taking account of (3.27) and (3.28), we have $c=0$ if $K=(n/(n+1))\ 'K$. Therefore, taking account of Propositions 3.7, 3.8, 3.9 and (2.6), we have

PROPOSITION 3.10. *In a Kählerian manifold of non-zero constant holomorphic sectional curvature, there exists no complex hypersurface V satisfying one of the following two conditions* (a) *and* (b):

(a) *V is compact and of constant curvature scalar. $\mathfrak{N}(V)$ is locally trivial.*

(b) *$\mathfrak{N}(V)$ is locally parallelizable.*

In a locally flat Kählerian manifold, a complex hypersurface V is totally geodesic if V satisfies one of the conditions (a) *and* (b) *mentioned above.*

§4. The Gauss map.

Let E^{2n+2} be a Euclidean space of even dimension $2n+2$ with the natural Kählerian structure (G, F), where G and F have respectively the following components:

$$(G_{CB})=I_{2n+2}, \qquad (F_B{}^A)=\begin{pmatrix} 0 & I_{n+1} \\ -I_{n+1} & 0 \end{pmatrix}$$

with respect to certain rectangular coordinates (x^A), where I_m denotes the $m\times m$ unit matrix. Denote by S^{2n+1} the unit sphere in E^{2n+2} defined by the equation

$$(x^1)^2+(x^2)^2+\cdots+(x^{2n+2})^2=1.$$

A 2-dimensional plane defined in E^{2n+2} by

$$x^A=\sigma A^A+\tau F_B{}^A A^B$$

is called a holomorphic plane, A^A being constant and σ, τ real parameters. The intersection of S^{2n+1} and a holomorphic plane is called a *holomorphic great circle*. The set of all holomorphic great circles forms, as is well known, a fibring of

S^{2n+1}, which determines the natural bundle structure π: $S^{2n+1}\to CP^n$ over the complex projective space CP^n. That is, each fibre of the bundle structure π: $S^{2n+1}\to CP^n$ is a holomorphic great circle.

Let there be given a complex hypersurface V in E^{2n+2} and denote by $\mathfrak{N}(V)$ the normal circle bundle of V. Let N be an element of $\mathfrak{N}(V)$. If we transport N parallelly to the origin, we get a unit vector N' at the origine and hence in S^{2n+1} a point $\psi(N)$ which is the terminal point of N'. In such a way, we can define a mapping ψ: $\mathfrak{N}(V)\to S^{2n+1}$. For a point P of V, the fibre $p^{-1}(V)$ of $\mathfrak{N}(V)$ over P is mapped onto a holomorphic great circle of S^{2n+1}, that is, $\psi(p^{-1}(\mathrm{P}))$ is a holomorphic great circle. Therefore, if we put $\varphi(\mathrm{P})=\pi(\psi(p^{-1}(\mathrm{P})))$, we can define a mapping φ: $V\to CP^n$, which is called the *Gauss map* of the complex hypersurface V in E^{2n+2}.

$$\begin{array}{ccc} \mathfrak{N}(V) & \xrightarrow{\psi} & S^{2n+1} \\ p\downarrow & & \downarrow\pi \\ V & \xrightarrow[\varphi]{} & CP^n \end{array}$$

We have here

$$\varphi\circ p=\pi\circ\psi. \tag{4.1}$$

Thus the Gauss map φ is regular if and only if the mapping ψ is so also.

We shall obtain a local expression of the mapping ψ: $\mathfrak{N}(V)\to S^{2n+1}$. Let U be a coordinate neighborhood of V and (u^a) coordinates defined in U. Then, as has been done in § 3, local coordinates (u^a, θ) are introduced in each neighborhood $\overline{W}$ $(\subset p^{-1}(U))$ of $\mathfrak{N}(V)$. We denote by $\{\bar{e}_i\}=\{\bar{e}_b, \bar{\xi}\}$ the natural frame of local coordinates $(\bar{q}^h)=(u^a, \theta)$ defined in $\overline{W}$, where $\bar{e}_{2n+1}=\bar{\xi}$.

