ON CONTINUOUS-TIME MARKOV PROCESSES WITH REWARDS, II

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1. Let X_t, $t \geqq 0$ be a continuous-time Markov process with the state space $\boldsymbol{S}=\{1, 2, \cdots, N\}$. The quantity a_{jk} is defined as follows: in a short time interval dt, the process that is now in state $j \in \boldsymbol{S}$ will make a transition to state $k \in \boldsymbol{S}$ with probability $a_{jk}dt+o(dt)$, $(j \neq k)$. The probability of two or more transitions is $o(dt)$. Then this Markov process is described by the transition-rate matrix $A=(a_{jk})$, where the diagonal elements of A are defined by $a_{jj}=-\sum_{k \neq j} a_{jk}$, $(j=1, 2, \cdots, N)$. Now, let us suppose that the system earns a reward at the rate of r_{jj} dollars per unit time during all the time it occupies state j. Suppose further that when the system makes a transition from state j to state k $(j \neq k)$, it receives a reward of r_{jk} dollars. In the previous paper, we have given a limiting property of the total reward $R(t)$ that the system will earn in a time t, by assuming that the multiplicity of every root of $\det(sI-A)=0$ is 1. In this paper, we shall prove this property in the case where the roots of the equation $\det(sI-A)=0$ are not necessarily simple.

2. Let $\varphi_{jt}(\theta) \overset{\text{def}}{=} E\{e^{i\theta R(t)} | X_0=j\}$ be the characteristic function of $R(t)$ given that $X_0=j$ and put

$$\Phi_j(\theta, s)=\int_0^\infty \varphi_{jt}(\theta)e^{-st}dt \qquad (j=1, 2, \cdots, N), \tag{1}$$

where s is a positive-valued variable. Introducing the $N \times N$ matrix

$$A(\theta) \overset{\text{def}}{=} \begin{pmatrix} a_{11}+i\theta r_{11} & a_{12}e^{i\theta r_{12}} & \cdots & a_{1N}e^{i\theta r_{1N}} \\ a_{21}e^{i\theta r_{21}} & a_{22}+i\theta r_{22} & \cdots & a_{2N}e^{i\theta r_{2N}} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{N1}e^{i\theta r_{N1}} & a_{N2}e^{i\theta r_{N2}} & \cdots & a_{NN}+i\theta r_{NN} \end{pmatrix}$$

and the vectors

$$\boldsymbol{\Phi}(\theta, s) \overset{\text{def}}{=} \begin{bmatrix} \Phi_1(\theta, s) \\ \vdots \\ \Phi_N(\theta, s) \end{bmatrix} \quad \text{and} \quad \boldsymbol{e} \overset{\text{def}}{=} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix},$$

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we have the following lemma the proof of which has been shown in section 3 of [1].

LEMMA. *We have that*

$$\Phi(\theta, s)=(sI-A(\theta))^{-1}\boldsymbol{e}, \tag{2}$$

where I is the $N\times N$ identity matrix.

We shall now prove the following

THEOREM. *If A is indecomposable, then the distribution function of the random variable $[R(t)-gt]/\sqrt{t}$ converges as $t\to\infty$ to a normal law, where g is a constant.*

Proof. Let $\alpha_0=0$, $\alpha_1, \cdots, \alpha_{N-1}$ be the roots of the equation $\det(sI-A)=0$ which need not all be distinct. It has been proved in Remark 1 of [1] that $\alpha_0=0$ is a simple root and $\Re(\alpha_j)<0$ $(j=1, 2, \cdots, N-1)$. Let $\zeta_0(\theta), \zeta_1(\theta), \cdots, \zeta_{N-1}(\theta)$ be the roots of det $(sI-A(\theta))=0$ such that

$$\zeta_0(\theta)\to 0, \quad \zeta_1(\theta)\to\alpha_1, \quad \cdots, \quad \zeta_{N-1}(\theta)\to\alpha_{N-1} \qquad \text{as } \theta\to 0.$$

Then there exist positive constants ε and θ_0 such that $\zeta_0(\theta)$ is analytic in θ for $|\theta|<\theta_0$, and

$$-\varepsilon\leqq\Re(\zeta_0(\theta)), \qquad \Re(\zeta_l(\theta))<-2\varepsilon \qquad (l=1, 2, \cdots, N-1) \text{ for } |\theta|<\theta_0. \tag{3}$$

From Lemma, the following expression for $\Phi(\theta, s)$ is obtained:

$$\Phi(\theta, s)=\frac{\boldsymbol{\sigma}(\theta)}{s-\zeta_0(\theta)}+\frac{\boldsymbol{g}(s, \theta)}{(s-\zeta_1(\theta))(s-\zeta_2(\theta))\cdots(s-\zeta_{N-1}(\theta))}, \tag{4}$$

where $\boldsymbol{\sigma}(\theta)$ is an N-dimensional vector-valued function which is continuous at $\theta=0$, and

$$\boldsymbol{g}(s, \theta)=\boldsymbol{g}_0(\theta)s^{N-2}+\boldsymbol{g}_1(\theta)s^{N-3}+\cdots+\boldsymbol{g}_{N-2}(\theta) \tag{5}$$

is a polynomial whose degree is at most $N-2$ and whose coefficients $\boldsymbol{g}_0(\theta)$, $\boldsymbol{g}_1(\theta), \cdots, \boldsymbol{g}_{N-2}(\theta)$ are vector-valued function of θ which are continuous at $\theta=0$. It is easy to see that the polynomial (5) may be written as follows:

$$\begin{aligned}\boldsymbol{g}(s, \theta)&=\boldsymbol{\tau}_1(\theta)\cdot(s-\zeta_2(\theta))(s-\zeta_3(\theta))\cdots(s-\zeta_{N-1}(\theta))\\&\quad+\boldsymbol{\tau}_2(\theta)\cdot(s-\zeta_3(\theta))\cdots(s-\zeta_{N-1}(\theta))\\&\quad+\cdots+\boldsymbol{\tau}_{N-2}(\theta)\cdot(s-\zeta_{N-1}(\theta))+\boldsymbol{\tau}_{N-1}(\theta),\end{aligned} \tag{6}$$

where the vector-valued functions $\boldsymbol{\tau}_1(\theta), \boldsymbol{\tau}_2(\theta), \cdots, \boldsymbol{\tau}_{N-1}(\theta)$ are continuous functions of $\boldsymbol{g}_0, \cdots, \boldsymbol{g}_{N-2}$ and $\zeta_1, \cdots, \zeta_{N-1}$. It follows from the continuity of $\boldsymbol{g}$'s and ζ's that $\boldsymbol{\tau}_j(\theta)$ is continuous at $\theta=0$ $(j=1, 2, \cdots, N-1)$, and therefore there exists a constant $K<+\infty$ such that

(7) $$|\tau_{jk}(\theta)| \leqq K \quad \text{for } |\theta| < \theta_0 \quad (j=1, 2, \cdots, N-1;\ k=1, 2, \cdots, N),$$

where $\tau_{jk}(\theta)$ is the k-th component of $\boldsymbol{\tau}_j(\theta)$. From (4) and (6), we have

(8) $$\boldsymbol{\Phi}(\theta, s) = \frac{\boldsymbol{\sigma}(\theta)}{s-\zeta_0(\theta)} + \frac{\boldsymbol{\tau}_1(\theta)}{s-\zeta_1(\theta)} + \frac{\boldsymbol{\tau}_2(\theta)}{(s-\zeta_1(\theta))(s-\zeta_2(\theta))} + \cdots + \frac{\boldsymbol{\tau}_{N-1}(\theta)}{(s-\zeta_1(\theta))(s-\zeta_2(\theta))\cdots(s-\zeta_{N-1}(\theta))}.$$

If we define $\gamma_k(t, \theta)\ (k=1, \cdots, N-1;\ t \geqq 0;\ |\theta| < \theta_0)$ to be

$$\gamma_1(t, \theta) = e^{\zeta_1(\theta)t}$$

and

$$\gamma_k(t, \theta) = \int_0^t e^{\zeta_k(\theta)u} \gamma_{k-1}(t-u, \theta)du \qquad (k \geqq 2),$$

then we have from (8) that

(9) $$\boldsymbol{\varphi}_t(\theta) \overset{\text{def}}{=} \begin{bmatrix} \varphi_{1t}(\theta) \\ \vdots \\ \varphi_{Nt}(\theta) \end{bmatrix} = e^{\zeta_0(\theta)t}\boldsymbol{\sigma}(\theta) + \gamma_1(t, \theta)\boldsymbol{\tau}_1(\theta) + \gamma_2(t, \theta)\boldsymbol{\tau}_2(\theta) + \cdots + \gamma_{N-1}(t, \theta)\boldsymbol{\tau}_{N-1}(\theta).$$

Since $\Re(\zeta_l(\theta)) < -2\varepsilon$ for $|\theta| < \theta_0$ and $l \neq 0$, it is easily verified by induction that

(10) $$|\gamma_k(t, \theta)| \leqq \frac{t^{k-1}}{(k-1)!} e^{-2\varepsilon t} \qquad (k=1, 2, \cdots, N-1).$$

It follows from (7), (9) and (10) that

(11) $$|\varphi_{jt}(\theta) - e^{\zeta_0(\theta)t}\sigma_j(\theta)| \leqq Ke^{-2\varepsilon t}\left(1+t+\cdots+\frac{t^{N-2}}{(N-2)!}\right) \qquad (j=1, \cdots, N;\ |\theta| < \theta_0),$$

where $\sigma_j(\theta)$ is the j-th component of $\boldsymbol{\sigma}(\theta)$. Since $\boldsymbol{\varphi}_t(0) = \boldsymbol{e}$, we have $\boldsymbol{\sigma}(0) = \boldsymbol{e}$. From (11), we have for every θ and sufficiently large t

$$\left|\varphi_{jt}\left(\frac{\theta}{t}\right) - e^{\zeta_0(\theta/t)t}\sigma_j\left(\frac{\theta}{t}\right)\right| \leqq Ke^{-2\varepsilon t}\left(1+t+\cdots+\frac{t^{N-2}}{(N-2)!}\right).$$

This shows that the characteristic function $\varphi_{jt}(\theta/t)$ converges as $t \to \infty$ for all θ to the continuous function $e^{\zeta_0'(0)\theta}$, because $\zeta_0(0)=0$ and so $\zeta_0(\theta/t)\cdot t$ converges to $\zeta_0'(0)\cdot\theta$ as $t \to \infty$. It follows that $e^{\zeta_0'(0)\theta}$ must be a characteristic function, and therefore that $\zeta_0'(0)$ is a pure imaginary. Now define the real number $g \overset{\text{def}}{=} -i\zeta_0'(0)$ and consider the family $\{[R(t)-gt]/\sqrt{t}\}_{t>0}$ of random variables. The characteristic function of the conditional distribution of the random variable $[R(t)-gt]/\sqrt{t}$ given that $X_0=j$ is

(12) $$\psi_{jt}(\theta) \overset{\text{def}}{=} E\{e^{i\theta[R(t)-gt]/\sqrt{t}} | X_0=j\} = e^{-ig\sqrt{t}\theta}\varphi_{jt}\left(\frac{\theta}{\sqrt{t}}\right).$$

From (11) and (12), we have for every θ

$$\left|\psi_{jt}(\theta)-e^{-ig\sqrt{t}\,\theta}e^{\zeta_0(\theta/\sqrt{t})t}\sigma_j\left(\frac{\theta}{\sqrt{t}}\right)\right|=O(t^{N-2}e^{-2\varepsilon t}) \tag{13}$$

as $t\to\infty$. On the other hand, we have for every θ

$$\begin{aligned} &e^{-ig\sqrt{t}\,\theta}e^{\zeta_0(\theta/\sqrt{t})t}\sigma_j\left(\frac{\theta}{\sqrt{t}}\right) \\ &=e^{-ig\sqrt{t}\,\theta+\zeta_0'(0)\sqrt{t}\,\theta+\zeta_0''(0)\theta^2/2+O(1/\sqrt{t})}\cdot\sigma_j\left(\frac{\theta}{\sqrt{t}}\right) \\ &\to e^{\zeta_0''(0)\theta^2/2} \end{aligned} \tag{14}$$

as $t\to\infty$, because $\zeta_0(0)=0$ and $g=-i\zeta_0'(0)$. It follows from (13) and (14) that

$$\psi_{jt}(\theta)\to e^{\zeta_0''(0)\theta^2/2} \qquad \text{as} \quad t\to\infty,$$

and, by the argument similar to the one used to show that $\zeta_0'(0)$ is pure imaginary, we know that $\zeta_0''(0)\leqq 0$. Therefore we have

$$\begin{aligned} \psi_t(\theta) &\overset{\text{def}}{=} E\{e^{i\theta[R(t)-gt]/\sqrt{t}}\}=\sum_{j=1}^{N}\psi_{jt}(\theta)\cdot P\{X_0=j\} \\ &\to e^{\zeta_0''(0)\theta^2/2} \qquad \text{as} \quad t\to\infty. \end{aligned}$$

This shows that $[R(t)-gt]/\sqrt{t}$ converges in distribution to the normal distribution $N(0,-\zeta_0''(0))$.

Reference

[1] Hatori, H., On continuous-time Markov processes with rawards, I. Kōdai Math. Sem. Rep. **18** (1966), 212–218.

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