KŌDAI MATH. SEM. REP.
19 (1967), 425—438

ON SLIT RECTANGLE MAPPINGS AND CONTINUITY OF EXTREMAL LENGTH

BY NOBUYUKI SUITA

§ 1. Introduction.

1. Let Ω be a plane domain and let α be its preassigned boundary component. When four curves defining vertices on α are given we discussed a conformal mapping of Ω onto a horizontally slit rectangle [11]. The mapping function could be constructed by means of an exhaustion of Ω in the directions to two opposite edges on α to be mapped into the horizontal sides of its image rectangle. In the present paper we shall deal with an alternative construction of such a mapping by means of its exhaustion in the direction to the other two edges. Indeed the limit function of a sequence of the normalized slit rectangle mappings of the members of its exhaustion gives a desired slit rectangle mapping, if the sequence of extremal distances of their two edges is positive and uniformly bounded. We think that these mapping problems should be discussed in connection with exhaustions or curve families. Such a consideration is found in Renglli [6]. In the proof of the present mapping theorem we shall use a conjugate family of the curve family clustering at the two edges of α which was first introduced by Andereian Cazacu [2] and used by Marden and Rodin in the circular-radial slit mappings [4].

Our mapping theorem has the following meaning in the problem of the continuity of extremal distances which was first discussed by Wolontis [12] and later by Strebel [9]. When we define two boundary parts on α by two defining sequences the sequence of the extremal distances of the relative boundaries of their members with the same indices is non-decreasing and its limit value, if of finite value, is not the extremal length of the family of curves joining these parts but that of the curve family clustering on them.

In the last section a few examples will be given in which the above phenomina really occur.

§ 2. Preliminary.

2. We begin with a definition of extremal length. Let Γ be a family of locally rectifiable curves, simply called a curve family in the sequel. Let $P(\Gamma)$ denote an admissible class of measurable metrics satisfying

$$(1) \qquad \int_\gamma \rho|dz| \geqq 1, \quad \gamma \in \Gamma.$$

Received April 13, 1967.

The module of Γ, denoted by $\mathrm{mod}\,\Gamma$, is the quantity

$$\inf_{\rho\in P(\Gamma)} ||\rho||^2$$

and its reciprocial is called the extremal length of Γ, denoted by $\lambda(\Gamma)$. It is often called the extremal distance of two sets, when Γ is the curve family joining them in a suitable sense.

The closure of the intersection of $P(\Gamma)$ with the space of l_2-metrics is denoted by $P^*(\Gamma)$ which is called the l_2-admissible class of Γ. There exists a unique metric ρ_0 in $P^*(\Gamma)$ such that $||\rho_0||^2 = \mathrm{mod}\,\Gamma$, if $P^*(\Gamma)\neq\phi$. The deviation of $\rho\in P^*(\Gamma)$ from ρ_0 is evaluated by an inequality [10]

$$(2) \qquad ||\rho-\rho_0||^2 \leqq ||\rho||^2 - ||\rho_0||^2.$$

3. The inequality (2) is useful for the existence proof of an extremal function. To this end we remark the following

LEMMA 1. *Let $\{\Omega_n\}_{n=1}^{\infty}$ be an increasing sequence of subdomains of Ω such that $\Omega=\cup\Omega_n$ and let $f_n(z)$ be an analytic function defined in Ω_n with finite norm $||f_n||$. Put $\rho_n=|f_n|$ in Ω_n and $=0$ outside of Ω_n. If $\{\rho_n\}$ makes a Cauchy sequence, we can select a convergent subsequence $\{f_{n_\nu}\}$ from $\{f_n\}$ such that $||f_{n_\nu}-f_0||_{\Omega_{n_\nu}}\to 0$.*

Proof. Since $\{f_n\}$ makes a normal family, we have a subsequence $\{f_{n_\nu}\}$ tending to a limit function f_0 uniformly on any compact subset of Ω. Then for any compact $K\subset\Omega$, $||f_{n_\nu}-f_0||^2_K$ is arbitrarily small for sufficiently large ν. Next we have

$$||f_{n_\nu}-f_0||_{\Omega_{n_\nu}-K} \leqq ||f_0||_{\Omega-K} + ||\rho_0||_{\Omega-K} + ||\rho_{n_\nu}-\rho_0||_{\Omega-K}$$

where ρ_0 is the strong limit of ρ_n. These inequalities complete the proof of the lemma.

REMARK. *From the above proof we know that any subsequence of $\{f_n\}$ has a strong limit and its absolute value is equal to ρ_0.*

A curve family with vanishing module is called an exceptional family and we say that a statement about Γ holds for almost all $\gamma\in\Gamma$, if it does except such a family.

4. We now state a definition of a boundary part. A boundary component α of Ω is defined by a defining sequence $\{\Delta_n\}$ [1, 11]. We first assume the relative boundary of Δ_n is an analytic curve. A boundary part β on α is a sequence of subdomains of Ω, denoted by $\{S_n\}$, such that any Δ_n contains an S_m, S_n has a single relative boundary, $S_n\supset S_{n+1}$ and $\cap\bar{S}_n=\phi$. A topological representation of α (resp. β) on the Riemann sphere is given by $\cap\mathrm{Cl}(\Delta_n)$ (resp. $\cap\mathrm{Cl}(S_n)$), where $\mathrm{Cl}(*)$ denotes the closure taken in the sphere. They are written by the same notations α and β, if no confusions occur. Their images under a topological mapping are defined by the images of their defining sequences.

A sequence of domains $\Omega_n=\Omega-\bar{S}_n$ is called an exhaustion of Ω in the direction to (or simply "towards") β. Two boundary parts are said to be *disjoint*, if the closures of two suitable members of their defining sequences taken in Ω are disjoint.

A curve tending to α (resp. β) means a curve whose suitable end arc is contained in every member of its defining sequence.

§ 3. Slit rectangle mapping.

5. Let γ_j $(1\leqq j\leqq 4)$ be four disjoint analytic curves starting from a point of Ω and tending to α. Then we can replace the defining sequence by a new sequence $\{\Delta_n\}$ such that the relative boundary of its every member intersects each γ_j precisely once. If the intersections $p_j^{(n)}$ are arranged in the negative orientation with respect to Δ_n, we say that γ_j's define vertices on α. Then we can construct four defining sequences from Δ_n and γ_j's as follows. Let $S_{j\,j+1}^{(n)}$ (modulo 4) be a subdomain of Δ_n whose relative boundary consists of end parts of γ_j and $\gamma_{j+1}\subset\Delta_n$ and the arc of the relative boundary of Δ_n between them. Thus we have two set of boundary parts α_{12} and α_{34} and α_{23} and α_{41} which are disjoint each other.

6. For simplicity's sake we assume that $\Omega\not\ni\infty$ and α is the outer boundary. Let $\{T_n\}$ be an exhaustion of Ω towards α_{23} and α_{41}. Then there exists a slit rectangle mapping φ_n such that

i) the image of the outer boundary of T_n under φ_n is the periphery of the rectangle $0<\mathrm{Re}\,\varphi_n<1$, $0<\mathrm{Im}\,\varphi_n<h_n$ with possible horizontal incisions emanating from its vertical sides, where the incisions are the sets $\varphi_n(\alpha_{12})-[0, ih_n]$ and $\varphi_n(\alpha_{34})-[1, 1+ih_n]$,

ii) the images of the relative boundaries of $S_{23}^{(n)}$ and $S_{41}^{(n)}$ under φ_n are closed subarcs in the lower and upper horizontal sides respectively,

iii) the image of the boundary components other than its outer boundary is a minimal[1] set of horizontal slits and

iv) the module of the family Γ_n of curves joining α_{12} and α_{34} within T_n is equal to h_n.

The construction of φ_n is achieved by the duplication of T_n with respect to its relative boundary [11]. The sequence $\{h_n\}$ is an increasing sequence, since Γ_n is increasing. Suppose $\lim h_n=h_0<\infty$. Then φ_n tends to a function φ_0 in the sense that $\|\varphi_n'-\varphi_0'\|_{T_n}\to 0$. φ_0 possesses the above properties i), iii) and iv) when we use h_0 and $\Gamma_0=\cup\Gamma_n$ instead of h_n and Γ_n respectively. The images $\varphi_0(\alpha_{23})$ and $\varphi_0(\alpha_{41})$ are continua (or points) on the horizontal sides. As a consequence of this property *the value* $\mathrm{Im}\,\varphi_0(z)$ *tends to* h_0 *along* γ_1 *and* γ_4, *to zero along* γ_2 *and* γ_3. These properties are shown in [11]. The last remark is easily verified from the following

LEMMA 2. *Suppose that any* $z\in\Omega$ *is contained in a curve of* Γ *and a modified curve of* γ *such that every compact subarc of it is replaced by any arc with the same end points remains in* Γ. *Let* ρ_0 *be a continuous extremal metric belonging*

1) We call a quasi-minimal set in [9] a minimal set simply.

to $P(\Gamma)$ and let Γ_z be its subfamily whose member runs through z. Then we get

$$\inf_{\gamma\in\Gamma_z}\int_\gamma \rho_0|dz|=1.$$

The lemma was proven in [9]. In fact the metric $\rho_0=|\varphi_0'|$ is an extremal metric in $P^*(\Gamma_0)$ and we know that its subfamily Λ of curves along which $\overline{\lim}\,\mathrm{Re}\,\rho_0(z)>0$ as z tends to α_{12} or $\underline{\lim}\,\mathrm{Re}\,\rho_0(z)<1$ as z tends to α_{34} is exceptional, whence ρ_0 is extremal in $P(\Gamma_0-\Lambda)$ [11]. Then in the image domain $\varphi_0(\Omega)$ $\varphi_0(\Gamma_0-\Lambda)$ is the family of curves satisfying $\lim \mathrm{Re}\, w=0$ and $\lim \mathrm{Re}\, w=1$ along them. Since $\varphi_0(\Omega)$ is dense in the rectangle $0<\mathrm{Re}\, w<1$, $0<\mathrm{Im}\, w<h_0$, we can take a point w with $\mathrm{Im}\, w=\varepsilon$ for arbitrarily small ε. In the w-plane the metric 1 is extremal for the image curve family and there exists a curve of $\varphi_0(\Gamma_0-\Lambda)$ through w and with length arbitrarily close to one satisfying $\sup \mathrm{Im}\, w<2\,\varepsilon$ on it. It belongs to some $\varphi_0(\Gamma_n)$ and hence $\varphi_0(S_{23}^{(m)})$, $m\geqq n$, lies under the line $\mathrm{Im}\, w=2\,\varepsilon$, which implies the above remark about γ_2 and γ_3.

This construction goes if only two boundary parts α_{23} and α_{41} are given. When the curve family Γ_n is replaced by such a curve family that its member intersects all the curves joining the relative boundaries of $S_{23}^{(n)}$ an $S_{41}^{(n)}$, the results in [11] are valid. The family Γ_0 is called the dividing curve family of α_{23} and α_{41} and φ_0 is called an *extremal slit rectangle mapping* with respect to the dividing curve family.

7. We now deal with an alternative construction of another type of such a mapping. A special case in which the same φ_0 occurred was discussed in [11].

Let $\{\hat{T}_n\}$ be an exhaustion of Ω towards α_{12} and α_{34} and let $\{\hat{T}_{nm}\}$ be an exhaustion of $\hat{T}_n$ towards the boundary components other than its outer boundary and α_{23} and α_{41} whose member is relatively compact and finitely connected. There exists a horizontally slit rectangle mapping ψ_{nm} such that the subarcs of the relative boundaries of $S_{12}^{(n)}$ and $S_{34}^{(n)}$ which are the boundaries of $\hat{T}_{nm}$ correspond to the vertical sides $[0, i\hat{h}_{nm}]$ and $[1, 1+i\hat{h}_{nm}]$ respectively. Here the image of the relative boundary of $\hat{T}_{nm}$ under ψ_{nm} is given by its extension over the closure of $\hat{T}_{nm}$. The function ψ_{nm} induces two extremal metrics for the following module problems. The first is the family of curves joining two edges corresponding to the vertical sides within $\hat{T}_{nm}$, say $\hat{\Gamma}_{nm}$. The standard method shows that the metric $\rho_{nm}=|\psi_{nm}'|$ is extremal for $\hat{\Gamma}_{nm}$ and $\mathrm{mod}\,\hat{\Gamma}_{nm}=\hat{h}_{nm}$.