The restriction of the mapping ψ to $\overline{W}$ is expressed by equations

$$x^A=x^A(u^a, \theta), \tag{4.2}$$

where $x^A(u^a, \theta)=C^A\cos\theta+D^A\sin\theta$. Thus, on putting $\tilde{q}^a=u^a$, $\tilde{q}^{2n+1}=\theta$, we can regard $(\tilde{q}^h)=(u^a, \theta)$ as coordinates defined in $\widetilde{W}$. Thus, differentiating (4.2), we have $2n+1$ local vector fields $\tilde{B}_b$ and $\tilde{B}_\theta$ in $\widetilde{W}=\psi(\overline{W})$. They are tangent to S^{2n+1} and have respectively components of the form

$$\begin{aligned} \tilde{B}_b{}^A&=\frac{\partial x^A(u^a, \theta)}{\partial u^b}=-(h_b{}^a\cos\theta+k_b{}^a\sin\theta)B_a{}^A+l_bY^A, \\ \tilde{B}_\theta{}^A&=\frac{\partial x^A(u^a,\theta)}{\partial\theta}=Y^A \end{aligned} \tag{4.3}$$

by virtue of (1.13), where we have put

$$Y^A=-C^A\cos\theta+D^A\sin\theta. \tag{4.4}$$

These $2n+1$ local tangent vector fields $\tilde{B}_b$ and $\tilde{B}_\theta$ form the natural frame of coordinates $(\tilde{q}^h)=(u^a, \theta)$ defined in $\tilde{W}$, when the mapping ψ is regular. On putting

$$d\psi(\bar{e}_b)=\tilde{e}_b, \qquad d\psi(\bar{\xi})=\tilde{\xi}, \tag{4.5}$$

we find

$$\tilde{e}_b=\tilde{B}_b-l_b\tilde{B}_\theta, \qquad \tilde{\xi}=\tilde{B}_\theta \tag{4.6}$$

as consequences of (3.3) and (3.6).

The mapping ψ is regular if and only if the vector fields $\tilde{B}_b$ and $\tilde{B}_\theta$ are linearly independent. Thus, according to (4.3), ψ is regular if and only if

$$\begin{vmatrix} -(h_b{}^a\cos\theta+k_b{}^a\sin\theta) & l_b \\ 0 & 1 \end{vmatrix} \neq 0,$$

i.e., if and only if

$$|h_b{}^a\cos\theta+k_b{}^a\sin\theta|\neq 0, \tag{4.7}$$

since the vector fields $B_a{}^A$ and Y^A are linearly independent. On the other hand, taking account of (1.18), we obtain

$$h_b{}^a\cos\theta+k_b{}^a\sin\theta=h_b{}^c(\delta_c^a\cos\theta+f_c{}^a\sin\theta).$$

Therefore the condition (4.7) is equivalent to the condition

$$|h_b{}^a|\neq 0 \tag{4.8}$$

because of $|\delta_c^a\cos\theta+f_c{}^a\sin\theta|\neq 0$. Taking account of (2.5) with vanishing c, we see that the condition (4.8) is equivalent to the condition

$$|K_{cb}|\neq 0.$$

Thus we have

PROPOSITION 4.1. *For a complex hypersurface V of E^{2n+2}, the Gauss map φ: $V\to CP^n$ is regular, or equivalently, the mapping ψ: $\mathcal{N}(V)\to S^{2n+1}$ is regular, if and only if $|h_b{}^a|\neq 0$, or, if and only if $|K_{cb}|\neq 0$, where K_{cb} are the components of the Ricci tensor of V.*

The metric tensor $\tilde{\gamma}$ induced naturally in S^{2n+1} has in $\tilde{W}$ the components of the form

$$(\tilde{\gamma}_{ji})=\begin{pmatrix} G_{CB}\tilde{B}_c{}^C\tilde{B}_b{}^B & G_{CB}\tilde{B}_c{}^C\tilde{\xi}^B \\ G_{CB}\tilde{\xi}^C\tilde{B}_b{}^B & G_{CB}\tilde{\xi}^C\tilde{\xi}^B \end{pmatrix}$$

with respect to coordinates $(\tilde{q}^h)=(u^a, \theta)$. Thus, according to (4.3) and (4.4), we obtain

$$(\tilde{\gamma}_{ji})=\begin{pmatrix}\gamma_{cb}+l_c l_b & l_c\\ l_b & 1\end{pmatrix},\qquad \gamma_{cb}=h_c{}^e h_{be}. \tag{4.9}$$

