NON-LINEAR CONNECTION IN VECTOR BUNDLES

By Mitsue Ako

Introduction.

The differential geometry of tangent bundle of Riemannian or Finslerian manifold has been studied by various authers. (Cf. Bibliography in Yano and Ishihara [6].) An almost complex structure induced on the tangent bundle of a differentiable manifold by a linear connection of the base manifold played an important role in these papers.

On the other hand, a vector bundle, or more precisely a hypertangent bundle, with a non-linear connection whose fibre has dimension larger than that of the underlying maifold admits a structure F characterized by the equation $F^3+F=0$. Such a structure first studied by Yano [2] contains as a special case an almost complex structure as well as an almost contact structure. Recently, Yano and Ishihara [6] studied such a structure induced on submanifolds in an almost complex space.

We consider, in this paper, a vector bundle $\mathcal{V}(M, \pi, Y, G)$, or $\mathcal{V}(M)$ simply, on a differentiable manifold M, where we denote by π the projection from the bundle space $\mathcal{V}$ to the base space M, by Y the fibre which is an m-dimensional vector space and by G the group of the bundle which is a Lie subgroup of $GL(m)$. We assume, in the sequel, that $\dim M=n$ and the dimension of each fibre of $\mathcal{V}(M)$ is $m>n$. (In the case where $m=n$, see Kandatu [1], Yano and Ishihara [5].)

In §1, non-linear connection is defined as a distribution in $\mathcal{V}(M)$ or as an operator on the set of all cross-sections on M.

We discuss, in §2, vectors in $\mathcal{V}(M)$ and then we introduce a special frame of reference which is suitable for our present theme.

The last §3 is devoted to the study of the structure F stated above. We also refer, in this section, the almost complex structure induced by F on the integral manifold of the distribution defined by the characteristic vectors corresponding to the characteristic roots other than zero.

§1. Non-linear connection in $\mathcal{V}(M)$.

Let M be a differentiable manifold of dimension n and Y be a vector space of dimension m which is larger than n. We denote by $\mathcal{V}(M)$ the differentiable vector

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bundle over M with fibre Y. (Differentiability is always assumed to be C^∞.) We introduce a local coordinate system in $\pi^{-1}(U)$, where π is the bundle projection and U an open neighbourhood of M. We denote by φ_U the coordinate function, that is, φ_U is a diffeomorphism from $U\times Y$ to $\pi^{-1}(U)$. For $p\in U$ and $y\in Y$ we define

$$\varphi_U(p, y)=(\xi^1, \cdots, \xi^n, \eta^1, \cdots, \eta^m),$$ [1]

where $(\xi^1, \cdots, \xi^n)$ are local coordinates of p in U and $(\eta^1, \cdots, \eta^m)$ cartesian coordinates of y in the vector space Y. If we denote by $\{y_k\}$ the basis of Y, then

$$\text{(1. 1)} \qquad \boldsymbol{Z}^U_\kappa(p)\overset{\text{def}}{=}\varphi_U(p, y_k), \qquad p\in U$$

form basis of Y_p, where Y_p is the fibre over p.

For $U\cap U'\neq\phi$, we denote the coordinate transformation by

$$g_{U'U}\colon\ U\cap U'\to GL(m).$$

The law of transformation of the frame $\boldsymbol{Z}^U_\kappa(p)$ at p is given by

(1. 2) $\quad \boldsymbol{Z}^U_\kappa(p)=\boldsymbol{Z}^{U'}_{\kappa'}A^{\kappa'}_\kappa(g_{U'U}(p))$, $p\in U\cap U'$; and $A^{\kappa'}_\kappa$ is a non-singular real $m\times m$ matrix,

if we take account of (1. 1). The facts above show that the law of coordinate transformation on $\mathcal{V}(M)$ is expressed as follows with respect to the local coordinate system (ξ^i, η^κ):

$$\text{(1. 3)} \qquad \xi^{i'}=\xi^{i'}(\xi), \qquad \eta^{\kappa'}=M^{\kappa'}_\kappa(\xi)\eta^\kappa,$$

where $M^{\kappa'}_\kappa(\xi)$ is a non-singular real $m\times m$ matrix.

