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ON ALMOST CONTACT STRUCTURES

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Introduction.

Let Γ be a pseudogroup of differentiable transformations of a manifold V and let M be a differentiable manifold. A Γ-*atlas* on M is a collection of local diffeomorphisms $\{\lambda_i; U_i\}$ of M into V which satisfies $\cup U_i = M$ and $\lambda_i \circ \lambda_j^{-1} \in \Gamma$ for all i and j such that $U_i \cap U_j \neq \phi$.

Two Γ-atlases are said to be equivalent if their union is a Γ-atlas. An equivalence class of Γ-atlases is called a Γ-*structure* on M.

By an *almost* Γ-*structure* on a manifold M we mean, roughly speaking, a structure on M which is identified with a Γ-structure up to a certain order of contact at each point. It is a G-structure of a certain order.

Let Γ be a pseudogroup of contact transformations. Then a Γ-structure is a contact structure and an almost Γ-structure is an almost contact structure. An almost contact structure is a G-structure of order 1.

Sasaki defined in [3] a (ϕ, ξ, η)-structure. The structure is closely related to an almost contact structure, but, precisely speaking, it is not an almost contact structure.

The relation between a (ϕ, ξ, η)-structure and an almost contact structure is similar to that between an almost complex structure and an almost homogeneous contact structure [2]. In fact, a (ϕ, ξ, η)-structure is a G-structure of order 1 and the Lie algebra of the structure group is the linear Lie algebra

$$\left\{ \left(\begin{array}{c|c} 0 & 0 \cdots 0 \\ \hline 0 & \\ \vdots & A \\ 0 & \end{array} \right) \middle| \; A \in \mathfrak{gl}(n, \mathrm{C}) \right\}.$$

An almost contact structure is, however, a G-structure of order 1 and the Lie algebra of the structure group is the linear Lie algebra

$$\left\{ \left(\begin{array}{c|c} 0 & 0 \cdots 0 \\ \hline * & \\ \vdots & A \\ * & \end{array} \right) + \begin{pmatrix} 2\lambda & & & 0 \\ & \lambda & & \\ & & \ddots & \\ 0 & & & \lambda \end{pmatrix} \middle| \; A \in \mathfrak{sp}(n), \; \lambda \in \mathrm{R} \right\},$$

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where

$$\mathfrak{sp}(n)=\left\{A\in\mathfrak{gl}(2n,\mathrm{R})\,|\,{}^tAJ+JA=0 \quad \text{for } J=\begin{pmatrix}0 & I_n\\ -I_n & 0\end{pmatrix}\right\}.$$

Let $\mathcal{L}$ be the sheaf of germs of Γ-vector fields on V, infinitesimal automorphisms of a Γ-structure, and $\mathcal{L}(0)$ the stalk of $\mathcal{L}$ at the distinguished point 0 of V. Then $\mathcal{L}(0)$ is a filtered Lie algebra. As to the most of classical examples, $\mathcal{L}(0)$'s are *flat* filtered Lie algebras, that is, they are isomorphic with graded Lie algebras. But the filtered Lie algebra associated with a contact structure is infinite and *non-flat*.

§ 1. Preliminaries.

Let M be a differentiable manifold of dimension $2n+1$ and $F(M)$ the bundle of linear frames of M. Then $F(M)$ is a principal fibre bundle over M with structure group $GL(2n+1,\mathrm{R})$.

Let G be a subgroup of $GL(2n+1,\mathrm{R})$. A *G-structure* on M is a reduction of $F(M)$ to the group G.

Let $P_G(M)$ be a G-structure on M and let U be a coordinate neighborhood in M with a local coordinate system $x^0, x^1, \cdots, x^{2n}$. We denote by X_α the vector field $\partial/\partial x^\alpha$, $\alpha=0,1,\cdots,2n$, defined in U. Every linear frame at a point x of U can be uniquely expressed by

$$(\sum X_0^\alpha(X_\alpha)_x, \cdots, \sum X_{2n}^\alpha(X_\alpha)_x),^{1)}$$

where (X_β^α) is a non-singular matrix. We take $(x^\alpha, X_\beta^\alpha)$ as a local coordinate system in $\pi^{-1}(U)\subset P_G(M)$, where π denotes the projection $P_G(M)\to M$. Let (Y_β^α) be the inverse matrix of (X_β^α) so that $\sum X_\gamma^\alpha Y_\beta^\gamma=\sum Y_\gamma^\alpha X_\beta^\gamma=\delta_\beta^\alpha$. Let $e_0, e_1, \cdots, e_{2n}$ be the natural basis for R^{2n+1}. Ler u be a point of $P_G(M)$ with coordinates $(x^\alpha, X_\beta^\alpha)$ so that u maps e_α into $\sum X_\alpha^\beta(X_\beta)_x$, where $x=\pi(u)$.

If $X\in T_x(M)$ and if

$$X=\sum \xi^\alpha(X_\alpha)_x,$$

then

$$u^{-1}(X)=\sum Y_\beta^\alpha\xi^\beta e_\alpha.$$

This implies that the components of a vector X with respect to a frame u is given by

$$(\sum Y_\beta^0\xi^\beta, \cdots, \sum Y_\beta^{2n}\xi^\beta).$$

Let $\mathfrak{g}$ be the Lie algebra of G. The cohomology class c in $\mathrm{Hom}\,(\mathrm{R}^{2n+1}\wedge\mathrm{R}^{2n+1},$

1) To simplify notation we adopt the convention that all repeated indices under a summation sign are summed.

$R^{2n+1})/\partial \operatorname{Hom}(R^{2n+1}, \mathfrak{g})$ determined by the torsion form of a local G-connection is called the *first order structure tensor* of the G-structure $P_G(M)$.

§ 2. Contact structures and almost contact structures.

Let $y^0, y^1, \cdots, y^{2n}$ be the natural coordinate system of R^{2n+1}. Let

$$\alpha = dy^0 - \frac{1}{2}\sum (y^{i+n}dy^i - y^i dy^{i+n}).^{2)}$$

Let $\mathcal{L}$ be the sheaf of germs of all vector fields X on R^{2n+1} which satisfy

$$L_X\alpha = \lambda\alpha,$$

where λ is a function depending on X. Let $\mathcal{L}(0)$ be the stalk of $\mathcal{L}$ at the origin 0. Then $\mathcal{L}(0)$ is a non-flat filtered Lie algebra of infinite dimensions. The linear isotropy algebra $\mathfrak{g}$ of $\mathcal{L}(0)$ is the linear Lie algebra

$$\left\{ \left(\begin{array}{c|c} 0 & 0 \cdots 0 \\ \hline * & \\ \vdots & A \\ * & \end{array} \right) + \begin{pmatrix} 2\lambda & & & 0 \\ & \lambda & & \\ & & \ddots & \\ 0 & & & \lambda \end{pmatrix} \;\middle|\; A \in \mathfrak{sp}(n),\ \lambda \in R \right\}.$$

PROPOSITION 2. 1. *$\mathfrak{g}$ is involutive.*

Proof. Let $e_0, e_1, \cdots, e_{2n}$ be the natural basis for R^{2n+1}. Let

$$d_k = \dim \{t \in \mathfrak{g} \mid [t, e_0] = \cdots = [t, e_k] = 0\}.$$

Then we have

$$d_k = (n+1)(2n+1) - (k+1)(2n+1) + \frac{k(k+1)}{2},$$

and hence

$$\sum_{k=0}^{2n-1} d_k = \frac{2}{3} n(n+1)(2n+1).$$

On the other hand, since $\mathfrak{g}^{(1)} \cong \mathfrak{sp}(n)^{(1)} + \mathfrak{g}$, we have

$$\dim \mathfrak{g}^{(1)} = \frac{1}{3}(n+1)(2n+1)(2n+3).$$

Therefore

2) Indices $i, j, k, \ldots$ run over the range $1, 2, \ldots, n$.

$$\dim \mathfrak{g}^{(1)} = \dim \mathfrak{g} + \sum_{k=0}^{2n-1} d_k.$$

This implies that $\mathfrak{g}$ is involutive. (Q.E.D.)

A diffeomorphism $f\colon U \to U'$, where U and U' are open subsets of R^{2n+1}, is called a *contact transformation* if it satisfies

$$f^*\alpha = \lambda\alpha,$$

where λ is a non-zero function on U. The collection, Γ, of all such contact transformations forms an infinite, continuous pseudogroup.

Let M be a differentiable manifold of dimension $2n+1$. A Γ-structure on M is called a *contact structure*. Giving a contact structure on M is the same as giving a 1-form ω *up to a scalar factor* on M which satisfies

$$\omega \wedge (d\omega)^n \neq 0.$$

The theorem of Darboux states that a 1-form ω satisfying $\omega \wedge (d\omega)^n \neq 0$ can locally be written as

$$\omega = dx^0 - \frac{1}{2} \sum (x^{i+n} dx^i - x^i dx^{i+n}).$$

A local coordinate system in which the form ω can be written as above will be called an *admissible* coordinate.

Let Γ_0 be the subset of Γ consisting of the elements which leave the origin 0 invariant. Let $j\colon \Gamma_0 \to GL(2n+1, R)$ be defined as follows: for $f \in \Gamma_0$, $j(f)$ is the 1-jet determined by f.

Let $G = j(\Gamma_0)$. Then G is a subgroup of $GL(2n+1, R)$ whose Lie algebra is $\mathfrak{g}$, the linear isotropy algebra of $\mathcal{L}(0)$.

Let M be a differentiable manifold of dimension $2n+1$. An *almost contact structure* on M is, by definition, a reduction of the bundle of linear frames $F(M)$ to G, that is, a G-structure $P_G(M)$ on M.

Given a G-structure $P_G(M)$ on M, we can define, up to scalar factors, a pair of a 1-form $\{\omega\}$ and a 2-form $\{\Omega\}$ which satisfy $\{\omega\} \wedge \{\Omega\}^n \neq 0$. In fact, for each $x \in M$, let u be a point of $P_G(M)$ with $\pi(u) = x$. For any tangent vectors X and Y at x, set

$$\omega_x(X) = \rho \cdot \alpha_0(u^{-1}X)$$

$$\Omega_x(X, Y) = \sigma \cdot (d\alpha)_0(u^{-1}X, u^{-1}Y),$$

where α_0 and $(d\alpha)_0$ denote, respectively, the values of α and $d\alpha$ at the origin $0 \in R^{2n+1}$, and ρ and σ are scalars. From the properties of G, this definition is independent of the choice of u.

Conversely, given, up to scalar factors, a pair of a 1-form $\{\omega\}$ and a 2-form $\{\Omega\}$, let $P_G(M)$ be the set of all linear frames u satisfying

$$\{\omega\}_x(X)=\alpha_0(u^{-1}X),$$

$$\{\Omega\}_x(X, Y)=(d\alpha)_0(u^{-1}X, u^{-1}Y)$$

for any vectors X and Y at $x=\pi(u)$. Then $P_G(M)$ is a G-structure on M.

Thus giving a G-structure on M is the same as giving a pair of a 1-form up to a scalar factor $\{\omega\}$ and a 2-form up to a scale factor $\{\Omega\}$ which satisfy $\{\omega\}\wedge\{\Omega\}^n \neq 0$ at every point of M.

Let M_0 be a manifold with a contact structure. Since every Γ-structure gives rise canonically to an almost Γ-structure, M_0 has a G-structure $P_G(M_0)$, an almost contact structure.

THEOREM 2. 1. *Let $P_G(M_0)$ be the almost contact structure associated with a contact structure on M_0. Then the first order structure tensor c has the following representative:*[3)]

$$(c^0_{\alpha\beta})=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & I_n\\ 0 & -I_n & 0\end{pmatrix},$$

$$(c^i_{\alpha\beta})=(c^{i+n}_{\alpha\beta})=0.$$[4)]

Proof. A representative of c is given by the torsion tensor of a G-connection.

Let Π be a connection and ∇ the covariant differentiation with respect to Π. Then Π is a G-connection if and only if

$$(*)\qquad \nabla\omega=0.$$

Let T be the torsion tensor of Π and $T^\alpha_{\beta\gamma}$ the components of T with respect to an admissible coordinate system $(x^0, x^1, \cdots, x^{2n})$. Then the equation (*) implies

$$T^0_{j0}-\frac{1}{2}\sum x^{i+n}T^i_{j0}+\frac{1}{2}\sum x^iT^{i+n}_{j0}=0,$$

$$T^0_{j+n,0}-\frac{1}{2}\sum x^{i+n}T^i_{j+n,0}+\frac{1}{2}\sum x^iT^{i+n}_{j+n,0}=0,$$

$$T^0_{jk}-\frac{1}{2}\sum x^{i+n}T^i_{jk}+\frac{1}{2}\sum x^iT^{i+n}_{jk}=0,$$

$$T^0_{j+n,k+n}-\frac{1}{2}\sum x^{i+n}T^i_{j+n,k+n}+\frac{1}{2}\sum x^iT^{i+n}_{j+n,k+n}=0,$$

3) Since the cohomology class c is an element of $\mathrm{Hom}(R^{2n+1}\wedge R^{2n+1}, R^{2n+1})/\partial\,\mathrm{Hom}(R^{2n+1}, \mathfrak{g})$, a representative of c is in $\mathrm{Hom}(R^{2n+1}\wedge R^{2n+1}, R^{2n+1})$. In other words, a representative of c is a *torsion-type* tensor.

4) Indices $\alpha, \beta, \gamma, \cdots$ run over the range $0, 1, 2, \cdots, 2n$.

$$\delta_{jk}+T^0_{j+n,k}-\frac{1}{2}\sum x^{i+n}T^i_{j+n,k}+\frac{1}{2}\sum x^i T^{i+n}_{j+n,k}=0.$$

We can take T as follows:

$$T^0_{j+n,k}=-\delta_{jk}$$

and the other components are all zero.

Since the first order structure tensor c is independent of the choice of a G-connection, our assertion is now clear. (Q.E.D.)

§ 3. The integrability problem for almost contact structures.

Let M be a differentiable manifold of dimension $2n+1$ and $P_G(M)$ a G-structure, an almost contact structure, on M. $P_G(M)$ is said to be *integrable* if it determines a contact structure on M.

THEOREM 3. 1. *Let c_0 be the structure tensor of the almost contact structure associated with a contact structure and c the structure tensor of $P_G(M)$. Then $P_G(M)$ is integrable if and only if $c=c_0$ at every point.*

Proof. The necessity is clear. We shall prove the sufficiency.

Since $\mathfrak{g}$ is reductive, there is an invariant complement C to $\partial\,\mathrm{Hom}(R^{2n+1},\mathfrak{g})$ in $\mathrm{Hom}(R^{2n+1}\wedge R^{2n+1},R^{2n+1})$.[5] Let $\tilde{c}$ be the element in C which corresponds to c under the isomorphism $\mathrm{Hom}(R^{2n+1}\wedge R^{2n+1},R^{2n+1})/\partial\,\mathrm{Hom}(R^{2n+1},\mathfrak{g})\cong C$. Then there exists a G-connection Π on $P_G(M)$ whose torsion is $\tilde{c}$. More precisely, let τ be an element of $\mathrm{Hom}(R^{2n+1}\wedge R^{2n+1},R^{2n+1})$ whose components $(\tau^\gamma_{\alpha\beta})$ are given by

$$(\tau^0_{\alpha\beta})=\begin{pmatrix}0&0&0\\0&0&I_n\\0&-I_n&0\end{pmatrix},$$

$$(\tau^i_{\alpha\beta})=(\tau^{i+n}_{\alpha\beta})=0.$$

Then it is easily seen that τ belongs to C. This, together with Theorem 2. 1, implies that τ is just $\tilde{c}$.

Let σ: $U\to P_G(M)$, $u=\sigma(x)$, be a local cross section. If we set

$$\Theta_x(X,Y)=\tau(u^{-1}X,u^{-1}Y),$$

where $X,Y\in T_x(M)$, then Θ is a R^{2n+1}-valued 2-form on M defined in U.

Let $\tilde{\sigma}$: $U\to P_G(M)$, $\tilde{u}=\tilde{\sigma}(x)$, be an another local cross section and set

$$\tilde{\Theta}_x(X,Y)=\tau(\tilde{u}^{-1}X,\tilde{u}^{-1}Y).$$

5) C is a $(1/3)n(2n-1)(2n+1)$-dimensional subspace of $\mathrm{Hom}(R^{2n+1}\wedge R^{2n+1},R^{2n+1})$.

Then $\tilde{\Theta}$ diffrs from Θ by a scalar factor. Hence we have a global 2-form Θ up to a scalar factor.

Let T be a tensor field of type $(1,2)$ on M determined by Θ. The dimension of the space of G-connections with torsion tensor T is equal to $\dim \mathfrak{g}^{(1)}=(1/3)(n+1)(2n+1)(2n+3)$. On the other hand, let α be a 1-form on M. Then the dimension of the space of G-connections satisfying $\nabla\alpha=0$ is equal to $\dim\{t\in \mathrm{Hom}(\mathrm{R}^{2n+1}, \mathfrak{g}) \mid \alpha\circ t=0\}=2n(n+1)(2n+1)$. Since $\dim \mathrm{Hom}(\mathrm{R}^{2n+1}, \mathfrak{g})=(n+1)(2n+1)^2$, there exists a G-connection, with torsion tensor T, which satisfies $\nabla\alpha=0$.

Let $\{\omega\}$ and $\{\Omega\}$ be the classes of 1-forms and 2-forms on M determined by $P_G(M)$. Then we can find locally a 1-form ω in $\{\omega\}$ and a G-connection with torsion tensor T which satisfy

$$\nabla\omega=0.$$

The 1-form ω satisfies

$$2d\omega(X, Y)=\omega(T(X, Y))$$

for all X and Y. In fact, for all X and Y, we have

$$0=(\nabla_X\omega)(Y)=X\cdot\omega(Y)-\omega(\nabla_X Y)$$

and

$$0=(\nabla_Y\omega)(X)=Y\cdot\omega(X)-\omega(\nabla_Y X).$$

Hence we obtain

$$\begin{aligned} X\cdot\omega(Y)-Y\cdot\omega(X)-\omega([X, Y]) \\ =\omega(\nabla_X Y)-\omega(\nabla_Y X)-\omega([X, Y]), \end{aligned}$$

that is,

$$2d\omega(X, Y)=\omega(T(X, Y)).$$

If $X=\sum \xi^\alpha X_\alpha$ and $Y=\sum \eta^\alpha X_\alpha$, then

$$\begin{aligned} \omega(T(X, Y)) &=\rho\cdot\alpha_0(\Theta(X, Y)) \\ &=\rho\cdot\alpha_0(\tau(u^{-1}X, u^{-1}Y)) \\ &=\rho\cdot dy^0\left(\sum \tau^\alpha_{\beta\gamma}(u^{-1}X)^\beta(u^{-1}Y)^\gamma e_\alpha\right) \\ &=\rho\cdot\sum \tau^0_{\beta\gamma}(u^{-1}X)^\beta(u^{-1}Y)^\gamma \\ &=\rho\cdot\sum (Y^i_\beta Y^{i+n}_\gamma - Y^{i+n}_\beta Y^i_\gamma)\xi^\beta\eta^\gamma. \end{aligned}$$

On the other hand

$$2(d\alpha)_0(u^{-1}X, u^{-1}Y)$$
$$=2\sum (dy^i\wedge dy^{i+n})(u^{-1}X, u^{-1}Y)$$
$$=\sum \{dy^i(u^{-1}X)\cdot dy^{i+n}(u^{-1}Y)-dy^i(u^{-1}Y)\cdot dy^{i+n}(u^{-1}X)\}$$
$$=\sum (Y^i_\beta\xi^\beta\cdot Y^{i+n}_\gamma\eta^\gamma-Y^i_\gamma\eta^\gamma\cdot Y^{i+n}_\beta\xi^\beta)$$
$$=\sum (Y^i_\beta Y^{i+n}_\gamma-Y^{i+n}_\beta Y^i_\gamma)\xi^\beta\eta^\gamma.$$

Therefore we have

$$d\omega(X, Y)=\rho\cdot(d\alpha)_0(u^{-1}X, u^{-1}Y).$$

This implies that $d\omega\in\{\Omega\}$ and hence ω satisfies

$$\omega\wedge(d\omega)^n\neq 0.$$

Hence $\{\omega\}$ defines a contact structure on M. (Q.E.D.)

Appendix. Cosymplectic structures and almost cosymplectic structures.

Let $y^0, y^1, \cdots, y^{2n}$ be the natural coordinate system of R^{2n+1}. Let

$$\alpha=dy^0 \qquad \text{and} \qquad \beta=\sum dy^i\wedge dy^{i+n}.$$

Let $\mathcal{L}$ be the sheaf of germs of all vector fields X on R^{2n+1} which satisfy

$$L_X\alpha=0 \qquad \text{and} \qquad L_X\beta=0.$$

Let $\mathcal{L}(0)$ be the stalk of $\mathcal{L}$ at the origin 0. Then $\mathcal{L}(0)$ is a *flat* filtered Lie algebra of infinite dimensions. The linear isotropy algebra $\mathfrak{g}$ of $\mathcal{L}(0)$ is

$$\left\{\left(\begin{array}{c|c} 0 & 0\cdots 0 \\ \hline 0 & \\ \vdots & A \\ 0 & \end{array}\right) \middle| \quad A\in\mathfrak{sp}(n)\right\}.$$

$\mathcal{L}(0)$ is isomorphic with

$$R^{2n+1}+\mathfrak{g}+\mathfrak{g}^{(1)}+\mathfrak{g}^{(2)}+\cdots,$$

where $\mathfrak{g}^{(k)}$ denotes the k-th prolongation of $\mathfrak{g}$.

A local diffeomorphism f of R^{2n+1} is called a cosymplectic transformation if it satisfies

$$f^*\alpha=\alpha,$$
$$f^*\beta=\beta.$$

The collection, Γ, of all such cosymplectic transformations forms an infinite, continuous pseudogroup.

Let M be a differentiable manifold of dimension $2n+1$. A Γ-structure on M is called a *cosymplectic structure*. Giving a cosympletic structure is the same as giving a pair of a *closed* 1-form ω and a *closed* 2-form Ω which satisfy $\omega\wedge\Omega^n\neq0$. Let M be a differentiable manifold of dimension $2n+1$. Let G be a subgroup of $GL(2n+1, \mathrm{R})$ whose Lie algebra is $\mathfrak{g}$.

An *almost cosymplectic structure* on M is, by definition, a reduction of $F(M)$ to G, that is, a G-structure $P_G(M)$ on M. Giving an almost cosymplectic structure on M is the same as giving a pair of a 1-form ω and a 2-form Ω on M which satisfy $\omega\wedge\Omega^n\neq0$. The answer to the integrability problem for an almost cosymplectic structure is the following

PROPOSITION. *An almost cosymplectic structure whose structure tensor of the first order vanishes is cosymplectic.*

Proof. Let $P_G(M)$ be an almost cosymplectic structure on M and (ω, Ω) the associated pair.

Let Π be a linear connection and ∇ the covariant differentiation with respect to Π. Then Π is a G-connection if and only if

$$\nabla\omega=0 \quad \text{and} \quad \nabla\Omega=0.$$

Since the first order structure tensor of $P_G(M)$ vanishes, there exists a torsionfree G-connection.

In general, let Π be a torsionfree linear connection and η a differential form. Then

$$d\eta=\mathcal{A}(\nabla\eta),$$

where $\mathcal{A}$ is the alternation operator. Hence, let Π be a torsionfree G-connection. Then we have $d\omega=0$ and $d\Omega=0$. This proves the Proposition. (Q.E.D.)

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