In the next step, assuming that the mapping ψ is regular, we shall obtain the contact metric structure $(\tilde{f}, \tilde{\xi}, \tilde{\eta}, \tilde{\gamma})$ induced naturally in S^{2n+1} (Cf. Sasaki and Hatakeyama [5]). Applying the operator F to (4.3), we get

$$\begin{aligned} F_B{}^A\tilde{B}_b{}^B&=f_b{}^a\tilde{B}_a{}^A-(f_b{}^a l_a)\tilde{B}_\theta{}^A-l_b X^A,\\ F_B{}^A\tilde{B}_\theta{}^B&=-X^A \end{aligned} \tag{4.10}$$

by means of (1.6) and (1.17), $f_b{}^a$ being the functions defining the components of the complex structure f induced in V with respect to coordinates (u^a), where we have put

$$X^A=-F_B{}^A Y^B=C^A\cos\theta+D^A\sin\theta, \tag{4.11}$$

which is normal to S^{2n+1} everywhere. Taking the tangential parts in (4.10), we find in S^{2n+1} a tensor field $\tilde{f}$ of type (1,1) with components

$$(\tilde{f}_i{}^h)=\begin{pmatrix} f_b{}^a & 0\\ -f_b{}^e l_e & 0\end{pmatrix} \tag{4.12}$$

with respect to the frame $\{\tilde{B}_i\}=\{\tilde{B}_a, \tilde{B}_\theta\}$. Thus we have

$$\tilde{f}\tilde{B}_b=f_b{}^a\tilde{B}_a-f_b{}^e l_e\tilde{B}_\theta,\qquad \tilde{f}\tilde{\xi}=0,$$

which imply

$$\tilde{f}\circ d\psi=d\psi\circ\bar{f} \tag{4.13}$$

by virtue of (3.11) and (4.6). If we put

$$\tilde{\eta}=\bar{\eta}\circ d\psi, \tag{4.14}$$

we see that $\tilde{\eta}$ has components of the form

$$(\tilde{\eta}_i)=(\tilde{\eta}_b, \tilde{\eta}_{2n+1})=(l_b, 1),\quad \text{i.e.,}\quad d\tilde{\eta}=l_b du^b+d\theta$$

with respect to coordinates $(\tilde{q}^h)=(u^a, \theta)$. Summing up (4.5), (4.13) and (4.14), we have

$$d\psi\circ\bar{f}=\tilde{f}\circ d\psi,\qquad d\psi(\bar{\xi})=\tilde{\xi},\qquad \bar{\eta}\circ d\psi=\tilde{\eta}. \tag{4.15}$$

Taking account of (1.7), (1.21), (1.22), (4.9), (4.12) and (4.14), we find

$$\begin{aligned} &\tilde{f}_j{}^i\tilde{f}_i{}^h=-\delta_j^h+\tilde{\eta}_j\tilde{\xi}^h, && \tilde{f}_j{}^h\tilde{\xi}^j=0,\\ &\tilde{f}_j{}^h\tilde{\eta}_h=0, && \tilde{\eta}_i\tilde{\xi}^i=1,\\ &\tilde{f}_j{}^k\tilde{f}_i{}^h\tilde{\gamma}_{kh}=\tilde{\gamma}_{ji}-\tilde{\eta}_j\tilde{\eta}_i, && \tilde{\eta}_j=\tilde{\gamma}_{ji}\tilde{\xi}^i, \end{aligned} \tag{4.16}$$

which show that $(\tilde{f}, \tilde{\xi}, \tilde{\eta}, \tilde{\gamma})$ is the almost contact metric structure induced naturally in S^{2n+1}. On the other hand, taking account of (4.14), we find $d\bar{\eta}=d(\tilde{\eta}\circ d\psi)={}^*\psi(d\tilde{\eta})$, which implies

$$
\begin{aligned}
d\bar{\eta}&=\frac{1}{2}(\nabla_d l_c-\nabla_c l_d)du^d\wedge du^c\\
&=(f_d{}^e\gamma_{ec})du^d\wedge du^c
\end{aligned}
\tag{4.17}
$$

by virtue of (3.19) and (3.20), $^*\psi$ denoting the mapping dual to $d\psi$. Thus, if we put

$$\tilde{f}_{ji}=\tilde{f}_j{}^h\tilde{\gamma}_{hi},$$

we obtain

$$d\tilde{\eta}=\tilde{f}_{ji}d\tilde{q}^j\wedge d\tilde{q}^i \tag{4.18}$$

as a consequence of (4.12) and (4.17). Consequently, taking account of (4.14) and (4.18), we see that $(\tilde{f}, \tilde{\xi}, \tilde{\eta}, \tilde{\gamma})$ is the contact structure induced naturally in S^{2n+1}, which is normal (Cf. Sasaki and Hatakeyama [4]). Thus we have

