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CURVATURE-PRESERVING TRANSFORMATIONS OF K-CONTACT RIEMANNIAN MANIFOLDS

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Let M be a contact Riemannian manifold with a contact form η, the associated vector field ξ, (1, 1)-tensor field ϕ and the associated Riemannian metric g. If ξ is a Killing vector field, M is said to be a K-contact Riemannian manifold. Further, M is said to be normal, if ϕ satisfies the relation

$$(\nabla_X\phi)(Y)=g(X, Y)\xi-\eta(Y)X$$

for any vector fields X and Y on M, where ∇ is the covariant differentiation with respect to g.

Recently Okumura [2] got the following result:

(A) *In a normal contact Riemannian manifold, any curvature-preserving infinitesimal transformation is an infinitesimal isometry.*

On the other hand, Sakai [3] got the result:

(B) *Any affine transformation of a K-contact Riemannian manifold is an isometry.*

In this note, we prove the next theorem which covers the above (A) and (B):

THEOREM. *Let M, N be K-contact Riemannian manifolds, then any curvature-preserving transformation of M to N is an isometry.*

The proof of our theorem has similar aspect to that in [3]. In an m-dimensional K-contact Riemannian manifold we have

$$R_1(\xi, X)=(m-1)\eta(X), \tag{1}$$

$$R(X, \xi)\xi=-X+\eta(X)\xi \tag{2}$$

for any vector field X on M, where R_1 and R denote the Ricci curvature and Riemannian curvature tensor [1].

§ Proof of the theorem.

We denote the corresponding tensors in N by " ′ ". Let φ be a curvature-preserving transformation of M to N and let x be an arbitrary point of M, and we put $y=\varphi x$. By X, Y, Z, W we denote vector fields on M. In any Riemannian manifold we have

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(3) $$'g_y('R(\varphi X, \varphi Y)\varphi Z, \varphi W)=-'g_y('R(\varphi X, \varphi Y)\varphi W, \varphi Z),$$

where φ stands for the differential of φ. As φ is curvature-preserving: $\varphi(R(X, Y)Z) ='R(\varphi X, \varphi Y)\varphi Z$, we have

(4) $$(\varphi^*{}'g)_x(R(X, Y)Z, W)=-(\varphi^*{}'g)_x(R(X, Y)W, Z).$$

If we put $Y=Z=W=\xi$, using (2) we have

(5) $$(\varphi^*{}'g)_x(X, \xi)=\sigma_x\eta_x(X),$$

where $\sigma_x=(\varphi^*{}'g)_x(\xi, \xi)$. Next we put $Y=Z=\xi$, then

(6) $$(\varphi^*{}'g)_x(-X+\eta(X)\xi, W)=-(\varphi^*{}'g)_x(R(X, \xi)W, \xi).$$

Replace X in (5) by W or $R(X, \xi)W$, then (6) turns to

$$-(\varphi^*{}'g)_x(X, W)+\sigma_x\eta_x(X)\eta_x(W)=-\sigma_x\eta_x(R(X, \xi)W).$$

On the other hand, as

$$\eta_x(R(X, \xi)W)=g_x(R(X, \xi)W, \xi)=g_x(X, W)-\eta_x(X)\eta_x(W),$$

we have $(\varphi^*{}'g)_x(X, W)=\sigma_x g_x(X, W)$. Namely φ is a conformal transformation. Next we prove $\sigma_x=1$. Put $X=\varphi\xi$ in (1), then we get

(7) $$'R_{1y}('\xi, \varphi\xi)=(m-1)'\eta_y(\varphi\xi).$$

Since φ also leaves R_1 invariant, we have

(8) $$'R_{1y}('\xi, \varphi\xi)=R_{1x}(\varphi^{-1}{}'\xi, \xi)=(m-1)\eta_x(\varphi^{-1}{}'\xi).$$

From (7) and (8), $'\eta_y(\varphi\xi)=\eta_x(\varphi^{-1}{}'\xi)$ follows. While we obtain

$$\begin{aligned}'\eta_y(\varphi\xi)&='g_y('\xi, \varphi\xi)='g_y(\varphi\cdot\varphi^{-1}{}'\xi, \varphi\xi)\\&=(\varphi^*{}'g)_x(\varphi^{-1}{}'\xi, \xi)=\sigma_x g_x(\varphi^{-1}{}'\xi, \xi)\\&=\sigma_x\eta_x(\varphi^{-1}{}'\xi).\end{aligned}$$

Thus we get $\sigma_x=1$ or $'\eta_y(\varphi\xi)=0$. Suppose that $'\eta_y(\varphi\xi)=0$, then we have $'R_y(\varphi\xi, '\xi)'\xi =-(\varphi\xi)_y$ by (2) and so

$$\begin{aligned}\sigma_x&='g_y(\varphi\xi, \varphi\xi)\\&='g_y('R('\xi, \varphi\xi)'\xi, \varphi\xi)\\&='g_y(\varphi\cdot R(\varphi^{-1}{}'\xi, \xi)\varphi^{-1}{}'\xi, \varphi\xi)\\&=(\varphi^*{}'g)_x(R(\varphi^{-1}{}'\xi, \xi)\varphi^{-1}{}'\xi, \xi)\\&=-\sigma_x g_x(R(\varphi^{-1}{}'\xi, \xi)\xi, \varphi^{-1}{}'\xi)\\&=\sigma_x g_x(\varphi^{-1}{}'\xi, \varphi^{-1}{}'\xi)=1.\end{aligned}$$

Therefore σ is equal to 1 on M, this completes the proof.

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