The fibre over a point p is the integral manifold through p of the distribution expressed by pfaffian equations

$$\text{(1. 4)} \qquad \omega^i\overset{\text{def}}{=}d\xi^i=0.$$

We shall define a non-linear connection which is given as a distribution complementary to the distribution (1. 4). First, let $'\mathcal{V}(M)$ be a vector bundle over M whose fibre $'Y$ consists of all non-zero vectors of Y. Clearly $'\mathcal{V}(M)$ is a sub-bundle of $\mathcal{V}(M)$. The bundle projection $'\mathcal{V}\to M$ is denoted by $'\pi$ and the fibre of $'\mathcal{V}(M)$ over $p\in M$ by $'Y_p={}'\pi^{-1}(p)$. Now we define a bundle transformation $R_a\colon \mathcal{V}\to\mathcal{V}$, a being a non-zero real number, by $R_a(\sigma)=a\sigma$, where σ is an arbitrary element of $\mathcal{V}$. More explicitly, if σ belongs to $\pi^{-1}(U)$ and is expressed by the local coordinates (ξ, η), then the local coordinates of $R_a(\sigma)$ is given by $(\xi, a\eta)$. Then $\pi\circ R_a$ is the identity mapping of M. The set of all such bundle transformations R_a forms a

1) We use, in this paper, different kinds of indices. Small roman indices $i, j, k, \cdots$ run over the range $1, 2, \cdots, n$ and large roman ones $A, B, C, \cdots$ the range $1, 2, \cdots, n+m$. On the other hand, greek indices in the latter half of alphabet $\kappa, \lambda, \mu, \cdots$ run over the range $1, 2, \cdots, m$.

group and we denote it by D. The group D is a group of bundle transformations of the subbundle $'\mathfrak{V}(M)$. A *non-linear connection* in $'\mathfrak{V}(M)$ is defined as a differentable distribution Π satisfying the following conditions.

$$(1.5)\qquad \begin{aligned} &\text{(a)} \qquad T_\sigma('\mathfrak{V}) = T_\sigma('Y_p) + \Pi_\sigma \qquad \text{(direct sum)},\\ &\text{(b)} \qquad dR_a(\Pi_\sigma) = \Pi_{R_{a(\sigma)}} \qquad \text{for any real number } a \neq 0, \end{aligned}$$

where $T_\sigma(\ )$ means the tangent space at σ of the space in parentheses and Π_σ is the value of the distribution Π at σ. (1.5) (a) says that $\dim \Pi = n$.

Π regarded as a distribution in $\mathfrak{V}(M)$ has singularities along the 0-cross-section. We consider (1.5) outside of the 0-cross-section and call Π a non-linear connection in $\mathfrak{V}(M)$.

Now we shall express the non-linear connections with respect to the local coordinate system (ξ, η) in $\pi^{-1}(U)$. We see, from (1.5) (a) that equations of Π is of the form

$$(1.6)\qquad \omega^\kappa = \Gamma^\kappa_j(\xi, \eta) d\xi^j + d\eta^\kappa = 0,$$

where $\Gamma^\kappa_j(\xi, \eta)$ are uniquely determined in $\pi^{-1}(U)$ except the domain containing $\eta = (0, \cdots, 0)$.