PROPOSITION 4.2. *Let V be a complex hypersurface in E^{2n+2} and suppose that the mapping ψ: $\mathfrak{N}(V)\to S^{2n+1}$ is regular. Then $(\bar{f}, \bar{\xi}, \bar{\eta}, \bar{\gamma})$ is a normal contact structure in $\mathfrak{N}(V)$, $\bar{\xi}, \bar{\eta}, \bar{f}$ and $\bar{\gamma}$ being respectively defined by* (3.3), (3.4) *and* (3.11), *and these tensor fields are respectively given by*

$$\bar{f}=d\psi^{-1}\circ\tilde{f}\circ d\psi,\qquad \bar{\xi}=d\psi^{-1}(\tilde{\xi}),\qquad \bar{\eta}=\tilde{\eta}\circ d\psi^{-1},\qquad \bar{\gamma}={}^*\psi^{-1}\tilde{\gamma},$$

where $(\tilde{f}, \tilde{\xi}, \tilde{\eta}, \tilde{\gamma})$ is the natural contact structure in S^{2n+1}.

We shall obtain a local expression of the natural projection π: $S^{2n+1}\to CP^n$. We assume that the mapping ψ: $\mathfrak{N}(V)\to S^{2n+1}$ is regular. Let $\widetilde{W}$ be a coordinate neighborhood in S^{2n+1} contained in $p^{-1}(U)$, U being a coordinate neighborhood of V. Then, as was shown above, coordinates $(\tilde{q}^h)=(u^a, \theta)$ are introduced in $\widetilde{W}$. Taking an arbitrary point P of $\widetilde{W}$, we can assign (u^a) to the point π(P), (u^a, θ) being coordinates of P, since holomorphic great circle in $\widetilde{W}$ are defined by equations $u^a=$const. Thus (u^a) are regarded as coordinates defined in $\overline{\overline{W}}=\pi(\widetilde{W})$. For the local vector fields $\tilde{e}_b$ and $\tilde{\xi}$ defined by (4.5), we find

$$d\pi(\tilde{\xi})=0. \tag{4.18}$$

If we put

$$d\pi(\tilde{e}_b)=\bar{\bar{e}}_b, \tag{4.19}$$

then $\{\bar{\bar{e}}_b\}$ forms the natural frame of coordinates (u^a) above defined in $\overline{\overline{W}}$.

Taking account of the definition of the Gauss map φ: $V\to CP^n$, we have $\varphi\circ p=\pi\circ\psi$ and hence $d\varphi\circ dp=d\pi\circ d\psi$. Thus, taking account of (4.5) and (4.19), we get

$$d\varphi\circ dp(\bar{e}_b)=d\pi\circ d\psi(\bar{e}_b)=d\pi(\tilde{e}_b)=\bar{\bar{e}}_b.$$

Thus we obtain

$$d\varphi(B_b)=\bar{\tilde{e}}_b, \tag{4.20}$$

since $dp(\bar{e}_b)=B_b$ because of (3.7).

We have easily $\mathcal{L}_{\tilde{\xi}}\tilde{\gamma}=0$, $\mathcal{L}_{\tilde{\xi}}\tilde{\eta}=0$ and $\tilde{\eta}(\tilde{\xi})=1$. Therefore $(\tilde{\xi}, \tilde{\eta}, \tilde{\gamma})$ defines in S^{2n+1} a structure of a fibred space with invariant metric $\tilde{\gamma}$ in the sense of [11] with respect to the projection π: $S^{2n+1}\to CP^n$. Since we easily have $\mathcal{L}_{\tilde{\xi}}\tilde{f}=0$, $\tilde{f}$ is an invariant tensor field in S^{2n+1}. Thus the projections $\bar{\tilde{\gamma}}$ of $\tilde{\gamma}$ and $\bar{\tilde{f}}$ of $\tilde{f}$ have in $\bar{\tilde{W}}$ respectively components of the form

$$\begin{aligned} \bar{\tilde{\gamma}}_{cb}&=\tilde{\gamma}_{ji}\tilde{e}^j{}_c\tilde{e}^i{}_b=\gamma_{cb}, \\ \bar{\tilde{f}}_b{}^a&=\tilde{f}_j{}^h\tilde{e}^i{}_b\tilde{e}_h{}^a=f_b{}^a \end{aligned} \tag{4.21}$$

with respect to (u^a) defined in $\bar{\tilde{W}}$, as consequences of (4.9) and (4.12), where $\tilde{e}^i{}_b$ and $\tilde{e}_h{}^a$ are given respectively by

$$(\tilde{e}^i{}_b)=\begin{pmatrix} \delta_b^a \\ -l_b \end{pmatrix}, \qquad (\tilde{e}_h{}^a)=(\delta_b^a, 0)$$

which are respectively the components of $\tilde{e}_b$ and $\tilde{e}^a$ with respect to (u^a, θ) defined in $\tilde{W}$, $\tilde{e}_b$ being defined by (4.6) and $\{\tilde{e}^a, \tilde{\eta}\}$ the coframe dual to the frame $\{\tilde{e}_b, \tilde{\xi}\}$ (Cf. [11]). The pair $(\bar{\tilde{\gamma}}, \bar{\tilde{f}})$ thus introduced, as was proved in [10], defines in CP^n the natural Kählerian structure of constant holomorphic sectional curvature 1. It is easily verified that

$$d\pi\circ\tilde{f}=\bar{\tilde{f}}\circ d\pi \tag{4.22}$$

by means of (4.12), (4.18), (4.19) and (4.21). Therefore, taking account of (4.15), we get

$$\begin{aligned} (d\varphi\circ dp)\circ\bar{f}&=(d\pi\circ d\psi)\circ\bar{f} \\ &=d\pi\circ\tilde{f}\circ d\psi=\bar{\tilde{f}}\circ(d\pi\circ d\psi)=\bar{\tilde{f}}\circ(d\varphi\circ dp) \end{aligned}$$

which implies

$$d\varphi\circ f=\bar{\tilde{f}}\circ d\varphi \tag{4.23}$$

because of (3.13). Thus we have from (4.23)

PROPOSITION 4.3. *When one of the three conditions*

$$\text{(a)}\quad |h_b{}^a|\neq 0, \qquad \text{(b)}\quad |K_{cb}|\neq 0, \qquad \text{(c)}\quad |\gamma_{cb}|\neq 0$$

is satisfied, the Gauss map φ: $V\to CP^n$ is regular and φ is an analytic mapping.

When the Gauss map φ is not regular, we can prove the fact that, if the rank of φ is constant, the image $\varphi(V)$ is an analytic submanifold of CP^n and the mapping φ: $V\to\varphi(V)$ is analytic, i.e., $d\varphi\circ f=\bar{\tilde{f}}\circ d\varphi$, $\bar{\tilde{f}}$ being the complex structure induced in $\varphi(V)$ from the natural complex structure of CP^n.

Denote by $|k|$ and $|g|$ the determinants $|K_{cb}|$ and $|g_{cb}|$ respectively, where K_{cb} and g_{cb} are respectively the components of the Ricci tensor and the induced metric g of V. Let $\varDelta$ be a compact domain of the complex hypersurface V in E^{2n+2}. Then, taking account of (2.5) with vanishing c and (4.21), we easily see that, if the Gauss map is a homemorphism in $\varDelta$, the volume $\mathcal{V}$ of the image $\varphi(\varDelta)$ of $\varDelta$ is given by the formula

$$\mathcal{V}=2^n\int_\varDelta\sqrt{\frac{|k|}{|g|}}\,dv,$$

dv denoting the volume element of V, where $\dim V=2n$.

Taking account of (4.21), we see that the Riemannian metric γ defined in V by (1.23) is the one induced from $\bar{\bar{\gamma}}$ by the Gauss map φ: $V\to CP^n$, if φ is regular. If we suppose that the Gauss map φ is regular and is a conformal mapping, then we obtain by definition

$$\gamma_{cb}=Ag_{cb},$$

which implies

$$K_{cb}=-2Ag_{cb}$$

as a consequence of (2.5) with vanishing c. That is, the complex hypersurface V is necessarily an Einstein space. Conversely, if V is an Einstein space, then the Gauss map φ is a conformal mapping. Therefore, *a complex hypersurface in a Euclidean space E^{2n+2} is an Einstein space if and only if the Gauss map φ is a conformal mapping.*

§ 5. Einstein complex hypersurface in a Euclidean space.

Let V be a complex hypersurface in an Euclidean space E^{2n+2} and suppose that V is an Einstein space. Then the relation

$$\gamma_{cb}=Ag_{cb}\qquad(A\geqq 0)\tag{5.1}$$

holds and the Gauss map φ: $V\to CP^n$ is a conformal mapping, if φ is regular. However, any conformal mapping between two Kählerian manifolds is necessarily homothetic if the mapping is analytic. Thus the Gauss map φ is homothetic because φ is analytic. Therefore the function A appearing in (5.1) is necessarily constant.

When A is zero in (5.1), we find $\gamma_{cb}=0$, which implies $h_b{}^a=0$ because of $\gamma_{cb}=h_c{}^e h_{be}$. Thus, in this case, the given hypersurface V is totally geodesic.

When A is not zero in (5.1), the Gauss map φ is regular and homothetic. Therefore we see that the curvature tensor $K_{dcb}{}^a$ of the metric g_{cb} induced in V coincides with the tensor $\bar{\bar{K}}_{dcb}{}^a$ induced from the curvature tensor of $\bar{\bar{\gamma}}$ by φ, where $\bar{\bar{\gamma}}$ is the natural Kählerian metric of CP^n. However, $\bar{\bar{\gamma}}$ is of constant holomorphic sectional curvature 1. Therefore the induced metric g is necessarily an Einstein metric with positive curvature scalar. This fact contradicts Proposition 2.1. Consequently, the constant A should be zero. Thus we have

THEOREM 5.1. *Let V be a complex hypersurface in a locally flat Kählerian space. If V is an Einstein manifold, then V is totally geodesic* (Cf. Smyth [6] for complete complex hypersurfaces V).

Combining Proposition 3.4 and Theorem 5.1, we have

PROPOSITION 5.1. *In a locally flat Kählerian manifold, there exists no complex hypersurface V such that the almost contact metric structure $(\bar{f}, \bar{\xi}, \bar{\eta}, \bar{g})$ defined in the normal circle bundle $\mathfrak{N}(V)$ is essentially a normal contact structure.*

Taking account of Theorem 2.1, we see that Theorem 5.1 is established for complex hypersurfaces, locally symmetric or with parallel Ricci tensor.

REFERENCES

[1] AKO, M., Submanifolds in Fubinian manifolds. Kōdai Math. Sem. Rep. **19** (1967), 103–128.

[2] HATAKEYAMA, Y., On the existence of Riemannian metrics associated with a 2-form of rank $2r$. Tôhoku Math. J. **14** (1962), 162–166.

[3] SASAKI, S., On differentiable manifolds with certain structures which are closely related to almost contact structure I. Tôhoku Math. J. **12** (1960), 459-476.

[4] SASAKI, S., AND Y. HATAKEYAMA, On differentiable manifolds with certain structures which are closely related to almost contact structure II. Tôhoku Math. J. **13** (1961), 281–294.

[5] SASAKI, S., AND Y. HATAKEYAMA, On differentiable manifolds with contact metric structures. J. Math. Soc. Japan **14** (1962), 249–271.

[6] SMYTH, S., Differential geometry of complex hypersurfaces. Ann. of Math. **85** (1967), 246–266.

[7] SCHOUTEN, J. A., AND K. YANO, On invariant subspace in the almost complex X_{2n}. Ind. Math. **17** (1955), 261–269.

[8] TAKAHASHI, T., Hypersurface with parallel Ricci tensor in a space of constant holomorphic sectional curvature. J. Math. Soc. Japan **19** (1967), 199–204.

[9] YANO, K., Differential geometry on complex and almost complex spaces. Pergamon Press (1965).

[10] YANO, K., AND S. ISHIHARA, Differential geometry of fibred spaces. Kōdai Math. Sem. Rep. **19** (1967), 257–288.

[11] YANO, K., AND S. ISHIHARA, Fibred spaces with invariant Riemannian metric. Kōdai Math. Sem. Rep. **19** (1967), 317–360.

DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY.