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REMARKS ON HAYMAN'S THEOREMS

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1. Introduction.

In this paper we shall give two results related to interesting theorems given by Hayman [3]. First he obtained a theorem of Picard type by mapping the unit circle onto a sector as follows:

THEOREM A. *Suppose that* $f(z)$ *is meromorphic and of finite order in the plane. Let* $1/2\leqq\rho<\infty$ *and let* $z_\nu(a)=r_\nu e^{i\theta_\nu}$ *be the roots of the equation* $f(z)=a$, *lying in the sector* $|\arg z|<\pi/2\rho$, *Then either*

(I) $f(z)$ *has bounded characteristic in* $|\arg z|<\pi/2\rho$, *in which case*

$$(*)\qquad \sum_\nu \frac{\cos\rho\theta_\nu}{r_\nu^\rho}$$

converges for every a; *or*

(II) $f(z)$ *has unbounded characteristic in* $|\arg z|<\pi/2\rho$, *in which case the series* (*) *diverges for every* a *with at most two exceptions.*

Also he proved the following theorem:

THEOREM B. *Suppose that* $f(z)$ *is meromorphic in* $|z|<1$ *and of finite order and that* $n(r)$ *is the total number of roots, contained in* $|z|\leqq r$, *of the equations* $f(z)=a_\nu$, $\nu=1$ *to* q, *where the* a_ν *are* $q\geqq 3$ *distinct complex numbers one of which may be infinite. Then if*

$$\varlimsup_{r\to1}(1-r)n(r)\leqq k<\infty,$$

we have

$$\varlimsup_{r\to1}\frac{(1-r)\log M(r,f)}{\log\dfrac{1}{1-r}}\leqq 2\lambda$$

where $\lambda=k/(q-1)$ *or* $k/(q-2)$ *according as the* a_ν *are all finite or not.*

To formulate our theorems, we define $T(r,f,\Delta)$, the characteristic function of $f(z)$ in the sector Δ: $|\arg z|<\pi/2\rho$, as follows (the definition is due to Tsuji [5]):

$$S(r,f,\Delta)=\frac{1}{\pi}\int_{-\pi/2\rho}^{\pi/2\rho}\int_0^r\frac{|f'(te^{i\theta})|^2}{(1+|f(te^{i\theta})|^2)^2}tdtd\theta,$$

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$$T(r, f, \Delta)=\int_0^r \frac{S(t, f, \Delta)}{t} dt.$$

If $T(r, f, \Delta)=O(1)$ as $r\to\infty$, $f(z)$ is called to be of bounded characteristic in the sector Δ. Mapping Δ onto the unit circle $|w|<1$ by the function

$$w=\frac{z^\rho-1}{z^\rho+1},$$

$F(w)=f[z(w)]$ is of bounded characteristic in $|w|<1$ if and only if $f(z)$ is of bounded characteristic in Δ. It is easy to see that if $T(r, f, \Delta)=O(\log r)$, $T(R, F) =O(\log(1/(1-R)))$ where $T(R, F)$ is the characteristic in the Ahlfors-Shimizu sense, i.e.

$$S(R, F)=\frac{1}{\pi}\int_0^{2\pi}\int_0^R \frac{|F'(te^{i\varphi})|^2}{(1+|F(te^{i\varphi})|^2)^2} t dt d\varphi,$$

$$T(R, F)=\int_0^R \frac{S(t, F)}{t} dt.$$

We also write $M(r, f)=\sup_{|z|=r}|f(z)|$.

Now we give extensions of theorem A and theorem B in the following.

THEOREM 1. *Suppose that $f(z)$ is meromorphic and of finite order in the plane. Let $1/2\leqq\rho<\infty$ and let $g(z)$ be a meromorphic function in the plane and $z_\nu(f=g) =r_\nu e^{i\theta_\nu}$ be the roots of the equation $f(z)=g(z)$, lying in the sector Δ: $|\arg z|<\pi/2\rho$. Then either*

(I) *$f(z)$ has bounded characteristic in $|\arg z|<\pi/2\rho$, in which case*

$$(*)\qquad \sum_\nu \frac{\cos\rho\theta_\nu}{r_\nu^\rho}$$

converges for every $g(z)$, if $g(z)$ has bounded characteristic in $|\arg z|<\pi/2\rho$; or

(II) *$f(z)$ has unbounded characteristic in $|\arg z|<\pi/2\rho$, in which case the series (*) diverges for every $g(z)$ with at most two exceptions, if $f(z)$ satisfies $T(r, f, \Delta) =O(\log r)$ and $g(z)$ has bounded characteristic in $|\arg z|<\pi/2\rho$; or $f(z)$ satisfies*

$$\overline{\lim_{r\to\infty}} \frac{T(r, f, \Delta)}{\log r}=\infty$$

and $g(z)$ satisfies $T(r, g, \Delta)=O(\log r)$.

THEOREM 2. *Suppose that $f(z)$ is a meromorphic function of finite order satisfying $\bar{N}(r, f)=o[T(r, f)]$ in $|z|<1$ and $n_q(r)$ is the total number of roots, contained in $|z|\leqq r$, of the equations $f(z)=g_\nu(z)$, $\nu=1$ to q, roots of order being counted p times if $p\leqq q$, and q times if $q<p$, where the $g_\nu(z)$ are $q\geqq3$ distinct meromorphic functions satisfying $T(r, g_\nu)=o[T(r, f)]$ in $|z|<1$, one of which may be constantly infinite. Then if*

$$\overline{\lim_{r\to1}}(1-r)n_q(r)\leqq k<\infty,$$

we have

$$\varliminf_{r\to 1}\frac{(1-r)\log M(r,f)}{\log\dfrac{1}{1-r}}\leqq 2\lambda,$$

where $\lambda=k/(q-2)$ *or* $k/(q-1)$ *according as one of the* $g_\nu(z)$ *is constantly infinite or not.*

2. Some lemmas.

The notations $n(r,f)$, $n(r,1/(f-a))$, $\bar{n}(r,f)$ and $m(r,f)$ are used in the sense of Nevanlinna [2], [4]. We shall suppose for simplicity that $f(0)=0$ and also write

$$N(r,f)=\int_0^r\frac{n(t,f)}{t}\,dt,\qquad N\left(r,\frac{1}{f-a}\right)=\int_0^r\frac{1}{t}\,n\left(t,\frac{1}{f-a}\right)dt.$$

Then $T(r,f)=m(r,f)+N(r,f)$ is the Nevanlinna characteristic of $f(z)$. The notation $\bar{N}_q(r,f)$ is defined as follows [1]:

$$\bar{N}_q(r,f)=\int_0^r\frac{\bar{n}_q(t,f)}{t}\,dt,$$

where $\bar{n}_q(t,f)$ denotes the number of poles of $f(z)$ in the circle $|z|\leqq t$, poles of order p being counted p times if $p\leqq q$, and q times if $q<p$ for a positive integer q. And particularly $\bar{N}_1(r,f)=\bar{N}(r,f)$.

First in order to prove our theorem 1, we divide meromorphic functions in the unit circle into following three classes:

(1°) $T(r,f)=O(1)$ as $r\to 1$, in which case $f(z)$ is called to be of bounded type;

(2°) $$T(r,f)=O\left(\log\frac{1}{1-r}\right)$$

and $f(z)$ does not belong to (1°);

(3°) $$\varlimsup_{r\to 1}\frac{T(r,f)}{\log\dfrac{1}{1-r}}=\infty$$

i.e. $f(z)$ belongs neither to (1°) nor (2°).

If $f(z)$ belongs to the class (3°), $f(z)$ is called to be admissible in $|z|<1$ (for the Nevanlinna theory [2]).

Now we need some lemmas on which the proof of our theorem 1 is based. Next lemma is classical and is independent of the behaviour of the growth of $T(r,f)$.

LEMMA 1. *Let* $z_\nu(a)$ *be zeros of* $f(z)-a$ *and* $r_\nu(a)=|z_\nu(a)|$, *then*

$$\lim_{r\to 1}N(r,a),\qquad \lim_{r\to 1}\int^r n(t,a)dt,\qquad \lim_{r\to 1}\sum_{r_\nu\leqq r}(1-r_\nu(a))$$

are convergent or divergent at the same time.

We remark that in general this lemma is true if $z_\nu(f=g)$ are zero points of $f(z)-g(z)$, where $f(z)$ and $g(z)$ are meromorphic functions.

LEMMA 2. *Suppose that $f(z)$ and $g(z)$ are meromorphic in $|z|<1$ and $T(r,f)=O(1)$, then*

$$\lim_{r\to 1}\sum_{r_\nu\leqq r}(1-r_\nu(f=g))$$

converges for every $g(z)$, if $g(z)$ belongs to the class (1°).

Proof. Put $F=f-g$, we have

$$T(r,F)+O(1)=T\left(r,\frac{1}{F}\right).$$

This gives

$$T(r,f)+T(r,g)+O(1)\geqq N\left(r,\frac{1}{f-g}\right).$$

Since $T(r,f)=O(1)$ and $T(r,g)=O(1)$, we get

$$N\left(r,\frac{1}{f-g}\right)=O(1).$$

It is easy to see that if $f(z)$ and $g(z)$ are meromorphic in $|z|<1$ and the number of zero points of $f(z)-g(z)$ is finite for any $g(z)$ belonging to the class (1°), then $T(r,f)=O(1)$.

We have the following result of Nevanlinna [4].

THEOREM C. *If $f(z)$ is meromorphic and admissible in $|z|<1$ and $g_\nu(z)$, $\nu=1,2,3$, are distinct meromorphic functions satisfying*

$$T(r,g_\nu(z))=o[T(r,f)] \quad \textit{as} \quad r\to 1,$$

then

$$\text{(1)} \qquad [1+o(1)]T(r,f)\leqq\sum_{\nu=1}^{3}N\left(r,\frac{1}{f(z)-g_\nu(z)}\right)+S(r,f) \quad \textit{as} \quad r\to 1,$$

where

$$S(r,f)=O\left[\log^+T(r,f)+\log\frac{1}{1-r}\right]$$

as $r\to 1$ outside a set E such that

$$\int_E\frac{1}{1-r}\,dr<\infty.$$

Using this theorem, we can now conclude that if $f(z)$ is admissible, then

$$\lim_{r\to 1}\sum_{r_\nu\leqq r}(1-r_\nu(f=g))=\infty \qquad \text{for every } g(z),$$

with two possible exceptions.

In fact, if

$$\lim_{r\to 1}\sum_{r_\nu \leqq r}(1-r_\nu(f=g_i))<\infty \qquad (i=1,2,3),$$

then

$$\lim_{r\to 1} N\left(r, \frac{1}{f-g_i}\right)<\infty,$$

so that

$$\lim_{r\to 1} T(r,f)<\infty$$

in view of (1). This contradicts the admissibility of $f(z)$.

Now, we remark that if $f(z)$ is not admissible in $|z|<1$, we cannot apply theorem C to our purpose. Hence we need the following lemma, due to Hayman [3], which will ignore the admissiblity of $f(z)$.

LEMMA 3. *Suppose D is a bounded domain containing $|z|<1$ and properly containing a set of arcs $z=e^{i\theta}$, $\alpha_\nu<\theta<\beta_\nu$, where*

$$\sum(\beta_\nu-\alpha_\nu)=2\pi, \qquad \sum(\beta_\nu-\alpha_\nu)\log\frac{1}{\beta_\nu-\alpha_\nu}<\infty.$$

Suppose further that $f(z)$ is meromorphic of finite order in D. Then, we have

$$S(r,f)=O[\log T(r,f)]+O(1) \qquad \text{as } r\to 1, \tag{2}$$

where

$$S(r,f)=m\left(r,\frac{f'}{f}\right)+m\left(r,\sum_{\nu=1}^{q}\frac{f'}{f-a_\nu}\right)+O(1),$$

and the a_ν, $\nu=1$ to q, are distinct finite complex numbers.

The phrase 'of finite order in D' we used above has the same meaning as given in Hayman [3].

Next in order to prove our theorem 2, we define counting function $N(r,f)$ after Hayman as follows:

$$N(r,f)=\int_{r/2}^{r}\frac{n(t,f)}{t}\,dt, \qquad \bar{N}_q(r,f)=\int_{r/2}^{r}\frac{\bar{n}_q(t,f)}{t}\,dt.$$

We remark that our results will not be affected by this definition.

Also we define $m(z_0,r,f(z))=m(r,f(z_0+z))$, with similar definitions for $N, \bar{N}, T$ etc., where the circle $|z-z_0|<r$ moves in $|z|<1$ and r remains greater than a positive constant, so that z_0 lies well inside $|z|<1$.

Now we have the following generalization of theorem C which has been proved by Chuang [1] for more than three functions $g_\nu(z)$ in slightly modified form.

THEOREM D. *Suppose that $f(z)$ is meromorphic in $|z|<1$ and $g_\nu(z)$, $\nu=1$ to q $(q\geqq 3)$, are distinct meromorphic functions satisfying $T(r,g_\nu)=o[T(r,f)]$. Then we have*

$$[q-1-o(1)]T(r,f)<\sum_{\nu=1}^{q}\bar{N}_q\left(r,\frac{1}{f-g_\nu}\right)+q\bar{N}(r,f)+S(r,f),$$

where

$$S(r,f)=O\left[\log T(r,f)+\log\frac{1}{1-r}\right]$$

as $r\to 1$, *for all* $r<1$ *if* $f(z)$ *is of finite order, and outside a set* E *such that*

$$\int_E \frac{1}{1-r}dr<\infty$$

if its order is infinite.

Particularly, if $\bar{N}(r,f)=o[T(r,f)]$, then we have

$$[q-1-o(1)]T(r,f)<\sum_{\nu=1}^{q}\bar{N}_q\left(r,\frac{1}{f-g_\nu}\right)+S(r,f)$$

which is analogous to the inequality of theorem C.

Further this result is expressible in the following form:

THEOREM D'. *Suppose that* $f(z)$ *is meromorphic in* $|z|<1$ *and* $g_\nu(z)$, $\nu=1$ *to* q $(q\geqq 3)$, *are distinct meromorphic functions satisfying* $T(r,g_\nu)=o[T(r,f)]$. *Then*

$$[q-1-o(1)]T(z_0,r,f)<\sum_{\nu=1}^{q}\bar{N}_q\left(z_0,r,\frac{1}{f-g_\nu}\right)+q\bar{N}(z_0,r,f)+S(z_0,r,f). \tag{3}$$

We need some lemmas.

LEMMA 4. *Under the assumption of theorem* 2, *given* $\varepsilon>0$, *we can find* $r_0<1$ *such that if* $r_0<r<1$, $|z_0|<1-r$, *we have*

$$\sum_{\nu=1}^{q}\bar{N}_q\left(z_0,r,\frac{1}{f-g_\nu}\right)<(k+\varepsilon)\log\frac{1}{1-|z_0|-r}.$$

Proof. If we write

$$n_q(z_0,r)=\sum_{\nu=1}^{q}\bar{n}_q\left(z_0,r,\frac{1}{f-g_\nu}\right),$$

then for $r>r_1(\varepsilon)$, $|z_0|<1-r$, we have

$$n_q(z_0,r)\leqq n_q(0,|z_0|+r)<\frac{k+\varepsilon}{1-|z_0|-r}.$$

And the subsequent proof of lemma 4 can be discussed similarly as in the proof of lemma 13 in [3] and so we omit.

LEMMA 5. *If* r_0 *is defined as in lemma* 4 *and*

$$\lambda_1=\frac{k}{q-2-o(1)}\quad \text{or}\quad \frac{k}{q-1-o(1)}$$

according as one of the $g_\nu(z)$ *is constantly infinite or not, then we have for* $r_0<r<1$, $|z_0|<1-r$

$$m(z_0,r,f)<(\lambda_1+\varepsilon)\log\frac{1}{1-|z_0|-r}+K_1\log\frac{1}{1-r}, \tag{4}$$

$$N(0,r,f)<K_2\log\frac{1}{1-r}. \tag{5}$$

Proof. We have by theorem D′

$$[q-1-o(1)]T(z_0,r,f)<\sum_{\nu=1}^{q}\bar{N}_q\left(z_0,r,\frac{1}{f-g_\nu}\right)+S(z_0,r,f) \tag{6}$$

where

$$S(z_0,r,f)=\sum_{k=1}^{q}m\left(z_0,r,\frac{f^{(k)}}{f}\right)+\sum_{\nu=1}^{q}\sum_{k=1}^{q}m\left(z_0,r,\frac{(f-g_\nu)^{(k)}}{f-g_\nu}\right)+O(1).$$

Since $f(z)$ has finite order in $|z|<1$ we have

$$m\left(z_0,r,\frac{f^{(k)}}{f}\right)<K_3\log\frac{1}{r(1-r)}$$

and consequently we get

$$S(z_0,r,f)=O\left(\log\frac{1}{r(1-r)}\right).$$

Hence, combining this with (6) and taking $z_0=0$ we can deduce

$$N(0,r,f)<(k+\varepsilon)\log\frac{1}{1-r}+K_4\log\frac{1}{1-r},$$

in view of lemma 4 for $r>r_0$.

Also (3) yields, using lemma 4,

$$[q-2-o(1)]m(z_0,r,f)<\sum_{\nu=1}^{q-1}\bar{N}_{q-1}\left(z_0,r,\frac{1}{f-g_\nu}\right)+(q-1)\bar{N}(z_0,r,f)+S(z_0,r,f),$$

namely, noting that $\bar{N}_{q-1}\leqq\bar{N}_q$,

$$[q-2-o(1)]m(z_0,r,f)<(k+\varepsilon)\log\frac{1}{1-r-|z_0|}+K_5\log\frac{1}{1-r}.$$

This proves (4) if one of the $g_\nu(z)$ is constantly infinite, so that $\lambda_1=k/(q-2-o(1))$. If $g_1(z)$ to $g_q(z)$ are all non-constant meromorphic, we apply (6). This gives

$$[q-1-o(1)]T(z_0,r,f)<\sum_{\nu=1}^{q}\bar{N}_q\left(z_0,r,\frac{1}{f-g_\nu}\right)+K_6\log\frac{1}{1-r},$$

hence, we get also

$$[q-1-o(1)]m(z_0,r,f)<(k+\varepsilon)\log\frac{1}{1-r-|z_0|}+K_6\log\frac{1}{1-r}.$$

Now again (4) follows, since $\lambda_1=k/(q-1-o(1))$ in this case, where K_i ($i=1$ to 6) are absolute constants.

3. Proofs of theorem 1 and theorem 2.

Proof of Theorem 1. Following Hayman we put

$$w=\frac{z^\rho-1}{z^\rho+1},\qquad z=\left(\frac{1+w}{1-w}\right)^{1/\rho},$$

so that the sector $|\arg z|<\pi/2\rho$ corresponds to $|w|<1$.

The functions

$$f(z)=f[z(w)]=F(w), \qquad g(z)=g[z(w)]=G(w)$$

are meromorphic in the plane cut from -1 to $-\infty$ and $+1$ to $+\infty$, along the real axis. Now we can take for D in lemma 3 the part of this cut plane lying in $|w|<2$.

The same argument as in [3] will show that $F(w)$ also has finite order in D and satisfies the assumption of lemma 3. Hence, if $G_\nu(w)$, $\nu=1,2,3$, are distinct meromorphic functions satisfying $T(r,G_\nu)=O(1)$ in $|w|<1$, then we have

$$[1+o(1)]T(r,F)<\sum_{\nu=1}^{3} N\left(r,\frac{1}{F-G_\nu}\right)+S(r,F) \qquad \text{as} \quad r\to 1,$$

where $S(r,F)$ denotes the term in (2). This inequality enables us to deduce that if $F(w)$ has unbounded characteristic in D and satisfies

$$T(r,F)=O\left(\log\frac{1}{1-r}\right)$$

in $|w|<1$, then we have

$$\lim_{r\to 1}\sum_{r_\nu\leqq r}(1-r_\nu(F=G))=\infty \tag{7}$$

for every $G(w)$, with two possible exceptions, if $G(w)$ has bounded characteristic in $|w|<1$. Next, if $F(w)$ is admissible in $|w|<1$ and $G(w)$ satisfies

$$T(r,G(w))=o[T(r,F)] \qquad \text{as} \quad r\to 1,$$

then (7) holds for every $G(w)$, with two possible exceptions, in virtue of theorem C. In view of lemma 2, if both $F(w)$ and $G(w)$ have bounded characteristic in $|w|<1$, then $\lim_{r\to 1}\sum_{r_\nu\leqq r}(1-r_\nu(F=G))$ converges for every $G(w)$.

Let $w_\nu(F=G)$ be the roots of $F(w)=G(w)$ in $|w|<1$. Put now $z=re^{i\theta}$. Then

$$|w|^2=\left|\frac{z^\rho-1}{z^\rho+1}\right|^2,$$

so that

$$1-|w|^2=\frac{4r^\rho\cos\rho\theta}{r^{2\rho}+2r^\rho\cos\rho\theta+1}.$$

Also w_ν runs over the roots of $F(w)=G(w)$ in $|w|<1$, while z runs over the roots $r_\nu e^{i\theta_\nu}$ of $f(z)=g(z)$ in $|\arg z|<\pi/2\rho$. The convergence of $\sum_\nu(1-|w_\nu|)$ is equivalent to that of $\sum_\nu(1-|w_\nu|^2)$ and hence to that of $\sum_\nu(\cos\rho\theta_\nu)/r_\nu^\rho$ since $r_\nu\to\infty$. This completes the proof of Theorem 1.

Proof of Theorem 2. We first take a fixed $j\geqq 4$ and for given μ, $0<\mu<1$, we define μ_j by $1-\mu_j=(1-\mu)/j$. Then for any fixed j and $\varepsilon>0$ we can find μ as near 1 as we please such that

$$n(0,\mu_j,f)<\frac{K}{1-\mu_j}=\frac{jK}{1-\mu}. \tag{8}$$

For if (8) is false for all $\mu'<\mu_j$ then for $\mu'<\mu<1$ we confront the contradiction with (5). Next we define after Hayman

$$f^*(z)=f(z)2^{-N}\prod_{i=1}^{N}\left(\frac{z-b_i}{1-\bar{b}_i z}\right),$$

where b_i ($i=1$ to N) are the poles of $f(z)$ lying in $|(z-b_i)/(1-\bar{b}_i z)|\leqq 1/2$. We now choose $z=z(\mu)$ so that $|z(\mu)|=\mu$ and $|f^*(z)|=M(\mu,f^*(z))$. We suppose $\mu>r$ and choose z_0 so that $\arg z_0=\arg z(\mu)$, $1-|z_0|-r=1-\mu_j$. Then we deduce, using (4), for $t_0=\max(r/2, |z(\mu)-z_0|)=|z(\mu)-z_0|$

$$\begin{aligned}
\log M(\mu,f^*(z)) &< \frac{r+|z(\mu)-z_0|}{r-|z(\mu)-z_0|}\left[m(z_0,r,f)+K'\int_{t_0}^{r} n(z_0,t,f)/t\,dt\right]\\
&< \frac{2j}{(j-1)(1-\mu)}\left[(\lambda_1+\varepsilon)\log\frac{1}{1-|z_0|-r}+K'\frac{r-t_0}{t_0}n(0,|z_0|+r,f)\right] \qquad (9)\\
&< \frac{2j}{(j-1)(1-r)}\left[(\lambda_1+\varepsilon)\log\frac{1}{1-r}+O(1)\right]
\end{aligned}$$

for all r such that $\mu\leqq r\leqq(1+\mu)/2$ where μ is sufficiently near 1. Now there is an r in this range for which

$$\log M(r,f)<\log M(r,f^*(z))+K''n\left(0,\frac{3+\mu}{4},f\right),$$

and we get $\mu_j\geqq(3+\mu)/4$ in view of (8). Since (9) holds for some r arbitrarily near 1 we can deduce

$$\varlimsup_{r\to1}\frac{(1-r)\log M(r,f)}{\log\dfrac{1}{1-r}}\leqq\varlimsup_{r\to1}\frac{2j(\lambda_1+\varepsilon)}{j-1}=\frac{2j(\lambda+\varepsilon)}{j-1},$$

and since j is large and ε small as we please, the result as required follows.

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