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THE SPHERICAL DERIVATIVE OF REGULAR AND MEROMORPHIC FUNCTIONS OF BOUNDED CHARACTERISTIC

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1. Introduction. Let D be the open unit disc in the complex plane. If $f(z)$ is a meromorphic function in D, we denote the spherical derivative of $f(z)$ by

$$\rho(f(z))=\frac{|f'(z)|}{1+|f(z)|^2}.$$

Lehto and Virtanen, and Noshiro obtained the following results [2], [4].

THEOREM A. *A non-constant $f(z)$, meromorphic in D, is normal if and only if it satisfies an inequality*

$$\sup_{|z|<1}(1-|z|)\rho(f(z))\leqq C,$$

where C is a finite constant.

COROLLARY A. *If (z) is normal meromorphic in D, its characteristic function $T(r)$ fulfills the following relation:*

$$T(r)=O\left(\log\frac{1}{1-r}\right).$$

Let $h(r)$ be a positive function such that $h(r)=o(r)$ $(r\to 0)$. The connection between $\rho(f(z))$ and Picard's Theorem is shown by the following result of Gavrilov [1].

THEOREM B. *Let $f(z)$ be meromorphic in D. If for a sequence $\{z_n\}$, $\lim_{n\to\infty}|z_n|=1$ and*

$$\lim_{n\to\infty}h[(1-|z_n|^2)]\rho(f(z_n))=+\infty,$$

then, Picard's Theorem holds for $f(z)$ in the union of any infinite subsequence of the discs

$$D_n=\{z\in D\,|\,\sigma(z,z_n)<\varepsilon(1-|z_n|^2)^{-1}h[(1-|z_n|^2)]\},$$

for each $\varepsilon>0$, where $\sigma(z,z_n)$ is the non-Euclidean hyperbolic distance between z and z_n in D.

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The aim of this paper is to show that for regular functions in D the boundedness of $T(r)$ imposes a restriction on the growth of $\rho(f(z))$, but does not for meromorphic functions.

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2. THEOREM 1. *Suppose that $f(z)$ is regular and of bounded type in D. Then*

$$(1-|z|)\rho(f(z))\leqq e^{c/(1-|z|)}, \tag{1}$$

where c is a positive constant.

The result (1) is sharp in such a sence that no improvement is possible on the order of $1/(1-|z|)$ in (1).

THEOREM 2. *Let $\varphi(r)$, $0\leqq r<1$, denote any positive monotonically increasing function with*

$$\lim_{r\to 1}\varphi(r)=+\infty.$$

Then, there exists a meromorphic function $f(z)$ of bounded type in D such that

$$\lim_{\substack{r\to 1\\ |z|=r}}\frac{1-|z|}{\varphi(r)}\rho(f(z))=+\infty.$$

Proof of Theorem 1. By the condition of $f(z)$ we can assume that

$$f(z)=\frac{\pi_2(z)}{\pi_1(z)},$$

where $\pi_i(z)$ is regular, $|\pi_i(z)|<1$, $(i=1,2)$, and $\pi_1(z)\neq 0$. Schwarz's lemma yields inequalities

$$|\pi_i'(z)|\leqq\frac{1}{1-|z|}\qquad (i=1,2).$$

Hence

$$|\pi_2(z)\pi_1'(z)|\leqq\frac{1}{1-|z|},\quad |\pi_2'(z)\pi_1(z)|\leqq\frac{1}{1-|z|}. \tag{2}$$

On the other hand if $\pi(z)$ is regular and of bounded type in D, $\pi(z)$ satisfies

$$|\pi(z)|\leqq e^{c/(1-|z|)}, \tag{3}$$

where c is a fixed positive constant [5]. From (2) and (3)

$$(1-|z|)\rho(f(z))\leqq(1-|z|)\frac{|\pi_2(z)\pi_1'(z)|+|\pi_2'(z)\pi_1(z)|}{|\pi_1(z)|^2}$$

$$\leqq e^{c/(1-|z|)},$$

since $1/(\pi_1(z))^2$ is regular and of bounded type in D. We require a lemma to prove the latter of theorem 1 and theorem 2. Let

$$x_n = 1-e^{-n} \quad (n=1, 2, \cdots).$$

Consider the Blaschke-product

$$B(z) = \prod_{n=1}^{\infty} \frac{x_n - z}{1-x_n z}. \tag{4}$$

LEMMA. *For the product* (4) *we have*

$$\varliminf_{n\to\infty} (1-x_n)|B'(x_n)| \geqq B, \tag{5}$$

where B *is a positive constant.*

Proof. Let

$$B_p(z) = \prod_{n=1}^{p-1} \frac{x_n - z}{1-x_n z} \prod_{n=p+1}^{\infty} \frac{x_n - z}{1-x_n z}.$$

Then,

$$B'(x_p) = -\frac{1}{1-x_p{}^2} B_p(x_p).$$

We write

$$\begin{aligned} |B_p(x_p)| &= \prod_{n=1}^{p-1} \frac{x_p - x_n}{1-x_p x_n} \prod_{n=p+1}^{\infty} \frac{x_n - x_p}{1-x_p x_n} \\ &\equiv T_1(p) T_2(p). \end{aligned} \tag{6}$$

Since, for $n=1, 2, \cdots, p-1$

$$\frac{x_p - x_n}{1-x_p x_n} > \frac{1-e^{n-p}}{1+e^{n-p}},$$

we infer that

$$T_1(p) > \prod_{n=1}^{p-1} \frac{1-e^{-n}}{1+e^{-n}},$$

and hence

$$\varliminf_{p\to\infty} T_1(p) \geqq \prod_{n=1}^{\infty} \frac{1-e^{-n}}{1+e^{-n}} \equiv A > 0.$$

Similarly,

$$\varliminf_{p\to\infty} T_2(p) \geqq A.$$

Therefore

$$\varliminf_{p\to\infty} |B_p(x_p)| \geqq B > 0.$$

By (6) this establishes (5).

Completion of Theorem 1. We form the function

$$f(z)=B(z)e^{(1+z)/(1-z)}.$$

$f(z)$ is evidently regular and of bounded type in D. For $p=1, 2, \cdots$,

$$f'(x_p)=B'(x_p)e^{(1+x_p)/(1-x_p)}$$

and hence, from the lemma

$$\varliminf_{p\to\infty} \frac{(1-x_p)\rho(f(x_p))}{e^{(1+x_p)/(1-x_p)}} \geqq B>0.$$

Therefore we have the remainder in theorem 1.

Proof of Theorem 2. Choose a natural number K_1 such that

$$K_1^2|B'(x_1)|\left\{1-\left(x_1+\frac{1}{K_1^2}\right)x_1\right\}>\varphi(x_1), \quad e^{-1}>\frac{1}{K_1^2}. \tag{7}$$

After natural numbers $K_1, K_2, \cdots, K_{p-1}$ are defined, we choose K_p with inequalities

$$\begin{gathered} K_p^2|B'(x_p)|\left\{1-\left(x_p+\frac{1}{K_p^2}\right)x_p\right\}>\varphi(x_p)\cdot p, \quad e^{-p}>\frac{1}{K_p^2}, \\ K_i<K_p, \qquad i=1, 2, \cdots, p-1. \end{gathered} \tag{8}$$

By this process we have a increasing sequence $\{K_p\}, p=1, 2, \cdots$, satisfying (8). Let

$$B_1(z)=\prod_{n=1}^{\infty}\frac{(x_n+1/K_n^2)-z}{1-(x_n+1/K_n^2)z}.$$

With $B(z)$ in the above lemma we form the function

$$f(z)=\frac{B(z)}{B_1(z)}.$$

It is obvious that $f(z)$ is meromorphic and of bounded type in D. Now,

$$\rho(f(x_p))=\left|\frac{B'(x_p)}{B_1(x_p)}\right|.$$

Hence, from (8) we obtain

$$\rho(f(x_p))>\varphi(x_p)\cdot p$$

and

$$\varlimsup_{\substack{|z|\to 1\\ |z|=r}} \frac{\rho(f(z))}{\varphi(r)} \geqq \varlimsup_{p\to\infty} \frac{\rho(f(x_p))}{\varphi(x_p)}=+\infty.$$

This holds too, when we take as $\varphi(r)$, $\varphi(r)/(1-r)$.

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