Condition (1.5) (b) requires that $\Gamma^\kappa_j(\xi, \eta)$ are homogeneous functions of degree one with respect to the arguments η, i.e.

$$\Gamma^\kappa_j(\xi, a\eta) = a\Gamma^\kappa_j(\xi, \eta) \qquad (a \neq 0).$$

$\Gamma^\kappa_j(\xi, \eta)$ are called *components of non-linear connection.*

We sometimes call Π_σ the *horizontal plane at* σ and Π the *horizontal plane field.*

Under the coordinate transformation (1.3) $\Gamma^\kappa_j(\xi, \eta)$ are transformed in to $\Gamma^{\kappa'}_{j'}(\xi', \eta')$ by the following law:

$$(1.7)\qquad \Gamma^{\kappa'}_{j'} \frac{\partial \xi^{j'}}{\partial \xi^k} = M^{\kappa'}_\kappa \Gamma^\kappa_k - \frac{\partial M^{\kappa'}_\kappa}{\partial \xi^k} \eta^\kappa.$$

This is easily verified by the use of (1.5) (a).

A non-linear connection on $\mathfrak{V}$ is defined in other ways. We shall give one of them in below.

Let $\mathfrak{F}$ be the set of all C^∞-functions on M and $\mathfrak{X}$ the set of all C^∞-cross-sections on M regarded as a vector space over the ring $\mathfrak{F}$. We denote by $\tilde{\mathfrak{X}}$ the sub-space of $\mathfrak{X}$ consisting of all C^∞-cross-sections $M \to \mathfrak{T}$, where $\mathfrak{T}$ is the bundle space of fhe tangent bundle $\mathfrak{T}(M)$ of M.

DEFINITION 1.1. A non-linear connection is defined as an operator

$$\nabla: \mathfrak{X} \times \tilde{\mathfrak{X}} \to \mathfrak{X}$$

which assigns $\nabla(V, v)\in\mathfrak{X}$, denoted by $\nabla_v V$ to $(V, v)\in\mathfrak{X}\times\tilde{\mathfrak{X}}$ and satisfies following conditions:

(i) $\nabla_{fv}V=f\nabla_v V$,

(ii) $\nabla_{(v+v')}V=\nabla_v V+\nabla_{v'}V$

and

(iii) $\nabla_v(fV)=f\nabla_v V+(vf)V$,

where $f\in\mathfrak{F}$, v, $v'\in\tilde{\mathfrak{X}}$ and $V\in\mathfrak{X}$. We further assume, by denoting V_p the value of V at $p\in M$, that

(iv) if $V_p=0$, then $(\nabla_v V)_p=(\mathring{\nabla}_v V)_p$, where $\mathring{\nabla}$ is an arbitrarily given linear connection on $\mathfrak{V}$, and that

(v) if $V_p=V'_p$, then $(\nabla_v(V-V'))_p=(V_v V)_p-(\nabla_v V')_p$.

If such an operator is defined in each of coverings $\{U\}$ of M and ∇ defined on U and ∇' on U' coincide on $U\cap U'\neq\phi$, then one can define $\bar{\nabla}$ on M which induces ∇ on U and ∇' on U'.

We see, as a direct result of Definition 1. 1, that a linear connection is a non-linear connection.

Now if we represent ∇ in the local coordinate system (ξ, η), then the equivalency of two definitions of a non-linear connection follows automatically.

We shall use, as the basis in $\pi^{-1}(U)$, the basis $Z_\lambda^U(p)$ on Y_p and the basis $d\varphi_U(\partial/\partial\xi^i)_p$ for an arbitrary point $p\in M$, where $d\varphi_U$ denotes the differential of φ_U and $\partial/\partial\xi^i$ are the basis of the tangent space M at p. These basis are denoted by $\boldsymbol{e}_A=\{\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n, \boldsymbol{e}_{n+1}, \cdots, \boldsymbol{e}_{n+m}\}$, where $\boldsymbol{e}_i=d\varphi_U(\partial/\partial\xi^i)$ and $\boldsymbol{e}_{n+\lambda}=\boldsymbol{Z}_\lambda^U(p)$. Let there be given ∇ on U. Conditions (ii) and (iii) show that it is sufficient to express $\nabla_{e_i}V$ in order to express ∇. We may put, by the condition (iv),

$$\nabla_{e_i}V=\left\{\frac{\partial V^B}{\partial\xi^i}+\tilde{\Gamma}_i{}^B(\xi, V)\right\}\boldsymbol{e}_B. \tag{1. 8}$$

The quantities $\tilde{\Gamma}_i{}^B$ appearing in (1. 8) seem to depend on p and V, but conditions (iv) and (v) show that $\tilde{\Gamma}_i{}^B$ are function of p and $V(p)$. Then we can write them as $\tilde{\Gamma}_i{}^B(\xi, \eta)$, where $(\xi, \eta)=V(p)$. Condition (i) requires that $\tilde{\Gamma}_i{}^B(\xi, \eta)$ are homogeneous functions of degree 1 with respect to η, i.e.

$$\tilde{\Gamma}_i{}^B(\xi, a\eta)=a\tilde{\Gamma}_i{}^B(\xi, \eta)$$

for any real number $a\neq 0$.

A vector field V is said to be *horizontal in the second sense* if $\nabla_{e_i}V$ is a linear combination of $\boldsymbol{e}_j$, that is

$$\left(\frac{\partial V^B}{\partial\xi^i}+\tilde{\Gamma}_i{}^B(\xi, \eta)\right)\boldsymbol{e}_B=\psi_i{}^j\boldsymbol{e}_j,$$

which is equivalent to

$$\frac{\partial V^{\kappa}}{\partial \xi^{i}}+\tilde{\Gamma}_{i}{}^{\kappa}(\xi, \eta)=0. \tag{1.9}$$

Now, we can define a horizontal plane field H by the condition that $H_{\sigma}\in H$ is the set of all vector fields which are horizontal at σ.

On the other hand, if there is given a non-linear connection in the first sense, then an operator $\tilde{\nabla}$: $\mathfrak{X}\times\tilde{\mathfrak{X}}\to\mathfrak{X}$ is defined by

$$\tilde{\nabla}_{e_i}V\overset{\text{def}}{=}\left(\frac{\partial V^{\kappa}}{\partial \xi^{i}}+\Gamma_{i}{}^{\kappa}\right)e_{\kappa}+\frac{\partial V^{j}}{\partial \xi^{i}}e_{j},$$

where $\Gamma_{i}{}^{\kappa}$ are given in (1. 6).

We conclude this section by the following theorem which will be obtained by a straightforward calculation.

THEOREM. *The non-linear connection is integrable if and only if*

$$K_{kj\lambda}{}^{\kappa}\eta^{\lambda}=0, \tag{1.10}$$

$K_{kj\lambda}{}^{\kappa}$ being defined by

$$2K_{kj\lambda}{}^{\kappa}=\partial_{[k}\Gamma^{\kappa}_{j]\lambda}-\Gamma^{\alpha}_{[k}\partial_{|\alpha|}\Gamma^{\kappa}_{j]\lambda}+\Gamma^{\kappa}_{[k|\alpha|}\Gamma^{\alpha}_{j]\lambda}, \tag{1.11}$$

where $\Gamma^{\kappa}_{j\lambda}=\partial\Gamma_{j}{}^{\kappa}/\partial\eta^{\lambda}$.

§2. Vectors in $\mathcal{V}$(M).

DEFINITION 2. 1. A *horizontal vector* $'V$ is, by definition, a vector which is tangent to the distribution H.

Then $'V$ has the following components

$$'V=\begin{pmatrix} V^{i} \\ -\Gamma_{j}{}^{\kappa}V^{j} \end{pmatrix}.$$

We can easily see that V^{i} are components of a tangent vector V of the manifold M. This fact gives the reason that we call such a vector $'V$ a *horizontal lift* of V.

A *vertical vector* $''V$ is, by definition, a vector which is tangent to a fibre of $\mathcal{V}(M)$.

Then we have

$$''V=\begin{pmatrix} 0 \\ V^{\kappa} \end{pmatrix}.$$

A special vertical vector $''V$ whose components are zero except $''V^{i^*}$ [2] and $''V^{i^*}=w^i$ are components of a tangent vector w of M is called a *vertical lift* of the vector w.

We suppose, in the rest of this paper, that $\mathcal{V}(M)$ is a hypertangent bundle, that is, $\mathcal{V}(M)$ has a sub-bundle $\tilde{\mathcal{T}}(M)$ which is isomorphic to the tangent bundle $\mathcal{T}(M)$ of M. We choose the natural frame $\boldsymbol{e}_i=\partial/\partial\xi^i$ in $T_p(M)$, the tangent space of M at $p\in M$. Then, denoting by ι the isomorphism of $\mathcal{T}(M)$ to $\tilde{\mathcal{T}}(M)$, we have the image $\iota(\boldsymbol{e}_i)$ of $\boldsymbol{e}_i$ denoted by $\boldsymbol{C}_{i^*}$ on the fibre at p in $\mathcal{T}(M)$. We choose $\boldsymbol{C}_x\,(x, y, \cdots =2n+1, \cdots, n+m)$ on the fibre at p in $\mathcal{V}(M)$ in such a way that $\boldsymbol{C}_x$ are complement to $\boldsymbol{C}_{i^*}$. We denote by $\boldsymbol{B}_i$ the horizontal lifts of $\boldsymbol{e}_i$.

In the sequel, we use the special frame of reference given by

$$\boldsymbol{A}_\alpha=(\boldsymbol{B}_i, \boldsymbol{C}_{i^*}, \boldsymbol{C}_x)\ ^{3)}$$

and their inverse

$$\boldsymbol{A}^\alpha=(\boldsymbol{B}^i, \boldsymbol{C}^{i^*}, \boldsymbol{C}^x).$$

The components of their special frame become

$$\boldsymbol{B}_i=\begin{pmatrix} \delta_i^h \\ \\ -\Gamma_i^\kappa \end{pmatrix}, \qquad \boldsymbol{C}_{i^*}=\begin{pmatrix} 0 \\ \delta_{i^*}^{j^*} \\ 0 \end{pmatrix}, \qquad \boldsymbol{C}_x=\begin{pmatrix} 0 \\ \\ \delta_x^y \end{pmatrix},$$

$$\boldsymbol{B}^i=(\delta_h^i, 0), \qquad \boldsymbol{C}^{i^*}=(\delta_h^i, \Gamma_h^{i^*}, 0), \qquad \boldsymbol{C}^x=(0, \Gamma_h^x).$$

If we represent a vector V with respect to the adapted frame $\boldsymbol{A}_\alpha$, then we have

$$V=V^i\boldsymbol{B}_i+V^{i^*}\boldsymbol{C}_{i^*}+V^x\boldsymbol{C}_x.$$

It can be easily verified that $V^i\boldsymbol{B}_i$ is a horizontal vector and that $V^\kappa\boldsymbol{C}_\kappa$ is a vertical vector. We call them a *horizontal part* and a *vertical part* of the vector V respectively.

§3. The structure F in $\mathcal{V}(M)$.

Under the notations in §2 we define a linear transformation F in $\mathcal{V}(M)$ by the following conditions:

$$\text{(3.1)}\qquad \begin{cases} F\boldsymbol{B}_i=\boldsymbol{C}_{i^*}, \\ F\boldsymbol{C}_{i^*}=-\boldsymbol{B}_i, \\ F\boldsymbol{C}_x=0. \end{cases}$$

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2) The range of the indices $i^*, j^*, \cdots$ is the first n parts of the range of the indices $\kappa, \lambda, \cdots$. We often write $i^*=n+i$ and use it to mean that it corresponds to i.

3) We shall use the indices $\alpha, \beta, \gamma, \cdots$ instead of the indices $A, B, C, \cdots$ in the case where we are using the special frame above.

Since $\mathcal{V}(M)$ is a hypertangent bundle, so we can make correspondence between the distribution defined by $\boldsymbol{C}_{\iota*}$ and that defined by $\boldsymbol{B}_{\iota}$.

Direct computations show that

$$F^3+F=0. \tag{3. 2}$$

Thus we have

PROPOSITION 3. 1. *There always exists, in a hypertangent bundle with a non-linear connection, a structure F satisfying* (3. 2) *and of rank* $2n$.

The components of F with respect to the adapted frame $\boldsymbol{A}_\alpha$ are given by

$$\begin{pmatrix} 0 & -E_n & 0 \\ E_n & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3. 3}$$

where E_n denotes $n\times n$ unit matrix.

We shall need the components of the non-holonomic object which are important when we are using a frame of reference such as $\boldsymbol{A}_\alpha$ which is not the natural one associated with the coordinate system. They are

$$\Omega_{\gamma\beta}{}^{\alpha}=-\Omega_{\beta\gamma}{}^{\alpha}=A^{\alpha}{}_{A}(X_\gamma A_\beta{}^A-X_\beta A_\gamma{}^A),$$

where $X_\alpha=A_\alpha{}^A\partial_A$.

The only non-vanishing components of $\Omega_{\gamma\beta}{}^{\alpha}$ are

$$\begin{cases} \Omega_{j\lambda}{}^{\kappa}=\Gamma_{j\lambda}{}^{\kappa}, \\ \Omega_{kj}{}^{\kappa}=-\Gamma_{kj\lambda}{}^{\kappa}\eta^\lambda. \end{cases} \tag{3. 4}$$

The components of Nijenhuis tensor constructed from F are given by

$$N_{\gamma\beta}{}^{\alpha}=F_\gamma{}^\varepsilon F_\beta{}^\delta\Omega_{\varepsilon\delta}{}^{\alpha}-F_\varepsilon{}^\alpha F_\beta{}^\delta\Omega_{\gamma\delta}{}^{\varepsilon}-F_\gamma{}^\varepsilon F_\delta{}^\alpha\Omega_{\varepsilon\beta}{}^{\delta}+F_\delta{}^\varepsilon F_\varepsilon{}^\alpha\Omega_{\gamma\beta}{}^{\delta}$$

with respect to the frame $\boldsymbol{A}_\alpha$.

A simple calculation, taking account of (3. 3) and (3. 4), gives

$$N_{kj}{}^{h}=N_{k*j*}{}^{h}=-N_{kj*}{}^{h*}=\Gamma^{h*}_{kj*}-\Gamma^{h*}_{jk*},$$

$$N_{kj*}{}^{h}=N_{kj}{}^{h*}=-N_{k*j*}{}^{h*}=K_{kj\lambda}{}^{h*}\eta^\lambda,$$

$$N_{kj*}{}^{x}=\Gamma_{jk*}{}^{x},$$

$$N_{k*j*}{}^{x}=-K_{kj\lambda}{}^{x}\eta^\lambda,$$

$$N_{kx}{}^{h*}=-\Gamma_{kx}{}^{h*}.$$

PROPOSITION 3. 2. *A necessary and sufficient condition that F is integrable is that*

$$\Gamma^{h^*}_{kj^*}-\Gamma^{h^*}_{jk^*}=0,$$

$$K_{kj\lambda}{}^{\mu}\eta^{\lambda}=0,$$

$$\Gamma^{x}_{jk^*}=0,$$

$$\Gamma^{h^*}_{kx}=0.$$

If we denote by L_1 the $2n$-dimensional distribution defined by $\boldsymbol{B}_i$ and $\boldsymbol{C}_{i^*}$, then the almost complex structure is introduced from F on L_1. The distribution determined by $\boldsymbol{C}_x$ is denoted by L_2, that is, L_2 is spanned by the characteristic vectors of F corresponding to the characteristic root 0, then $\dim L_2=m-n$.

We define tensors l and m as follows

$$l=-F^2 \qquad \text{and} \qquad m=F^2+I,$$

where I is a unit tensor. It is easily verified that l and m are complementary projection tensors, that is, $l+m=I$, $l^2=l$, $m^2=m$ and $lm=ml=0$.

Yano and Ishihara [4] proved the following two theorems.

THEOREM ([4]). *A necessary and sufficient condition for the distribution L_1 to be integrable is that one of the following condition is satisfied*:

(i) $$N_{FE}{}^{D}l_{C}{}^{F}l_{B}{}^{E}m_{D}{}^{A}=0,$$

(ii) $$N_{CB}{}^{D}m_{D}{}^{A}=0.$$

If we write down the integrability condition of L_1 with respect to the adapted frame $\boldsymbol{A}_\alpha$, we have

$$\Gamma^{x}_{ji^*}=0 \qquad \text{and} \qquad K_{ji\lambda}{}^{x}\eta^{\lambda}=0.$$

THEOREM ([4]). *A necessary and sufficient condition that L_1 is integrable and besides the almost complex structure introduced on L_1 is integrable is that*

$$N_{FE}{}^{A}l_{C}{}^{F}l_{B}{}^{E}=0.$$

If we express this condition with respect to the adapted frame $\boldsymbol{A}_\alpha$, then we have

$$\Gamma^{h^*}_{ji^*}-\Gamma^{h^*}_{ij^*}=0, \quad K_{ji\lambda}{}^{\mu}\eta^{\lambda}=0 \quad \text{and} \quad \Gamma^{x}_{jk^*}=0.$$

Now let us consider infinitesimal transformations which preserve the structure F. First of all we calculate the Lie derivative of F with respect to a vector field V and represent it in the adapted frame $\boldsymbol{A}_\alpha$. Then we have

$$(\underset{V}{\mathcal{L}} F)^{\alpha}_{\beta} = V^{\varepsilon} X_{\varepsilon} F^{\alpha}_{\beta} - F_{\beta}{}^{\varepsilon} X_{\varepsilon} V^{\alpha} + F_{\varepsilon}{}^{\alpha} X_{\beta} V^{\varepsilon}$$

$$+ (\Omega_{\varepsilon\delta}{}^{\alpha} F_{\beta}{}^{\delta} - \Omega_{\varepsilon\beta}{}^{\delta} F_{\delta}{}^{\alpha}) V^{\varepsilon}.$$

PROPOSITION 3. 3. *If the horizontal lift $'V$ of a vector field v on M preserves the structure F, then we have*

$$K_{ji\lambda}{}^{\mu} \eta^{\lambda} v^{j} = 0, \qquad \Gamma^{x}_{ji*} = 0$$

and

$$\partial_i v^h + \Gamma^{h*}_{ji*} v^j = 0,$$

where v^i are the components of the vector v and ∂_i denotes the partial derivative with respect to ξ^i.

PROPOSITION 3. 4. *If the vertical lift $''V$ of a vector field v on M preserves the structure F, then we have*

$$\Gamma^{x}_{ij*} v^j = 0$$

and

$$\partial_i v^h + \Gamma^{h*}_{ij*} v^j = 0.$$

Next we consider the case in which the distribution L_1 is integrable. In such a case there exists an almost complex structure F induced by F. If $'V$ (or $''V$) is tangent to the integral manifold of L_1 and $\underset{'V}{\mathcal{L}} F = 0$ (or $\underset{''V}{\mathcal{L}} F = 0$), then $'V$ (or $''V$) is an almost analytic vector.

Together with Propositions 3. 3 and 3. 4, we have

PROPOSITION 3. 5. *A necessary and sufficient condition for the horizontal lift $'V$ of a vector v on M which is tangent to the integral manifold of L_1 to be almost analytic is that*

$$K_{jh\lambda}{}^{i*} \eta^{\lambda} v^h = 0$$

and

$$\partial_j v^i + \Gamma^{i}_{hj*} v^h = 0.$$

For the vertical lift $''V$ the above conditions should be

$$\partial_j v^i + \Gamma^{i}_{jh*} v^h = 0.$$

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DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY