# OKA'S PRINCIPLE FOR THE EXTENSION OF HOLOMORPHIC MAPPINGS

By Joji Kajiwara

**Introduction.**

Let $A$ be a subset of a set $X$ and $g$ be a mapping of $A$ in a set $Y$. We say that $g$ *can be extended* to a mapping $f$ of $X$ in $Y$ if $f=g$ in $A$. We shall give an example of a holomorphic mapping $g$ of $\{(z, w);\ 1-zw=0\}$ in $C-\{0\}$ which can not be extended to a holomorphic mapping of $C^2$ in $C-\{0\}$. We shall remark that the so called "Oka's principle" holds for the extension problem of holomorphic mappings in the following sense:

Let $A$ be an analytic subset of a simply connected Stein manifold $S$ and $g$ be a holomorphic mapping of $A$ in a connected abelian complex Lie group $L$. Then $g$ can be extended to a holomorphic mapping of $S$ in $L$ if and only if $g$ can be extended to a continuous mapping of $S$ in $L$.

**§1. Example.**

We put

$$A=\{(z, w);\ 1-zw=0\}$$

and

$$\lambda(z, w)=1/z$$

for $(z, w)\in A$. Then $\lambda$ is a biholomorphic mapping of $A$ on $C-\{0\}$.

Lemma. *$\lambda$ can not be extended to a holomorphic mapping of $C^2$ in $C-\{0\}$.*

*Proof.* Suppose that $\lambda$ can be extended to a holomorphic mapping $\mu$ of $C^2$ in $C-\{0\}$. We put

$$\xi=\lambda^{-1}\circ\mu.$$

Then $\xi$ is a holomorphic mapping of $C^2$ in $A$ whose restriction in $A$ is the identity. We put

$$R=\{(z, w);\ |zw-1|<1/2\}.$$

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Then $R$ is a Runge domain in $C^2$. Let $g$ be a holomorphic mapping of $R$ in $C-\{0\}$. We put

$$f=\lambda^{-1}\circ g.$$

Then $f$ is a holomorphic mapping of $R$ in $C^2$. Let $K$ be a compact subset of $R$. Since $R$ is a Runge domain, there exists a sequence $\{f_n;\ n=1, 2, 3, \cdots\}$ of holomorphic mappings of $C^2$ in $C^2$ which tends uniformly to $f$ in $K$. If we put

$$g_n=\mu\circ f_n$$

for any $n$, $\{g_n\}$ is a sequence of holomorphic mappings $g_n$ of $C^2$ in $C-\{0\}$ which tends uniformly to $g$ in $K$. But this contradicts Grauert [2]. Therefore $\lambda$ can not be extended to a holomorphic mapping of $C^2$ in $C-\{0\}$.

## § 2. Oka's principle.

Let $A$ be an analytic subset of a complex manifold $M$. A continuous mapping $g$ of $A$ in a complex manifold $M'$ is called a *holomorphic mapping* of $A$ in $M'$ if for any point $x$ of $A$ there exists an open neighbourhood $U$ of $x$ such that the restriction $g|U$ of $g$ in $U$ can be extended to a holomorphic mapping of $U$ in $M'$.

PROPOSITION 1. *Let $A$ be an analytic subset of a simply connected Stein manifold $S$ and $g$ be a holomorphic mapping of $A$ in a connected abelian complex Lie group $L$. Then $g$ can be extended to a holomorphic mapping of $S$ in $L$ if and only if $g$ can be extended to a continuous mapping of $S$ in $L$.*

*Proof*. From definition there exist an open covering $\mathfrak{U}=\{U_i;\ i\in I\}$ of $S$ and $\{g_i\}\in C^0(\mathfrak{U};\ \mathfrak{A}_L)$ with the following properties where $\mathfrak{A}_L$ is the sheaf of all germs of holomorphic mappings in $L$:

(1) $U_i$ is simply connected for any $i\in I$.

(2) $U_i\cap A$ and $U_i\cap U_j$ are connected for any $i$ and $j\in I$.

(3) $g|U_i\cap A$ can be extended to the holomorphic mapping $g_i$ of $U_i$ in $L$ for any $i$ with $U_i\cap A\neq\phi$.

Since $L$ is a connected abelian complex Lie group, there exists a holomorphic mapping $\chi$ of $C^p$ in $L$ with the following properties where $p$ is the complex dimension of $L$:

(1) $(C^p, \chi)$ is a covering manifold of $L$.

(2) $\chi$ is a homomorphism of $C^p$ in $L$.

(3) The kernel $N$ of $\chi$ is a discrete subgroup of $C^p$ isomorphic to the fundamental group $\pi(L)$ of $L$.

Since each $U_i$ is simply connected, there exists a holomorphic mapping $G_i$ in $C^p$ with $\chi\circ G_i=g_i$ for $i\in I$. Then we have

$$\{(G_i-G_j)|A\}\in Z^1(\mathfrak{U}\cap A,\ N)$$

where $\mathfrak{U}\cap A=\{U_i\cap A;\ i\in I\}$ is an open covering of $A$.

Now suppose that $g$ can be extended to a continuous mapping $f$ of $S$ in $L$. Since $S$ is simply connected, there exists a continuous mapping $F$ of $S$ in $C^p$ with $f=\chi\circ F$. Then $G_i-F$ takes a constant value $n_i$ belonging to $N$ in each $U_i\cap A$ which is connected. We put

$$n_i=0$$

for $i$ with $U_i\cap A=\phi$. Then we have

$$(G_i-G_j)|A=n_i-n_j$$

in $U_i\cap U_j\cap A$ which is not empty. We shall denote by $\mathfrak{F}$ the sheaf of all germs of holomorphic functions vanishing in $A$. From Oka [3] and Cartan [1] it is an analytic coherent sheaf over the Stein manifold $S$ and we have $H^1(S;\mathfrak{F})=0$. We consider

$$\{(G_i-n_i)-(G_j-n_j)\}\in Z^1(\mathfrak{U};\mathfrak{F})=B^1(\mathfrak{U};\mathfrak{F}).$$

There exists $\{H_i\}\in C^0(\mathfrak{U};\mathfrak{F})$ whose coboundary is the above cocycle. If we put

$$K=G_i-H_i-n_i$$

in $U_i$, $K$ is a well-defined holomorphic mapping of $S$ in $C^p$. If we put

$$k=\chi\circ K$$

in $S$, $k$ is a holomorphic mapping of $S$ in $L$ with

$$k|U_i\cap A=\chi\circ G_i=g$$

in any $U_i\cap A\neq\phi$. Hence $g$ can be extended to a holomorphic mapping $k$ of $S$ in $L$.

PROPOSITION 2. *Let $A$ be an analytic subset of a Stein manifold $S$ and $L$ be a connected abelian complex Lie group $L$ with the fundamental group $\pi(L)$. If $H^1(A;\pi(L))=0$, any holomorphic mapping of $A$ in $L$ can be extended to a holomorphic mapping of $S$ in $L$.*

*Proof.* Under the notations in the proof of Proposition 1, we have

$$\{(G_i-G_j)|A\}\in Z^1(\mathfrak{U}\cap A;N)=B^1(\mathfrak{U}\cap A;N).$$

Therefore we can prove quite similarly as in the proof of Proposition 1 that $g$ can be extended to a holomorphic mapping of $S$ in $L$.

## § 3. Remark.

Let $A$ be an analytic subset of a complex manifold $M$, $L$ be a complex Lie group and $\mathfrak{C}_L$ be the sheaf of all germs of continuous mappings in $L$. Consider the canonical commutative diagram

$$\begin{array}{ccccccc}
H^0(M, A; \mathfrak{A}_L) & \longrightarrow & H^0(M; \mathfrak{A}_L) & \longrightarrow & H^0(A; \mathfrak{A}_L) & \longrightarrow & H^1(M, A; \mathfrak{A}_L) \\
{\scriptstyle i_1}\downarrow & & {\scriptstyle i_2}\downarrow & & {\scriptstyle i_3}\downarrow & & {\scriptstyle i}\downarrow \\
H^0(M, A; \mathfrak{C}_L) & \longrightarrow & H^0(M; \mathfrak{C}_L) & \longrightarrow & H^0(A; \mathfrak{C}_L) & \longrightarrow & H^1(M, A; \mathfrak{C}_L)
\end{array}$$

where the horizontal sequences are exact sequences of relative cohomology sets and the vertical mappings $i_1$, $i_2$ and $i_3$ are injective. If we can prove the injectivity of $i$, we obtain "*Oka's principle*" for this triple $M$, $A$ and $L$.

## REFERENCES

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COLLEGE OF GENERAL EDUCATION,
NAGOYA UNIVERSITY.