Let $\hat{T}_{nm}^*$ be the Stoïlow compactification of $\hat{T}_{nm}$ [1] and let Γ_{nm}^* be the family of curves dividing the same edges within the complement of its outer boundary with respect to $\hat{T}_{nm}^*$. Then the metric $\mu_{nm}=|\psi_{nm}'/\hat{h}_{nm}|$ is extremal and $\mathrm{mod}\,\Gamma_{nm}^*=1/\hat{h}_{nm}$.

Put $\hat{\Gamma}_n=\cup\hat{\Gamma}_{nm}$. A continuity lemma of the extremal length given in [11] shows $\mathrm{mod}\,\hat{\Gamma}_n=\lim\,\mathrm{mod}\,\hat{\Gamma}_{nm}$. Suppose $\mathrm{mod}\,\hat{\Gamma}_n=\hat{h}_n<\infty$. Then $\hat{\rho}_{nm}$ tends to an extremal metric $\hat{\rho}_n$ for $\hat{\Gamma}_n$. From Lemma 1 and the normalization of ψ_{nm} we can deduce that ψ_{nm} tends to a function ψ_n such that $\|\psi_{nm}'-\psi_n'\|_{\hat{T}_{nm}}\to 0$ as $m\to\infty$. The ψ_n possesses the following properties:

i) The image of the outer boundary of $\hat{T}_n$ under ϕ_n is the periphery of the rectangle $0<\mathrm{Re}\,\phi_n<1$, $0<\mathrm{Im}\,\phi_n<\hat{h}_n$,

ii) the images of the relative boundaries of $S_{12}^{(n)}$ and $S_{34}^{(n)}$ are open vertical sides $(0, i\hat{h}_n)$ and $(1, 1+i\hat{h}_n)$ respectively,

iii) the image of the boundary components other than its outer boundary is a minimal set of horizontal slits and

iv) $\mathrm{mod}\,\hat{\Gamma}_n=\hat{h}_n$, where $\hat{\Gamma}_n$ is the family of curves joining the relative boundaries of $\hat{T}_n$ within it.

$P^*(\hat{\Gamma}_n)$ is decreasing with n. Suppose, furthermore, $\lim\mathrm{mod}\,\hat{\Gamma}_n=\hat{h}_0>0$. Then the inequality (2) shows that $\hat{\rho}_n$ tends to a metric $\hat{\rho}_0$. Again Lemma 1 guarantees the existence of a univalent function ϕ_0 such that $\|\phi_n'-\phi_0'\|_{T_n}\to 0$, $\|\phi_0'\|^2=\hat{h}_0$ and $\hat{\rho}_0=|\phi_0'|$.

The class $P^*(\Gamma_{nm}^*)$ is increasing with m. Hence μ_{nm} tends to a metric $\mu_n=|\phi_n'/\hat{h}_n|$ with $\|\mu_n\|^2=1/\hat{h}_n$. Let $\hat{T}_n{}^*$ be the campactificatian of $\hat{T}_n$ and let $\Gamma_n{}^*$ be the dividing curve family of its relative boundaries within the complement of its outer boundary with respect to $\hat{T}_n{}^*$. Then μ_n is an extremal metric of $P^*(\Gamma_n{}^*)$. In fact the segment L_x in the image plane; $\mathrm{Re}\,w=x$ $(0<x<1)$, $0<\mathrm{Im}\,w<\hat{h}_n$ belongs to $\phi_n(\Gamma_n{}^*)$. For any $\mu\in P(\phi_n(\Gamma_n{}^*))$ we have

$$\int_{L_x\cap\phi_n(\hat{T}_n)}\mu\,dy\geqq 1\quad\text{and}\quad\int_{L_x\cap\phi_n(\hat{T}_n)}\mu^2\,dy\geqq\frac{1}{\hat{h}_n}$$

from Schwarz's inequality. We get $\|\mu\|^2\geqq 1/\hat{h}_n$ and therefore μ_n is extremal.

Finally let Ω^* be the compactification of Ω and let $\Gamma_0{}^*$ be the family of curves dividing α_{12} and α_{34} within $\Omega^*-\alpha$. Since $\Gamma_0{}^*=\cup\Gamma_n{}^*$, $\mathrm{mod}\,\Gamma_n{}^*$ tends to $\mathrm{mod}\,\Gamma_0{}^*=\hat{h}_0$ and μ_n tends to $\mu_0=|\phi_0'/\hat{h}_0|$ which is extremal for $\Gamma_0{}^*$ [11]. We obtain

THEOREM 1. *Let ϕ_n be the slit rectangle mapping of $\hat{T}_n$ with height $\hat{h}_n$ constructed above. If the decreasing sequence $\hat{h}_n$ tends to a positive value $\hat{h}_0$, ϕ_n tends to a univalent function ϕ_0 such that $\|\phi_n'-\phi_0'\|_{\hat{T}_n}\to 0$ and $\|\phi_0'\|^2=\hat{h}_0$. Let $\Gamma_0{}^*$ be the family of curves dividing α_{12} and α_{34} within $\Omega^*-\alpha$. Then* $\mathrm{mod}\,\Gamma_0{}^*=1/\hat{h}_0$ *and the metric $\mu_0=|\phi_0'/\hat{h}_0|$ is extremal for $\Gamma_0{}^*$.*

Similar module problems as in the theorem was treated by Andreian Cazacu [2] and by Marden and Rodin [4].

8. We now investigate the image of Ω under ϕ_0. We say that a curve *clusters* on a boundary part β, if it intersects an infinite number of members of its defining sequence. We state

THEOREM 2. *Suppose* $\mathrm{mod}\,\hat{\Gamma}_1<\infty$ *and* $\lim\mathrm{mod}\,\hat{\Gamma}_n=\hat{h}_0>0$. *Then ϕ_0 constructed in no. 7 has the following properties:*

i) *$\phi_0(\alpha)$ is the periphery of the rectangle* $0<\mathrm{Re}\,\phi_0<1$, $0<\mathrm{Im}\,\phi_0<h_0$ *with possible horizontal incisions emanating from its vertical sides,*

ii) *$\phi_0(\alpha_{12})$ and $\phi_0(\alpha_{34})$ is the vertical sides* $[0, i\hat{h}_0]$ *and* $[1, 1+i\hat{h}_0]$ *with possible horizontal incisions emanating from them respectively,*

iii) $\phi_0(\partial\Omega-\alpha)$ *is a minimal set of horizontal slits and*

iv) *the module of the family of curves clustering on* α_{12} *and* α_{34}, *denoted by* $\hat{\Gamma}_0$ *is equal to* $\hat{h}_0$ and $\hat{\rho}_0=|\phi_0'|$ *is extremal for* $\hat{\Gamma}_0$.

Before proving the theorem we prepare two lemmas.

9. The first is a lemma on a conformal mapping. Let $\beta=\{S_n\}$ be a boundary part of the outer boundary component α of a finite domain Ω. Then we state

LEMMA 3. *Let* $\{f_\nu\}$ *be a convergent sequence of univalent functions in* Ω *with limit* f_0 *such that* $\|f_\nu'\|<\infty$, $\|f_\nu'-f_0'\|^2\to 0$, *and* $f_0\neq const$. *Suppose* $f_\nu(\Omega)$ *lies in the upper half plane and* $f_\nu(\beta)$ *on the real axis and* $f_\nu(\partial\Omega-\alpha)$ *is a minimal set of horizontal slits. If a point* ω *of* $f_0(\beta)$ *has a closed disc* $C: |w-\omega|\leqq r$, *such that* $\kappa=C\cap f_0(\alpha)$ *has a neighborhood* $U_n(\kappa)$ *whose intersection with* $f_0(\Omega)$ *is contained in* $f_0(S_n)$ *and if there exists a subcontinuum of* κ_ω *(not a point) containing* ω, *then* ω *is on the real axis.*

Proof. Let K be a closed disc in Ω. Put $\operatorname{Im} f_\nu(z)=v_\nu(z)$ $(\nu=0,1,\cdots)$. Let Ω^* denote the Stoïlow compactification of Ω. Consider a module problem for the family, denoted by Λ^*, of curves joining K and β within $\Omega^*-\alpha$ along which $\overline{\lim}\, v_0(z)>0$ as z tends to β. Let Λ_k^* be its subfamily consisting of the curves along which $\overline{\lim}\, v_0(z)>1/k$. Since $v_\nu(z)$ is continuous on any curve belonging to Λ^*, the metric $\rho_\nu=2k|\operatorname{grad}(v_0-v_\nu)|$ is admissible for Λ_k^* if ν is sufficient large and we get $\operatorname{mod}\Lambda_k^*=0$. Thus Λ^* is exceptional since $\Lambda^*=\cup\Lambda_k^*$.

Contrary to the assertion, suppose that there exists a $\omega\in f_0(\beta)$ with $\operatorname{Im}\omega>0$ and satisfying the conditions in the lemma. We may assume that the radius r of the disc C is less than $\operatorname{Im}\omega$ and the circle $|w-\omega|=r$ has a point of $f_0(\Omega)$ in the image w-plane. Let $U(f_0(K))$ be a simply connected neighborhood of $f_0(K)$ contained in $f_0(\Omega)$. We connect the disc $|w-\omega|<r$ and $U(f_0(K))$ by a canal within $f_0(\Omega)$ so that the union of these three domains may remain simply connected. Take a continuum κ_ω containing ω and contained the disc $|w-\omega|<r$ and consider a module problem for the curve family joining κ_ω and $f_0(K)$ within this domain. Its module has a positive value, say d. Every curve of it intersects $f_0(\beta)$ and from the assumption $f_0(\Omega)\cap U_n(\kappa)\subset f_0(S_n)$ we see that the curve contains the intersection of a image curve of Λ^* with $f_0(\Omega)$ as a subset. Hence we get $\operatorname{mod}\Lambda^*\geqq d$ which contradicts the exceptionality of Λ^*.

10. The next lemma is originally due to Strebel [7]. It was proven alternatively by means of a quasiconformal mapping [9] and the method is effective for the following

LEMMA 4. *Let* Ω *be a finite domain with outer boundary* α *and let* β_1 *and* β_2 *be disjoint two boundary parts of* α. *Suppose* $\partial\Omega-\alpha$ *is a minimal set of horizontal slits.*

We denote by Ω' *a component of* α^c *containing* Ω. *If the line* $\operatorname{Im} z=y$, $a<y<b$, *contains a subarc* L_y *with length* $l(y)$ *clustering on* β_1 *and* β_2 *within* Ω', *then we*

have

$$\text{mod}\,\hat{\Gamma} \geqq \int_a^b \frac{1}{l(y)}\,dy, \tag{3}$$

where $\hat{\Gamma}$ is the family of curves clustering on β_1 and β_2 within Ω.

Proof. Let $\{\Omega_n\}$ be an exhaustion of Ω towards α. We can make a countably connected subdomain Ω^ε of Ω with the same outer boundary α containing the relative boundaries of the defining sequences of β_1 and β_2 and $(1+\varepsilon)$-quasiconformal mapping $\Phi^\varepsilon(z)$ of Ω^ε such that $\Phi^\varepsilon(\partial\Omega^\varepsilon-\alpha)$ is a countable set of horizontal slits, $\Phi^\varepsilon(\alpha)=\alpha$ and

$$\sup_{\Omega-\Omega_n} |\Phi^\varepsilon(z)-z| \to 0 \quad \text{as} \quad n\to\infty.$$

Their construction was given in [9] and the last property not stated there is easily verified from the construction. Then the segment L_y except a countable number of y clusters on $\Phi^\varepsilon(\beta_1)$ and $\Phi^\varepsilon(\beta_2)$ which coincide with β_1 and β_2 as representations. We have

$$\text{mod}\,\hat{\Gamma} \geqq \frac{1}{1+\varepsilon}\int_a^b \frac{1}{l(y)}\,dy$$

since the curve L_y is the image of a curve of $\hat{\Gamma}$ under $\Phi^\varepsilon(z)$. Letting ε tend to zero we get the inequality (3).

11. *Proof* of Theorem 2. We first show that a point ω of $\psi_0(\alpha_{23})$ (resp. $\psi_0(\alpha_{41})$) disjoint from $\psi_0(\alpha_{12})\cup\psi_0(\alpha_{34})$, if any, lies on the real axis (resp. the line $\text{Im}\,w=\hat{h}_0$). In fact ω is not contained in $\text{Cl}(\psi_0(S_{12}^{(n_0)})\cup\psi_0(S_{34}^{(n_0)}))$ for some n_0 and we apply Lemma 3 to the sequence $\{\psi_\nu\}$ in the domain $S_{23}^{(n_0)}$ and the boundary part $\{S_{23}^{(n)}\}$ $(n\geqq n_0+1)$. ω has a closed disc $C\colon |w-\omega|\leqq r$ such that $C\cap\text{Cl}(\psi_0(S_{12}^{(n_0)})\cup\psi_0(S_{34}^{(n_0)}))=\phi$. Furthermore we may assume that C is disjoint from the closure of the image of the relative boundary of $S_{23}^{(n_0)}$. Let κ be the intersection of C with the ψ_0-image of the outer boundary of $S_{23}^{(n_0)}$. κ contains a subcontinuum $\kappa_\omega\ni\omega$ because the image is a continuum. κ has a positive distance d_n from the relative boundary of $\psi_0(S_{23}^{(n)})$, $n\geqq n_0$. Covering κ by a finite number of discs with radius $d_n/2$, we get their union as a neighborhood $U_n(\kappa)$ satisfying $\psi_0(S_{23}^{(n_0)})\cap U_n(\kappa)\subset\psi_0(S_{23}^{(n)})$. Thus the conditions in Lemma 3 are fulfilled and we obtain $\text{Im}\,\omega=0$ (resp. $\text{Im}\,\omega=\hat{h}_0$).

Next we see that $\psi_0(\alpha_{12})$ and $\psi_0(\alpha_{34})$ contain the vertical sides $[0, i\hat{h}_0]$ and $[1, 1+i\hat{h}_0]$ respectively. Put $\Delta=\psi_0(\Omega)$. Let Δ^* denote the Stoïlow compactification of Δ. Similarly as in the proof of Lemma 3 the family of curves dividing $\psi_0(\alpha_{12})$ and $\psi_0(\alpha_{34})$ within $\Delta^*-\psi_0(\alpha)$ along which $\overline{\lim}\,\text{Im}\,w>0$ as w tends to $\psi_0(\alpha_{23})$ or $\underline{\lim}\,\text{Im}\,w<\hat{h}_0$ as w tends to $\psi_0(\alpha_{41})$ is exceptional. On the other hand the module of the dividing curve family within $\Delta^*-\psi_0(\alpha)$ less this family is equal to $1/\hat{h}_0$ from Theorem 1. There exists a curve γ of the family dividing $\psi_0(\alpha_{12})$ and $\psi_0(\alpha_{34})$ and satisfying $\lim\,\text{Im}\,w=0$ and $\lim\,\text{Im}\,w=h_0$ as w tends to $\psi_0(\alpha_{23})$ and $\psi_0(\alpha_{41})$ along it respectively. The curve γ moving from the real axis to the line $\text{Im}\,w=\hat{h}_0$ divides Δ into two subdomains, one of which induces the positive orientation on it, say Δ_1, and the other is denoted by Δ_2. From the normalization of ψ_n we see $\psi_0(S_{12}^{(n)})\subset\Delta_1$ and $\psi_0(S_{34}^{(n)})\subset\Delta_2$. Thus $\mathrm{Cl}(\Delta_2)\cap(0, ih_0)=\phi$, since γ is a curve only clustering at the points on the horizontal sides. From the fact shown above there exists no point of the open sides $(0, ih_0)$ not belonging to $\psi_0(\alpha_{12})$.

12. *Continued.* We investigate the shape of each incision which is a component of $\psi_0(\alpha_{12})-[0, i\hat{h}_0]$ and $\psi_0(\alpha_{34})-[1, 1+i\hat{h}_0]$, if any. We prove that it is a harizontal segment emanating from a vertical side. To show it, suppose there exists a component E emanating from the side $[0, i\hat{h}_0]$ which is not a segment. E contains two points ω_1 and ω_2 such that $\mathrm{Im}\,\omega_1>\mathrm{Im}\,\omega_2$. Put $\mathrm{Im}\,\omega_1-\mathrm{Im}\,\omega_2=\delta$. There exists a subcontinuum E_1 of E containing ω_1 and ω_2. Since Δ is dense in the rectangle, we can take a point w_0 of Δ such that $\mathrm{Re}\,w_0<\mathrm{Re}\,\omega$ for all $\omega\in E_1$ and $\mathrm{Im}\,w_0$ is sufficiently close to $\mathrm{Im}\,(\omega_1+\omega_2)/2$. As in the proof of Lemma 4 we can construct a countably connected subdomain Δ^ε of Δ with the same outer boundary $\psi_0(\alpha)$ containing the relative boundaries of $\psi_0(S_{12}^{(n)})$ and $\psi_0(S_{34}^{(n)})$ and w_0, and a $(1+\varepsilon)$-quasi-conformal mapping Φ^ε of it such that Φ^ε fixes any boundary part, on $\psi_0(\alpha)$ of Δ^ε, $\Phi^\varepsilon(\partial\Delta^\varepsilon-\psi_0(\alpha))$ consists of a countable number of horizontal slits and the image of w_0, denoted by w_ε, is arbitrarily close to w_0. Then any curve running through the point w_ε within the complement of $\psi_0(\alpha)$ with respect to the compactification of $\Phi^\varepsilon(\Delta^\varepsilon)$ and dividing $\psi_0(\alpha_{12})$ and $\psi_0(\alpha_{34})$ has the length larger than $\hat{h}_0+\delta_0$, where δ_0 is a fixed positive number close to δ and independent of ε. We consider a module problem of the dividing curve family of $\psi_0(\alpha_{12})$ and $\psi_0(\alpha_{34})$ within the above domain in the compactification. There exists a disc $|w-w_\varepsilon|<\eta$, $\delta_1\leqq\eta/\hat{h}_0<\delta_0/2$, belonging to $\Phi^\varepsilon(\Delta^\varepsilon)$. Set

$$\rho_\varepsilon=\begin{cases} 0 & \text{in } |w-w_\varepsilon|<\eta, \\ 1/\hat{h}_0 & \text{elsewhere.}\end{cases}$$

Then ρ_ε is admissible and the module does not exceed the value $1/\hat{h}_0-\pi\delta_1^2$. Every curve of the dividing curve family $\Gamma_0{}^*$ of $\psi_0(\alpha_{12})$ and $\psi_0(\alpha_{34})$ within $\Delta^*-\psi_0(\alpha)$ contains the counter image of the restriction of a curve of the family in the domain $\Phi^\varepsilon(\Delta^\varepsilon)$. Hence letting ε tend to zero, we get $\mathrm{mod}\,\Gamma_0{}^*\leqq 1/\hat{h}_0-\pi\delta_1{}^2$, which contradicts Theorem 1. Thus we obtain the properties i) and ii).

The property iii) is a common property of minimal sets shown in [5,9].

Finally we show the property iv). Let Δ' be a component of the complement of $\psi_0(\alpha)$ containing Δ. Then the intersection of the line $\mathrm{Im}\,w=y$ $(0>y>h_0)$ with Δ' is an open segment whose two end points belong to $\psi_0(\alpha_{12})$ and $\psi_0(\alpha_{34})$ since the incisions are horizontal. We see that the segment clusters on these boundary parts considered in Δ'. To show this we prove

$$\overline{\lim}\,\mathrm{Im}\,w=\hat{h}_0 \qquad \text{along } \psi_0(\gamma_1) \text{ and } \psi_0(\gamma_4) \tag{4}$$

and

$$\underline{\lim}\,\mathrm{Im}\,w=0 \qquad \text{along } \psi_0(\gamma_2) \text{ and } \psi_0(\gamma_3). \tag{5}$$

For instance suppose $\overline{\lim}\,\mathrm{Im}\,w=h<\hat{h}_0$ along $\psi_0(\gamma_1)$. Then $\mathrm{Cl}(\psi_0(\gamma_1))$ contains a point ω of $\psi_0(\alpha_{12})$ with $\mathrm{Im}\,\omega=h$. Considering the orientation of γ_1, we can see that the arc $(ih, i\hat{h}_0)$ does not belong to $\psi_0(\alpha_{12})$ because the incisions are horizontal. This is contrary to the property ii). The relative boundaries of $\psi_0(S_{12}^{(n)})$ and $\psi_0(S_{34}^{(n)})$ cluster on the both horizontal sides and divide the boundary parts $\psi_0(\alpha_{12})$ and $\psi_0(\alpha_{34})$. Therefore the segment intersects them and hence clusters on these boundary parts. Then from Lemma 4 we have $\mathrm{mod}\,\hat{\Gamma}_0\geqq 1/\hat{h}_0$, since $l(y)\leqq\hat{h}_0$. We have seen in no. 8 that the metric $\hat{\rho}_0=|\psi_0'|$ is the limit of the extremal metrics $\hat{\rho}_n=|\psi_n'|\in P^*(\hat{\Gamma}_n)\subset P^*(\hat{\Gamma}_0)$. Hence $\hat{\rho}_0\in P^*(\hat{\Gamma}_0)$ and the equality $\|\hat{\rho}_0\|^2=1/\hat{h}_0$ proves $\mathrm{mod}\,\hat{\Gamma}_0=1/\hat{h}_0$ and $\hat{\rho}_0$ is extremal.

By the way the equalities (4) and (5) correspond to the remark stated in no. 5.

§ 4. Extremal property.

13. We call ψ_0 *an extremal horizontally slit rectangle mapping* of Ω with respect to the clustering curve family of α_{12} and α_{34}. We gave an extremal property to the extremal slit rectangle mapping with respect to the dividing curve family [11] and we show a similar extremal property for ψ_0.

Let $\mathfrak{F}(\hat{\Gamma}_0)$ (resp. $\mathfrak{F}(\Gamma_0)$) be the family of univalent functions $f(z)$ satisfying $0<\mathrm{Re}\,f(z)<1$, $\inf\,\mathrm{Im}\,f(z)=0$ $(z\in\Omega)$ and $\lim\,\mathrm{Re}\,f(z)=0$ and $\lim\,\mathrm{Re}\,f(z)=1$ along almost all $\gamma\in\hat{\Gamma}_0$ (resp. Γ) as z tends to α_{12} and α_{34} respectively, where $\hat{\Gamma}_0$ and Γ_0 are the clustering curve family of α_{12} and α_{34} and the dividing curve family of α_{23} and α_{41}. Put $H(f)=\sup\,\mathrm{Im}\,f(z)$ $(z\in\Omega)$. We state

THEOREM 3. *If* $\mathrm{mod}\,\hat{\Gamma}_0$ *(resp.* $\mathrm{mod}\,\Gamma_0$*) is finite and positive, the function* ψ_0 *(resp.* φ_0*) is the unique function which minimizes the quantity* $H(f)$ *within* $\mathfrak{F}(\hat{\Gamma}_0)$ *(resp.* $\mathfrak{F}(\Gamma_0)$*).*

Proof. The metric $\hat{\rho}_0=|\psi_0'|$ is extremal and $\rho=|f'|$ is admissible. The inequality (2) shows

$$\|\rho-\rho_0\|^2\leqq\|\rho\|^2-\|\rho_0\|^2\leqq H(f)-H(\psi_0).$$

A domain Ω is said *a minimal horizontally slit rectangle* with respect to $\hat{\Gamma}_0$ (resp. Γ_0) if it is the image of a domain under ψ_0 (resp. φ_0). For the minimal slit rectangle the extremal slit rectangle mapping coincides with the identity. We now give its characterization which is an analogue given in [10].

COROLLARY 1. *Let* Ω *be a domain whose outer boundary* α *is the periphery of the rectangle* $0<\mathrm{Re}\,z<1$, $0<\mathrm{Im}\,z<h$ *with possible horizontal incisions and let* α_{12} *and* α_{34} *be two disjoint boundary parts containing the vertical sides* $[0, ih]$ *and* $[1, 1+ih]$ *respectively. Suppose the extremal distance of the relative boundaries of the first member of their defining sequences is positive. Then any two of the following three imply the minimality of* Ω *with respect to the clustering curve family* $\hat{\Gamma}_0$ *of* α_{12} *and* α_{34}.

i) $\partial\Omega-\alpha$ *is minimal,*

ii) $\lim \mathrm{Re}\, z=0$ *and* $\lim \mathrm{Re}\, z=1$ *along almost all* $\gamma\in\hat{\Gamma}_0$ *as* z *tends to* α_{12} *and* α_{34} *respectively, and*

iii) $\mathrm{mod}\,\hat{\Gamma}_0=h$.

Conversely the minimal slit rectangle with respect to the clustering curve family has all the above properties.

Proof. The assumption about the extremal distance is needed for the construction of ψ_0. We first assume the conditions ii) and iii). From ii) the metric $\rho_0=1$ $(=|\mathrm{grad}\,\mathrm{Re}\, z|)$ is l_2-admissible. The equality $\|\rho_0\|^2=h$ shows the extremality of ρ_0 and we get $\psi_0=z$.

Next from i) and Lemma 4 we have $\mathrm{mod}\,\hat{\Gamma}_0\geqq h$. Then ii) implies the extremality of ρ_0 as before.

Finally suppose i) and iii). From iii) and Lemma 4 we can deduce that the projection of the set of all incisions into the imaginary axis has a vanishing measure. Let $\hat{\Gamma}'$ denote the subfamily of $\hat{\Gamma}_0$ satisfying the conditions in ii). Then Lemma 4 (slightly modified) is applied and we get $\mathrm{mod}\,\hat{\Gamma}'\geqq h$. The metric ρ_0 is extremal for $\hat{\Gamma}'$ and $\mathrm{mod}\,\hat{\Gamma}'=h$. Since $\hat{\Gamma}'\subset\hat{\Gamma}_0$, having the same module, both families have the common extremal metric ρ_0. The converse is a direct consequence of Theorem 2.

Similarly we have

COROLLARY 2. *Let* Ω *be the domain given in Corollary* 1 *and let* α_{23} *and* α_{41} *be two boundary parts of* α *contained in the horizontal sides* $[0,1]$ *and* $[ih, 1+ih]$ *respectively. Then any two of the three conditions in Corollary* 1 *imply the minimality of* Ω *with respect to the dividing curve family* Γ_0 *when* $\hat{\Gamma}_0$ *is replaced by* Γ_0.

Conversely the minimal slit rectangle has all the above properties.

§ 5. Examples.

14. Let γ_j's be four curves defining vertices on α. In general the slit rectangle mapping φ_0 with respect to the dividing curve family of α_{23} and α_{41} does not coincide with the function ψ_0 with respect to the clustering curve family of α_{12} and α_{34}. It is shown by the following

EXAMPLE 1. Let Ω be the square with vertices at i, 0, 1 and $1+i$ and let γ_j's be

$$\gamma_1: z_1(t)=i+\frac{1}{4}e^{i7\pi/4}t,\quad \gamma_2: z_2(t)=\frac{1}{4}(t+i(1-t)\sin t^{-1}+i),$$

$$\gamma_3: z_3(t)=1+\frac{1}{4}e^{i3\pi/4}t,\quad \gamma_4: z_4(t)=1+i+\frac{1}{4}e^{i5\pi/4}t \quad (0<t<1).$$

The domain Ω is a minimal slit rectangle (without slits) with respect to the clustering curve family of α_{12} and α_{34} since the conditions i) and ii) of Corollary 1 is complied. But it is not minimal with respect to the dividing curve family of α_{23} and α_{41}, because α_{23} is the union of the segments $[0, i/2]$ and $[0, 1]$ and the assumption $\alpha_{23}\subset[0, 1]$ does not hold. Thus $\varphi_0\neq\psi_0=z$.

This example shows a *discontinuity* of the extremal distance. To see it let $\{S_{12}^{(n)}\}$ and $\{S_{34}^{(n)}\}$ be defining sequences of α_{12} and α_{34}. Let χ be the family of curves joining α_{12} and α_{34} defined by these defining sequences. It is a subclass of the family Γ_0 defined in no. 6. Then we have

$$\operatorname{mod}\Gamma_0=\operatorname{mod}\chi<\lim\operatorname{mod}\hat{\Gamma}_n=\operatorname{mod}\hat{\Gamma}_0.$$

In fact we know from Theorems 2 and 3 that

$$\operatorname{mod}\Gamma_0=h_0<\lim\operatorname{mod}\hat{\Gamma}_n=\operatorname{mod}\hat{\Gamma}_0,$$

where $h_0=H(\varphi_0)$ is the height of the image rectangle of Ω. The equality $H(\varphi_0)=\operatorname{mod}\chi$ was proved in [11], since any curve of χ joins two edges in a member of $\{T_n\}$.

15. Next we construct a minimal slit rectangle with respect to the dividing curve family of α_{23} and α_{41}, one of which is a point.

EXAMPLE 2. Let E be a countable number of segments

$$\frac{1}{2}+s+\frac{i}{n+1},\quad |s|\leqq\frac{1}{2}\left(1-\frac{2}{n+1}\right)\qquad(n=1, 2,\cdots),$$

and let Ω be the complement of E with respect to the square in Example 1. Let γ_1 and γ_4 be the curves in Example 1 and let

$$\gamma_2: z_2(t)=e^{i2\pi/5}t/4\quad\text{and}\quad\gamma_3: z_3(t)=e^{i3\pi/10}t/4.$$

α_{23} is the point at the origin, α_{41} is the segment $[i, 1+i]$ and Ω is a minimal slit rectangle with respect to the dividing curve family of them. In fact the set of countable slits is minimal [9]. Let $\{T_n\}$ be an exhaustion of Ω towards α_{23} and α_{41}. Let Γ_n be the dividing curve family of the relative boundaries of T_n. Then any curve of Γ_n not tending to the both vertical sides clusters either at an inner point of the lower horizontal side or at the origin. The family of the former curves is exceptional since they have an infinite length while so is the latter family since the extremal distance of the vertical side $[0, i]$ and the boundary element at the origin in the Carathéodory sense not contained in α_{12} in the interior of the outer bounday of T_n is infinite. Then the conditions i) and ii) of Corollary 2 is satisfied and Ω is minimal.

For this domain Ω the extremal slit rectangle mapping with respect to the clustering curve family of α_{12} and α_{34} can not be constructed by the procedure in no. 7. Indeed let $\{\hat{T}_n\}$ be an exhaustion of towards α_{12} and α_{34} and let $\hat{\Gamma}_n$ be the family of curves joining the relative boundaries of $\hat{T}_n$. Any $\hat{T}_n$ contains a sector

$$\frac{3}{10}\pi<\arg z<\frac{2}{5}\quad\text{and}\quad 0<|z|<\delta_n,\ \delta_n>0.$$

For every $\rho \in P(\hat{\Gamma}_n)$ Schwarz's inequality shows

$$\int_{3\pi/10}^{2\pi/5} \rho^2(re^{i\theta})\, r d\theta \geqq \frac{\pi}{10\,r} \qquad (0<r<\delta_n).$$

Thus we get $\|\rho\|^2=\infty$ and $\operatorname{mod} \Gamma_n=\infty$, which denies the construction of ψ_0 by means of an exhaustion.

On the other hand we can see that the module of the clustering curve family $\hat{\Gamma}_0$ of α_{12} and α_{34} remains finite and is equal to one. To show it we have from Lemma 4 $\operatorname{mod} \hat{\Gamma}_0 \geqq 1$. We set for $r<e^{-1}$

$$\rho_r = \begin{cases} \,||z| \log |z||^{-1} & \text{in } \ |z|<r, \\ \,1 & \text{elsewhere.} \end{cases}$$

ρ_r is admissible for $\hat{\Gamma}_0$ since $\rho_r=1$, the length of a curve clustering at a point of $(0,1)$ is infinite and for a curve γ tending to the origin

$$\int_\gamma \rho_r |dz| \geqq \int_0^{r_0} \rho_r\, dr=\infty, \quad r_0=\sup_{z\in\gamma}|z|.$$

Hence we have $\operatorname{mod}\hat{\Gamma}_0=1$ since $\|\rho_r\|^2 \to 1$ as $r\to 0$.

This example shows the following *discontinuity*;

Although α_{12} and α_{34} are disjoint

$$\operatorname{mod}\hat{\Gamma}_0 < \lim \operatorname{mod}\hat{\Gamma}_n = \infty.$$

In the original formulation of Strebel's continuity lemma [8] the defining sequence of a boundary component satisfies $\Delta_n \supset \bar{\Delta}_{n+1}$ and in our definition of boundary part it is replaced by $\Delta_n \supset \Delta_{n+1}$. We can easily modify the defining sequences of the boundary parts of two examples in such a way that the above strictly decreasing condition is satisfied.

16. Our last example is a minimal slit rectangle such that α_{12} is the point at the origin, ψ_0 exists and coincides with φ_0.

EXAMPLE 3. Take a square with vertices at i, 0, 1 and $1+i$ and curves in Example 2. Put

$$A=\sum_{\nu=1}^{\infty} 2^{-\nu^3}, \ t_n=A-\sum_{\nu=1}^{n} 2^{-\nu^3} \quad \text{and} \quad a_n=z_3(t_n) \qquad (n=1, 2, \cdots)$$

Let L_n be a segment on γ_3 with center a_n such that L_n is disjoint from the other a_ν and the module of the family, say Γ_n^1, of curve joining L_n and γ_2 is less than $1/n^2$. We denote by L_n' the open complementary segment between L_n and L_{n+1} and its center by b_n. We take a point b_n' on the ray b_n+s $(s>0)$ in such a way that the distance of γ_2 and the circle $|z-b_n'|=2^{-n^3}$ is equal to 2^{-n}. We replace the segment L_n' by a rectilinear curve consisting of two horizontal segments between the end points of L_n' and the vertical diameter of the circle and its subarc between these horizontal segments. We consider two sufficiently short vertical segments emanating below and above from the upper and lower end points of L_n' such that the module of the family, say Γ_n^2, of the curves joining γ_2 and these segments is less than $1/n^2$. Take two horizontal slits between the end points of these vertical segments and the circle and let E_1 be the union of them. We denote by γ_3^* the union of these rectilinear curves and L_n's and connect the end points of γ_3^* and γ_4 in the square less E_1 by an analytic curve such that the union of these three is a Jordan arc. It divides the square into two domains. In the domain disjoint from E_1 we can take a set E_2 of a countable number of horizontal slits which is closed in it and makes every point of $(0,1)$ inaccessible.

Let Ω be the complement of the union of E_1 and E_2 with respect to the square and let γ_1, γ_2, γ_3^* and γ_4 be the curves defining the vertices. Then Ω is a minimal slit rectangle for the both family dividing α_{23} and α_{41} and clustering on α_{12} and α_{34}. From the construction α_{23} is the point at the origin.

In fact similarly as in Example 2 we have the minimality of Ω for the dividing curve family and $\mathrm{mod}\,\hat{\Gamma}_0=1$. We only verify the construction of ψ_0. By Hersch's lemma [3] it is sufficient to prove the module of the family $\hat{\Gamma}_1$ of curves joining γ_2 and γ_3^* is finite. Any curve of $\hat{\Gamma}_1$ intersects at least one of the L_n's, the auxiliary vertical segments and the circles $|z-b'|=2^{-n^3}$. The family of the third curves, denoted by Γ_n^3, has an admissible metric $\rho_n=(|z|\log 2^{n^3-n})^{-1}$. We get

$$\mathrm{mod}\,\Gamma_n^3 \leqq \frac{2\pi}{(n^3-n)\log 2}.$$

By the same Hersch's lemma we have

$$\mathrm{mod}\,\hat{\Gamma}_1 \leqq \sum \mathrm{mod}\,\Gamma_n^1 + \sum \mathrm{mod}\,\Gamma_n^2 + \sum \mathrm{mod}\,\Gamma_n^3.$$

The three series in the right hand are convergent. We obtain the minimality of Ω with respect to the clustering curve family of α_{12} and α_{34}.

REFERENCES

[1] AHLFORS, L., AND L. SARIO, Riemann surfaces. Princeton Univ. Press (1960).
[2] ANDREIAN CAZACU, C., Überlagerungseigenschaften Riemannscher Flächen. Rev. Math. Pures Appl. **6** (1961), 685–702.
[3] HERSCH, J., Longueurs extrémales et théorie des fonctions. Comment. Math. Helv. **29** (1955), 301–337.
[4] MARDEN, A., AND B. RODIN, Extremal and conjugate extremal distance on open Riemann surfaces with application to circular-radial slit mapping. Acta Math. **115** (1966) 237–269.
[5] OIKAWA, K., Minimal slit regions and linear operator method. Kōdai Math. Sem. Rep. **17** (1965), 187–190.
[6] RENGLLI, H., Zur konformen Abbidung auf Normalgebiete. Comment. Math. Helv. **31** (1956), 5–40.

[7] STREBEL, K., A remark on the extremal distance of two boundary components. Proc. Nat. Sci. U.S.A. **40** (1954), 942–844.

[8] ———, Die extremale Distanz zweier Enden einer Riemannschen Fläche. Ann. Acad. Sci. Fenn. **179** (1955), 21.

[9] SUITA, N., Minimal slit domains and minimal sets. Kōdai Math. Sem. Rep. **17** (1966), 166–186.

[10] ———, On radial slit disc mappings. Ibid. **18** (1965), 219–228.

[11] ———, On a continuity lemma of extremal length and its applications to conformal mappings. Ibid. **19** (1967), 125–137.

[12] WOLONTIS, W., Properties of conformal invariants. Amer. J. Math. **74** (1952), 587–606.

DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY.