ON A MATCHING METHOD FOR A LINEAR ORDINARY DIFFERENTIAL EQUATION CONTAINING A PARAMETER, II

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§ 1. Introduction.

In this paper, we consider the asymptotic behavior of the solution of a linear ordinary differential equation of the form

$$\varepsilon^{h}\frac{dy}{dx}=A(x,\varepsilon)y \tag{1.1}$$

as the parameter ε tends to zero. Here we suppose that

1) h is a positive integer;
2) x and ε are a complex variable and a complex parameter respectively;
3) y is an n-dimensional column vector;
4) $A(x,\varepsilon)$ is an n-by-n matrix function holomorphic and bounded in the domain of the x,ε space defined by the inequalities,

$$|x|\leqq x_0<1, \qquad 0<|\varepsilon|\leqq\varepsilon_0, \qquad |\arg\varepsilon|\leqq\delta_0; \tag{1.2}$$

5) when ε tends to zero in the domain

$$0<|\varepsilon|\leqq\varepsilon_0, \qquad |\arg\varepsilon|\leqq\delta_0, \tag{1.3}$$

$A(x,\varepsilon)$ admits for $|x|\leqq x_0$ a uniform asymptotic expansion in powers of ε:

$$A(x,\varepsilon)\simeq\sum_{\nu=0}^{\infty}A^{(\nu)}(x)\varepsilon^{\nu}, \tag{1.4}$$

where the coefficients $A^{(\nu)}(x)$ are n-by-n matrices whose components are functions holomorphic and bounded for $|x|\leqq x_0$;

6) the matrix $A(x,\varepsilon)$ has the form

$$A(x,\varepsilon)=\begin{bmatrix}A_{11}(x,\varepsilon) & & & \\ A_{21}(x,\varepsilon) & A_{22}(x,\varepsilon) & & 0 \\ \cdots & & \ddots & \\ A_{p1}(x,\varepsilon) & \cdots & & A_{pp}(x,\varepsilon)\end{bmatrix}, \tag{1.5}$$

where $A_{jk}(x,\varepsilon)$ are n_j-by-n_k matrices $(j,k=1,\cdots,p)$;

7) in particular, each of the matrices $A_{ii}(x,\varepsilon)$ has the form

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$$(1.6) \qquad A_{ii}(x, \varepsilon)=\begin{bmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ & & & 1 \\ a_{in_i} & a_{in_i-1} & \cdots \; a_{i2} & 0 \end{bmatrix},$$

where $a_{il}(x, \varepsilon)$ are the functions such that

$$(1.7) \qquad \begin{aligned} a_{il}(x, \varepsilon) &\simeq \sum_{\nu=0}^{\infty} a_{ik}{}^{(\nu)}(x)\varepsilon^{\nu} \qquad (i=1, \cdots, p;\ l=n_i, n_i-1, \cdots, 2), \\ a_{il}{}^{(\nu)}(x) &= \sum_{\mu=m_{il}{}^{(\nu)}}^{\infty} a_{il,\mu}^{(\nu)} x^{\mu}, \qquad a_{il,m_{il}^{(\nu)}}^{(\nu)} \neq 0, \end{aligned}$$

with $m_{il}{}^{(0)} \geqq 1$, for some index (i, l)

$$a_{il}{}^{(0)}(x) \neq 0;$$

8) let $\mathfrak{A}_{jk}(x, \varepsilon)$ be the (j, k) elements of the matrix $A(x, \varepsilon)$ which is not the elements of $A_{ii}(x, \varepsilon)$, and let

$$(1.8) \qquad \mathfrak{A}_{jk}(x, \varepsilon) \simeq \sum_{\nu=\mathfrak{N}_{jk}}^{\infty} \mathfrak{A}_{jk}{}^{(\nu)}(x)\varepsilon^{\nu}$$

be the asymptotic expansion in powers of ε with holomorphic coefficients:

$$\mathfrak{A}_{jk}{}^{(\nu)}(x) = \sum_{\mu=0}^{\infty} \mathfrak{A}_{jk,\mu}^{(\nu)} x^{\mu},$$

and assume without loss of generality that

$$(1.9) \qquad \nu/a - q(j+1-k)/m > 0 \qquad \text{for } j>k,\ \nu \geqq \mathfrak{N}_{jk},$$

where $a=mh/(m+q)$, m and q are some positive integers defined in 9).

Under these conditions, the characteristic equation of the matrix $A^{(0)}(x)$ in λ

$$\prod_{i=1}^{s} [\lambda^{n_i} - \lambda^{n_i-2} a_{i2}^{(0)}(x) - \cdots - a_{in_i}^{(0)}(x)] = 0$$

has only one n-ple root $\lambda=0$ for $x=0$ and at least two distinct roots for $x \neq 0$. Therefore $x=0$ is a turning point of the system (1.1).

About this equations, Iwano [1] developed his method to construct the characteristic polygon for the system (1.1) and to divide the domain (1.2) into a finite number of subdomains so that the solution behaves quite differently as ε tends to zero in each of these subdomains. But to know about the asymptotic character of the solution of (1.1), it is necessary for us to prove the existence of fundamental solution and find out the asymptotic expression of it in each of these subdomains and to determine the connection formula between two different asymptotic expressions. The purpose of this paper is to solve these problems by means of a matching method which is originally due to Wasow [7], [8].

To do this, we need a fundamental assumption concerning the characteristic polygon Π for the system (1.1). We consider a plane whose points are represented by the coordinates (X, Y), and plot the points

$$P_{il}{}^{(\nu)}=\left(\frac{\nu}{l}, \frac{m_{il}{}^{(\nu)}}{l}\right), \qquad R=(h, -1) \qquad \begin{pmatrix} l=n_i, n_i-1, \cdots, 2 \\ i=1, 2, \cdots, p \\ \nu=0, 1, \cdots \end{pmatrix}, \tag{1.10}$$

where $m_{il}{}^{(\nu)}$ are defined in (1.7), then Π is a polygonal line, convex downward, such that its vertices are some of the points (1.10) and none of the points (1.10) is located below Π (for details, see Iwano [1]). Here we assume that

9) Π consists of only a line connecting the point R and some point P_0 on Y-axis whose coordinate can be written $(0, q/m)$, or what is the same thing, for (1.7),

$$\frac{\nu}{a}+\mu-\frac{lq}{m}\geqq 0 \qquad \begin{pmatrix} l=n_i, n_i-1, \cdots, 2 \\ i=1, \cdots, p \\ \mu\geqq m_{il}{}^{(\nu)} \\ \nu=0, 1, \cdots \end{pmatrix}. \tag{1.11}$$

In the author's previous paper [4], we treated the same problem with $p=1$ and with a further assumption;

$$\frac{\nu}{a}+\mu-\frac{lq}{m}>0 \qquad \begin{pmatrix} l=n_1, n_1-1, \cdots, 2 \\ \mu\geqq m_{1l}{}^{(\nu)} \\ \nu=1, 2, \cdots \end{pmatrix}. \tag{1.11'}$$

In this paper, it will be removed the assumption (1.11′) and will be generalized to any positive integer of p.

In Sections 2 and 3, we calculate the two types of formal solutions, in Sections 4 and 5, it will be proved that there exist fundamental solutions whose asymptotic expansions coincide with the formal solutions in several subdomains which overlap the full neighborhood of the turning point.

§ 2. Formal solution for $x \neq 0$.

The linear transformation originally due to Iwano [1]

$$y=\Omega(x)u, \tag{2.1}$$

where

$$\Omega(t)=\begin{bmatrix} 1 & & & 0 \\ & t^{q/m} & & \\ & & \ddots & \\ 0 & & & t^{q(n-1)/m} \end{bmatrix} \tag{2.2}$$

changes the equation (1.1) into

$$[x^{-1/a}\varepsilon]^h x\frac{du}{dx}=\{B_1(x, \varepsilon)+B_2(x, \varepsilon)\}u, \tag{2.3}$$

where

$$B_1(x, \varepsilon)=\begin{bmatrix}\begin{bmatrix}0 & 1 & & 0\\ & & \ddots & \\ & 0 & & 1\\ b_{1n_1} & \cdots & b_{12} & 0\end{bmatrix} & & \mathbf{0} \\ & \ddots & \\ \mathfrak{B}_{jk} & & \begin{bmatrix}0 & 1 & & 0\\ & & \ddots & \\ & 0 & & 1\\ b_{pn_p} & \cdots & b_{p2} & 0\end{bmatrix}\end{bmatrix} \tag{2.4}$$

with

$$\begin{cases} b_{il}(x, \varepsilon)=(x^{q/m})^{-l}a_{il}(x, \varepsilon) \\ \mathfrak{B}_{jk}(x, \varepsilon)=(x^{q/m})^{-(j+1-k)}\mathfrak{A}_{jk}(x, \varepsilon) \quad (j>k),\end{cases} \qquad \begin{pmatrix} i=1, \cdots p \\ l=n_i, n_i-1, \cdots, 2\end{pmatrix}, \tag{2.5}$$

and

$$B_2(x, \varepsilon)=[x^{-1/a}\varepsilon]^h\begin{bmatrix}0 & & 0\\ & 1 & \\ & & \ddots \\ 0 & & n-1\end{bmatrix}. \tag{2.6}$$

From (1. 7), (1. 9) and (2. 5), we have

$$\begin{cases} b_{il}(x, \varepsilon) \simeq \sum_{\nu=0}^{\infty} \sum_{\mu=m_{il}^{(\nu)}}^{\infty} a_{il,\mu}^{(\nu)} x^{\nu/a+\mu-ql/m}[x^{-1/a}\varepsilon]^{\nu}, \\ \mathfrak{B}_{jk}(x \ \varepsilon) \simeq \sum_{\nu=\mathfrak{N}_{jk}}^{\infty} \sum_{\mu=0}^{\infty} \mathfrak{B}_{jk,\mu}^{(\nu)} x^{\nu/a+\mu-q(j+1-k)/m}[x^{-1/a}\varepsilon]^{\nu}.\end{cases} \tag{2.7}$$

Remembering the assumptions (1. 9) and (1. 11), we can write the equation (2. 3) as

$$[x^{-1/a}\varepsilon]^h x\frac{du}{dx}=B(x, \varepsilon)u, \tag{2.8}$$

where $B(x, \varepsilon)$ has an asymptotic expansion in powers of $(x^{-1/a}\varepsilon)$ in the domain (1. 2) such that

$$B(x, \varepsilon) \simeq \sum_{\nu=0}^{\infty} B^{(\nu)}(x)[x^{-1/a}\varepsilon]^{\nu}$$

with

$$B^{(0)}(x)=\begin{bmatrix}\begin{bmatrix}0 & 1 & & 0\\ & & \ddots & \\ & 0 & & 1\\ a_{1n_1,\mu_{n1}}^{(0)} & & \cdots & 0\end{bmatrix} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \begin{bmatrix}0 & 1 & & 0\\ & & \ddots & \\ & 0 & & 1\\ a_{pn_p,\mu_p}^{(0)} & & \cdots & 0\end{bmatrix}\end{bmatrix}+B_1{}^{(0)}(x). \tag{2.9}$$

Here we notice that $a_{il,\mu_l}^{(0)}$ are some constants and there exists at least one non-zero element, and for such element we must have $\mu_l=ql/m$. The matrices $B_1^{(0)}(x)$

and $B^{(\nu)}(x)$ ($\nu \geqq 1$) are holomorphic in $x^{1/mh}$, and $B_1^{(0)}(0)=0$.

The characteristic equation of the matrix $B^{(0)}(0)$ is

$$\prod_{i=1}^{p}[\lambda^{n_i}-\lambda^{n_i-2}a_{i2,\mu_2}^{(0)}-\cdots-a_{in_i,\mu_{ni}}^{(0)}]=0.$$

Let $\lambda_1, \cdots, \lambda_n$ be the roots of the above equation, and we must assume that

(2. 10) $$\lambda_j \neq \lambda_k, \qquad j \neq k, \qquad j, k=1, \cdots, n.$$

If we put

$$x=\tau^{mh},$$

then the equation (2. 8) becomes

(2. 11) $$[\tau^{-(m+q)}\varepsilon]^h \tau \frac{du}{d\tau}=C(\tau, \varepsilon)u,$$

where $C(\tau, \varepsilon)$ is holomorphic in τ and ε for

$$|\tau| \leqq \tau_0, \qquad 0<|\varepsilon| \leqq_0, \qquad |\arg \varepsilon| \leqq \delta_0,$$

and has an asymptotic expansion when ε tends to zero:

$$C(\tau, \varepsilon) \simeq \sum_{\nu=0}^{\infty} C^{(\nu)}(\tau)[\tau^{-(m+q)}\varepsilon]^{\nu}.$$

The matrices $C^{(\nu)}(\tau)$ are holomorphic for $|\tau| \leqq \tau_0$ and

$$C^{(0)}(\tau)=mhB^{(0)}(x).$$

The characteristic roots of the matrix $C^{(0)}(0)$ are all distinct with each other, and then we can prove the following lemma.

LEMMA 2. 1. *Under the condition* (2. 10), *there exists a linear transformation*

(2. 12) $$u=P(\tau, \varepsilon)z.$$

which changes the equation (2. 11) *into*

(2. 13) $$[\tau^{-(m+q)}\varepsilon]^h \tau \frac{dz}{d\tau}=D(\tau, \varepsilon)z,$$

with the following properties:

a) *$D(\tau, \varepsilon)$ is holomorphic in τ and ε for*

(2. 14) $$|\tau| \leqq \tau_0, \qquad |\arg \tau| \leqq \alpha_0, \qquad 0<|\varepsilon| \leqq \varepsilon_0, \qquad |\arg \varepsilon| \leqq \delta_0, \qquad 0<|\tau^{-(m+q)}\varepsilon| \leqq \mu_0,$$

for sufficiently small positive numbers τ_0 and μ_0, and arbitrary α_0;

b) *as $|\tau^{-(m+q)}\varepsilon|$ tends to zero, we have*

(2. 15) $$D(\tau\ \varepsilon) \simeq \sum_{\nu=0}^{\infty} D^{(\nu)}(\tau)[\tau^{-(m+q)}\varepsilon]^{\nu}$$

uniformly in (2. 14);

c) *the matrices $D^{(\nu)}(\tau)$ are diagonal and holomorphic for $|\tau| \leqq \tau_0$ and*

$$D^{(0)}(0)=mh\begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix};$$

d) *the matrix* $P(\tau, \varepsilon)$ *is holomorphic in* τ *and* ε *for* (2. 14) *and has an uniformly asymptotic expansion as* $|\tau^{-(m+q)}\varepsilon|\to 0$:

$$P(\tau, \varepsilon)\simeq \sum_{\nu=0}^{\infty} P^{(\nu)}(\tau)[\tau^{-(m+q)}\varepsilon]^{\nu}, \tag{2. 16}$$

where $P^{(\nu)}(\tau)$ *are holomorphic and* $P^{(0)}(\tau)$ *is nonsingular for* $|\tau|\leqq\tau_0$.

Proof. We give here only a brief proof, and the details are for example in [3]. At first, from the assumption (2. 10), there exists a nonsingular matrix $P^{(0)}(\tau)$ such that $P^{(0)}(\tau)^{-1}C^{(0)}(\tau)P^{(0)}(\tau)$ is diagonal. Thus the transformation

$$u=P^{(0)}(\tau)z^{(0)}$$

changes the equation (2. 11) into

$$[\tau^{-(m+q)}\varepsilon]^{h}\tau\frac{dz^{(0)}}{d\tau}=D_0(\tau, \varepsilon)z^{(0)}, \tag{2. 17}$$

where

$$D_0(\tau, \varepsilon)\simeq \sum_{\nu=0}^{\infty} D_0^{(\nu)}(\tau)[\tau^{-(m+q)}\varepsilon]^{\nu} \tag{2. 18}$$

with $D_0^{(0)}(\tau)$ diagonal. Next, by the usual method, we can construct the matrix $Q^{(k)}(\tau)$ $(k\geqq 1)$ such that the transformation

$$z^{(0)}=\{I+Q^{(k)}(\tau)[\tau^{-(m+q)}\varepsilon]^{k}\}z^{(k)},$$

where I is unit matrix, takes the equation (2. 17) into

$$[\tau^{-(m+q)}\varepsilon]^{h}\tau\frac{dz^{(k)}}{dt}=D_k(\tau, \varepsilon)z^{(k)},$$

where

$$D_k(\tau, \varepsilon)\simeq \sum_{\nu=0}^{\infty} D_k^{(\nu)}(\tau)[\tau^{-(m+q)}\varepsilon]^{\nu}$$

with the matrix $D_k^{(k)}(\tau)$ diagonal. Then we get a formal transformation

$$u\sim\tilde{P}(\tau, \varepsilon)z,$$

where

$$\tilde{P}(\tau, \varepsilon)\sim P^{(0)}(\tau)\prod_{k=1}^{\infty}\{I+Q^{(k)}(\tau)[\tau^{-(m+q)}\varepsilon]^{k}\},$$

such that

$$[\tau^{-(m+q)}\varepsilon]^{h}\tau\frac{d\hat{z}}{d\tau}\sim\hat{D}(\tau, \varepsilon)\hat{z},$$

where

$$\hat{D}(\tau, \varepsilon) \sim \sum_{\nu=0}^{\infty} \hat{D}^{(\nu)}(\tau)[\tau^{-(m+q)}\varepsilon]^{\nu}$$

with $\hat{D}^{(\nu)}(\tau)$ holomorphic for $|\tau| \leqq \tau_0$ and diagonal. The analytical meaning follows from a Borel-Ritt theorem.

Since all the matrices $D^{(\nu)}(\tau)$ of (2.15) are diagonal, we can easily calculate a formal series solution of the differential equation (2.13) and get a following theorem.

THEOREM 2.1. *The differential equation* (2.11) *possesses a formal matrix solution of the form*

$$u \sim \sum_{\nu=0}^{\infty} \varepsilon^{\nu} u^{(\nu)}(\tau) \exp\left[\sum_{\nu=0}^{h} \varepsilon^{\nu-h} F^{(\nu)}(\tau)\right] \tag{2.19}$$

with the following properties;

$$u^{(\nu)}(\tau) = \tau^{-(m+q)\nu} \hat{u}^{(\nu)}(\tau), \tag{2.20}$$

where $\hat{u}^{(\nu)}(\tau)$ *are polynomials of degree* ν, *at most, in* $\log \tau$, *whose coefficients are holomorphic in* $|\tau| \leqq \tau_0$, *and bounded in the domain* (2.14);

$$F^{(\nu)}(\tau) = \int D^{(\nu)}(\tau) \tau^{(m+q)(h-\nu)-1} d\tau \qquad (\nu \geqq 0), \tag{2.21}$$

if in this integral the determination of the integral is chosen such that the series expansion has no constant term, $F^{(\nu)}(\tau)$ *can be written*

$$\begin{cases} F^{(\nu)}(\tau) = \tau^{-(m+q)(\nu-h)} \hat{F}^{(\nu)}(\tau) & (\nu \leqq h-1), \\ F^{(\nu)}(\tau) = f^{(\nu)} \log \tau + \tau^{-(m+q)(\nu-h)} \hat{F}^{(\nu)}(\tau) & (\nu \geqq h), \end{cases} \tag{2.22}$$

where $\hat{F}^{(\nu)}(\tau)$ *are holomorphic for* $|\tau| \leqq \tau_0$, *and* $f^{(\nu)}$ *are constant matrices.*

§ 3. Formal solution in the neighborhood of $x=0$.

We transform the equation (1.1) by the stretching and shearing transformation of the form (see Iwano [1]),

$$x = \varepsilon^{a} s, \tag{3.1}$$

$$y = \Omega(\varepsilon^{a}) v, \tag{3.2}$$

where $\Omega(\varepsilon^{a})$ is defined by (2.2) with ε^{a} instead of t, and then becomes

$$\frac{dv}{ds} = B(s, \varepsilon) v, \tag{3.3}$$

where

$$B(s, \varepsilon) = \begin{bmatrix} B_{11}(s, \varepsilon) & & 0 \\ \cdots & \ddots & \\ B_{p1}(s, \varepsilon) & \cdots & B_{pp}(s, \varepsilon) \end{bmatrix} \tag{3.4}$$

with

(3. 5) $$B_{ii}=\begin{bmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ & & & 1 \\ b_{in_i}, & b_{in_i-1}, & \cdots, b_{i1} & 0 \end{bmatrix},$$

(3. 6) $$b_{il}(s,\varepsilon)=\varepsilon^{-\gamma l}a_{il}(\varepsilon^{a}s,\varepsilon) \qquad \begin{pmatrix} l=n_i, n_i-1, \cdots, 2 \\ i=1, \cdots, p \end{pmatrix}.$$

Here we denote the number aq/m by γ. If $\mathfrak{B}_{jk}(s,\varepsilon)$ is a $(j\text{-}k)$ element of the matrix $B(s,\varepsilon)$ which does not belong to the matrix $B_{ii}(s,\varepsilon)$ $(i=1,\cdots,p)$, we can write

(3. 7) $$\mathfrak{B}_{jk}(s\varepsilon)=\varepsilon^{-(j+1-k)\gamma}\mathfrak{A}_{jk}(\varepsilon^{a}s,\varepsilon) \qquad (j>k).$$

From (1. 7) and (1. 8), we have formally

(3. 8) $$b_{il}(s,\varepsilon)\sim\sum_{\nu=0}^{\infty}\sum_{\mu=m_{il}^{(\nu)}}^{\infty}a_{il,\mu}^{(\nu)}s^{\mu}\varepsilon^{\nu+a\mu-\gamma l},$$

(3. 9) $$\mathfrak{B}_{jk}(s,\varepsilon)\sim\sum_{\nu=\mathfrak{N}_{jk}}^{\infty}\sum_{\mu=0}^{\infty}\mathfrak{A}_{jk,\mu}^{(\nu)}s^{\mu}\varepsilon^{\nu+a\mu-\gamma(j+1-k)}.$$

Here we put $\varepsilon=\rho^{m+q}$ with $\rho>0$ for $\varepsilon>0$ and rearrange the formal series (3. 8) and (3. 9) by the series of ascending power of ε. Remembering the assumptions (1. 9) and (1. 11), we can write the equation (3. 3) as

(3. 10) $$\frac{dv}{ds}=H(s,\rho)v,$$

where the matrix $H(s,\rho)$ is holomorphic in s and ρ for (1. 2), and formally,

(3. 11) $$H(s,\rho)\sim\sum_{\nu=0}H^{(\nu)}(s)\rho^{\nu}$$

with

(3. 12) $$H^{(0)}(s)=\begin{bmatrix} H_{11}^{(0)}(0) & & 0 \\ & \ddots & \\ 0 & & H_{pp}^{(0)}(s) \end{bmatrix}, \qquad H_{ii}^{(0)}(s)=\begin{bmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & 1 \\ h_{in_i}^{(0)}(s) & \cdots & h_{i2}^{(0)}(s) & 0 \end{bmatrix},$$

(3. 13) $$H^{(\nu)}(s)=\begin{bmatrix} H_{11}^{(\nu)} & & 0 \\ & \ddots & \\ \mathfrak{H}_{jk}^{(\nu)}(s) & & H_{pp}^{(\nu)} \end{bmatrix}, \qquad H_{ii}^{(\nu)}(s)=\begin{bmatrix} & & 0 & \\ h_{in_i}^{(\nu)}(s) & \cdots & h_{i2}^{(\nu)}(s) & 0 \end{bmatrix}.$$

All of the elements of the matrices $H^{(\nu)}(s)$ are polynomials of s and can be written

(3. 14) $$h_{il}^{(\nu)}(s)=s^{ql/m+\nu/mh}\hat{h}_{il}^{(\nu)}(s) \qquad \begin{pmatrix} l=n_i, n_i-1, \cdots, 2 \\ i=1, 2, \cdots, p \\ \nu=0, 1, \cdots \end{pmatrix},$$

(3. 15) $$\mathfrak{H}_{jk}^{(\nu)}(s)=s^{q(j+1-k)/m+\nu/mh}\hat{\mathfrak{H}}_{jk}^{(\nu)}(s)\quad\begin{pmatrix} j>k \\ \nu=0,1,\cdots\end{pmatrix},$$

where $\hat{h}_{il}^{(\nu)}(s)$ and $\hat{\mathfrak{H}}_{jk}^{(\nu)}(s)$ are bounded at $s=\infty$, and in particular,

(3. 16) $$h_{il}^{(0)}(s)=a_{il,\mu_l}^{(0)}s^{\mu_l}+\cdots\quad\begin{pmatrix} l=n_i,\ n_{i-1},\cdots,2 \\ i=1\ 2,\cdots,p\end{pmatrix},$$

where $a_{il,\mu_l}^{(0)}$ are constants and not zero for which $a\mu_l=\gamma l$, and $\cdots$ denotes a polynomial of s of lower degree which comes from the indices of (ν,μ) such that $\nu+a\mu-\gamma l=0$ $(v\geqq 1)$.

Now we consider the analytic meaning of (3. 11). We denote by $D(|s|\leqq s_1)$ the domain such that s and ρ are contained in (1. 2) and $|s|\leqq s_1$, and denote by $D(|s|>s_1)$ the domain of s and ρ contained in (1. 2) and $|s|>s_1$. Then clearly for arbitrary s_1, we have uniformly asymptotic expansion

(3. 17) $$H(s,\rho)\simeq\sum_{\nu=0}H^{(\nu)}(s)\rho^\nu$$

in $D(|s|\leqq s_1)$. Next we must consider the asymptotic property of $H(s,\rho)$ in $D(|s|>s_1)$.

Let $H^{(\nu)}(s)^{**}$ be a matrix defined by

$$H^{(\nu)}(s)^{**}=\Omega(s)^{-1}H^{(\nu)}(s)\Omega(s)\qquad(\nu\geqq 1),$$

where $\Omega(s)$ is defined by (2. 2). From (3. 14) and (3. 15) we have

$$H^{(\nu)}(s)^{**}=s^{\nu/mh+q/m}H^{(\nu)}(s)^*,$$

where $H^{(\nu)}(s)^*$ is bounded at $s=\infty$, and then $H^{(\nu)}(s)$ can be written

(3. 18) $$H^{(\nu)}(s)=s^{\nu/mh+q/m}\Omega(s)H^{(\nu)}(s)^*\Omega(s^{-1})\qquad(\nu\geqq 1).$$

Now we can prove the following lemma.

LEMMA 3. 1. *For every $r\geqq 0$, there exists a matrix $F_{r+1}(s)$ bounded in $D(|s|>s_1)$ such that*

(3. 19) $$H(s,\rho)-\sum_{\nu=0}^{r}H^{(\nu)}(s)\rho^\nu=s^{q/m}\Omega(s)E_{r+1}(s,\rho)\Omega(s^{-1})[s^{1/mh}\rho]^{r+1}.$$

Proof. To prove this Lemma, it is sufficient for us to state that for every $r\geqq 0$, there exists a bounded function $e_{r+1}(s,\rho)$ in $D(|s|>s_1)$ such that

$$b_{il}(s,\varepsilon)-\sum_{\nu=0}^{r}h_{il}^{(\nu)}(s)\rho^\nu=s^{lq/m+(r+1)/mh}e_{r+1}(s,\rho)\rho^{r+1},$$

$$\mathfrak{B}_{jk}(s,\varepsilon)-\sum_{\nu=0}^{r}\mathfrak{H}_{jk}^{(\nu)}(s)\rho^\nu=s^{q(j+1-k)/m+(r+1)/mh}e_{r+1}(s,\rho)\rho^{r+1}.$$

But this is easily derived by considering the order of magnitude of the remainder terms.

Let

$$v \sim \sum_{\nu=0}^{\infty} v^{(\nu)}(s)\rho^{\nu}$$

be a formal series solution of (3. 10). Then $v^{(\nu)}(s)$ must satisfy the following equations:

$$\frac{dv^{(0)}}{ds}=H^{(0)}(s)v^{(0)}, \tag{3. 20}$$

$$\frac{dv^{(\nu)}}{ds}=H^{(0)}(s)v^{(\nu)}+\sum_{\mu=1}^{\nu} H^{(\mu)}(s)v^{(\nu-\mu)} \qquad (\nu\geqq 1). \tag{3. 21}$$

The asymptotic solution of (3. 20) in the neighborhood of $s=\infty$ can be obtained using a theorem of Hukuhara [2]. At first, it is convenient for us to transform the equation (3. 20) by

$$\eta=m^{1/(m+q)}s^{1/m} \qquad (m^{1/(m+q)}>0), \tag{3. 22}$$

$$v^{(0)}=\Omega(s)w^{(0)}, \tag{3. 23}$$

then we have

$$\frac{dw^{(0)}}{d\eta}=[K_1(\eta)+K_2(\eta)]w^{(0)}, \tag{3. 24}$$

where

$$K_1(\eta)=\begin{bmatrix} K_{11}(\eta) & & 0 \\ & \ddots & \\ 0 & & K_{pp}(\eta) \end{bmatrix}, \qquad K_{jj}(\eta)=\begin{bmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & 1 \\ k_{jn_j}(\eta) & \cdots, & k_{j2}(\eta) & 0 \end{bmatrix}, \tag{3. 25}$$

with

$$k_{il}(\eta)=m^{ql/(m+q)}\eta^{-q(l-1)+m-1}h_{il}^{(0)}(s) \qquad \begin{pmatrix} l=n_i, n_i-1, \cdots, 2 \\ i=1, 2, \cdots, p \end{pmatrix}, \tag{3. 26}$$

and

$$K_2(\eta)=-q\eta^{-1}\begin{bmatrix} 0 & & \\ & 1 & 0 \\ 0 & & \ddots & \\ & & & n-1 \end{bmatrix}. \tag{3. 27}$$

If we substitute the equations (3. 16) into (3. 26) and rearrange them in powers of η, we have

$$\frac{dw^{(0)}}{d\eta}=\{\eta^{q+m-1}B^{(0)}(0)+\eta^{q+m-2}L_1+\cdots\}w^{(0)}, \tag{3. 28}$$

where $B^{(0)}(0)$ is a constant matrix calculated from (2. 9) whose characteristic roots are λ_j and $L_1, \cdots$ are matrices of lower order. From the assumption (2. 10), we can calculate the asymptotic solutions of the equation (3. 28) in the neighborhood of $\eta=\infty$. It is easy to see that the equation (3. 28) has a formal matrix solution of the form

$$w^{(0)} \sim \left\{ \sum_{\nu=0}^{\infty} w_{\nu}^{(0)} \eta^{-\nu} \right\} \eta^{\hat{\Pi}} \exp [\hat{Q}(\eta)], \tag{3. 29}$$

where $w_{\nu}^{(0)}$ are constant matrices and $w_0(0)$ is nonsingular, $\hat{Q}(\eta)$ is a diagonal matrix such that

$$\hat{Q}(\eta) = \begin{bmatrix} \hat{q}_1(\eta) & & 0 \\ & \ddots & \\ 0 & & \hat{q}_n(\eta) \end{bmatrix}, \tag{3. 30}$$

where

$$\hat{q}_j(\eta) = \frac{\lambda_j}{m+q} \eta^{m+q} + \hat{q}_{j2} \eta^{m+q-1} + \cdots + q_{jm+q} \eta, \tag{3. 31}$$

and $\hat{\Pi}$ is a constant diagonal matrix such that

$$\hat{\Pi} = \begin{bmatrix} \hat{\pi}_1 & & 0 \\ & \ddots & \\ 0 & & \hat{\pi}_n \end{bmatrix}. \tag{3. 32}$$

Now let us apply the theory of Hukuhara [2] to the equation (3. 28) and (3. 29). At first we define the singular direction $\arg \eta = \theta_{jk}$ $(j, k=1, \cdots, n)$ in the η-plane for which

$$\cos \{(m+q)\theta_{jk} + \arg (\lambda_j - \lambda_k)\} = 0. \tag{3. 33}$$

For each fixed j $(j=1, 2, \cdots, n)$, there corresponds a formal solution of the equation (3. 20) of the form

$$w_{jk}^{(0)}(\eta) \sim \left\{ \sum_{\nu=0} w_{jk\nu}^{(0)} \eta^{-\nu} \right\} \eta^{\hat{\pi}_j} \exp [\hat{q}_j(\eta)] \qquad (k=1, 2, \cdots, n), \tag{3. 34}$$

where $w_{jk\nu}^{(0)}$ are constant n-dim. vectors. Let $\hat{\Sigma}_j$ be sector in the η-plane

$$\hat{\Sigma}_j\colon\ \theta_{j1} \leqq \arg \eta \leqq \theta_{j2} \tag{3. 35}$$

such that $\hat{\Sigma}_j$ contains at least one singular direction θ_{jk} $(k=1, 2, \cdots, n)$. Clearly a finite number of such sectors overlap the full neighborhood of $\eta=\infty$. Here we divide the indices k into two groups. The first is the indices $(k_1, \cdots, k_{n'})$ for which

$$\mathrm{Re} \left\{ \frac{d}{d\eta} \hat{q}_j(\eta) - \frac{d}{d\eta} \hat{q}_k(\eta) \right\} \leqq b < 0 \tag{3. 36}$$

for all sufficiently large $\eta \in \hat{\Sigma}_j$, and the second is the remainder indices. Then from a theorem of Hukuhara [2, p. 155], we can conclude that the equation (3. 28) has one and only one solution which is asymptotically developable in series (3. 34) as $\eta \to \infty$ in $\hat{\Sigma}_j$ such that

$$w_{jk}^{(0)}(\eta_1) = w_{jk}^{(0)} \qquad (k=k_1, \cdots, k_{n'}),$$

where η_1 is taken large enough in $\hat{\Sigma}_j$ and $w_{jk}^{(0)}$ are arbitrary numbers.

From this, the neighborhood of $\eta=\infty$ is divided into a finite number of sufficiently small sectors, and in each of such sectors there exists a fundamental matrix solution of the equation (3. 28) which can be expanded asymptotically in the form (3. 29). Now we pick up one of such sectors $\hat{\Sigma}$,

$$\hat{\Sigma}\colon \theta_1 \leqq \arg \eta \leqq \theta_2,$$

where θ_1 and θ_2 do not coincide with any singular direction, and let S be the inverse image of $\hat{\Sigma}$ under the transformation (3. 22). Thus we have the following lemma.

LEMMA 3. 2. *The equation* (3. 20) *has a fundamental solution which is asymptotically developable for sufficiently large s in the sector S such that*

$$v^{(0)} \simeq \Omega(s)\left\{\sum_{\nu=0}^{\infty} v_\nu^{(0)} s^{-\nu/m}\right\} s^{\Pi} \exp[Q(s)], \tag{3.37}$$

where $v_\nu^{(0)}$ are constant matrices and $v_0^{(0)}$ is nonsingular, Π is a constant diagonal matrix

$$\Pi = \begin{bmatrix} \pi_1 & & 0 \\ & \ddots & \\ 0 & & \pi_n \end{bmatrix},$$

and $Q(s)$ is a diagonal matrix

$$Q(s) = \begin{bmatrix} q_1(s) & & 0 \\ & \ddots & \\ 0 & & q_n(s) \end{bmatrix},$$

$$q_j(s) = \frac{m\lambda_j}{m+q} s^{(m+q)/m} + q_{j1} s^{(m+q-1)/m} + \cdots + q_{jk+q} s^{1/m}. \tag{3.38}$$

Next we must solve the equation (3. 21) all of which have the form

$$\frac{dt}{ds} = H^{(0)}(s)t + F(s) \tag{3.39}$$

with entire coefficients. The integral

$$t(s) = \int_{\Gamma(s)} v^{(0)}(s) v^{(0)}(\sigma)^{-1} F(\sigma) d\sigma \tag{3.40}$$

is a solution of (3. 39) if $\Gamma(s)$ designates a set of paths $\gamma_{jk}(s)$ in σ-plane ending at s for every scalar integral contained in (3. 40). The paths $\gamma_{jk}(s)$ will be given later.

Define $\hat{t}(s)$, $\hat{v}^{(0)}(s)$ and $\hat{F}(s)$ by the relations

$$\begin{aligned} t(s) &= \Omega(s)\hat{t}(s) s^{\Pi} \exp[Q(s)], \\ v^{(0)}(s) &= \Omega(s)\hat{v}^{(0)}(s) s^{\Pi} \exp[Q(s)], \\ F(s) &= \Omega(s)\hat{F}(s) s^{\Pi} \exp[Q(s)]. \end{aligned} \tag{3.41}$$

Then (3. 40) becomes

$$\hat{t}(s) = \hat{v}^{(0)}(s) \int_{\Gamma(s)} \{\exp[Q(s) - Q(\sigma)]\} (s/\sigma)^{\Pi} v^{(0)}(\sigma)^{-1} \tag{3.42}$$

$$\hat{F}(\sigma)(\sigma/s)^{\Pi}\{\exp[Q(\sigma)-Q(s)]\}d\sigma.$$

Assume for the moment that

(3. 43) $\hat{F}(s)s^{-b}$ is bounded in $S(s_1)$,

where b is a positive constant and $S(s_1)$ is a domain such that

$$s\in S, \qquad |s|>s_1$$

for sufficiently large positive number s_1, then we have

$$\hat{v}^{(0)}(s)^{-1}\hat{F}(s)s^{-b} \text{ is bounded in } S(s_1).$$

For simplicity, we introduce $q_{jk}(s)$ by

$$q_{jk}(s)=q_j(s)-q_k(s). \tag{3. 44}$$

Then every element of the matrix in the integrand of (3. 42) has a form

$$p_{jk}(\sigma)\sigma^b(s/\sigma)^{\pi_j-\pi_k}\exp[q_{jk}(s)-q_{jk}(\sigma)], \tag{3. 45}$$

where $p_{jk}(\sigma)$ is bounded in $S(s_1)$.

To calculate this integral it is convenient to introduce the auxiliary variables

$$\zeta=\frac{m}{m+q}\sigma^{(m+q)/m}, \qquad \xi=\frac{m}{m+q}s^{(m+q)/m}. \tag{3. 46}$$

Let the sector S in the σ-plane correspond to the sector Σ in the ζ-plane and let $S(s_1)$ correspond to $\Sigma(\xi_1)$ with $\xi_1=(m/(m+q))s_1^{(m+q)/m}$. We assume here that the central angle of Σ is not larger than π (this is always possible by subdividing the sector S if necessary). Now we determine the path of integration $\gamma_{jk}(s)$ for each pair of (j, k). First let $j\neq k$. Draw in the ζ-plane the sector Σ and the line $\arg\zeta=\alpha_{jk}$ for which the quantity

$$\cos\{\alpha_{jk}+\arg(\lambda_j-\lambda_k)\} \tag{3. 47}$$

equals to zero. Then the ζ-plane is divided into two half plane, in one of which the quantity (3. 47) is positive and in the other plane negative. If the sector Σ and the positive half plane have a common part, we can draw the line l_{jk} for which the quantity (3. 47) is positive. Then we can draw the line $\lambda_{jk}(\xi)$ such that the line $\lambda_{jk}(\xi)$ is parallel to l_{jk}, starts from ξ and extends to ∞ in Σ. In this case, we can choose a positive number β_{jk} such that

$$\operatorname{Re}[(\lambda_j-\lambda_k)(\xi-\zeta)]\leqq-\beta_{jk}|\xi-\zeta|$$

on the line $\lambda_{jk}(\xi)$ for all $\zeta\in\Sigma(\xi_1)$, and for each fixed ξ there exist constant numbers R and K uniformly in ξ such that

$$\begin{aligned}&\operatorname{Re}[q_{jk}(s)-q_{jk}(\sigma)]\leqq K \quad \text{for } |\xi-\zeta|\leqq R, \qquad |\xi|>\xi_1,\ |\zeta|>\xi_1,\\ &\operatorname{Re}[q_{jk}(s)-q_{jk}(\sigma)]\leqq-\beta_{jk}|\xi-\zeta| \quad \text{for } |\xi-\zeta|>R,\ |\xi|>\xi_1,\ |\zeta|>\xi_1.\end{aligned} \tag{3. 48}$$

Next, if the sector Σ and the positive half plane are disjoint, we can take $\lambda_{jk}(\xi)$ as a segment from some fixed point ξ_2 to ξ in $\Sigma(\xi_1)$ such that the inequality (3. 48) is also satisfied on it for some positive constant β_{jk}. Here we must choose the quantities $|\xi_2|$ and ξ_1 sufficiently large. Let the integral path $\gamma_{jk}(s)$ be the inverse image of $\lambda_{jk}(\xi)$ under the transformation (3. 46). For the paths $\gamma_{jj}(s)$ it is sufficient to take them as segments from some fixed point s_2 to s in $S(s_1)$. In order to make sure that all points of $\gamma_{jk}(s)$ lie in the domain $S(s_1)$ of the σ-plane, we must limit s to a domain $S(s_0)$, where s_0 is sufficiently large. By these determinations of the paths of integration, we can easily prove the following lemma.

LEMMA 3. 3. *If the differential equation* (3. 39) *satisfies the condition* (3. 43), *then it possesses a solution of the form*

$$t(s)=s^{b+1}\Omega(s)t^*(s)s^{\Pi}\exp[Q(s)] \tag{3. 49}$$

where $t^*(s)$ is bounded as s extends to ∞ in $S(s_0)$.

Proof. The integral (3. 45) along $\gamma_{jk}(s)$ has the form in terms of ζ and ξ,

$$\xi^{\pi_j-\pi_k}\int_{\lambda_{jk}(\xi)}\{\exp[q_{jk}(\xi)-q_{jk}(\zeta)]\}\tilde{p}_{jk}(\sigma)\zeta^{\pi_k-\pi_j}\zeta^{(mb-q)/(m+q)}d\zeta, \qquad j,k=1,2,\cdots,n.$$

Let us express ζ on $\lambda_{jk}(\xi)$ in the form

$$\zeta=\xi+\delta_{jk}\alpha, \qquad j,k=1,2,\cdots,n,$$

where δ_{jk} is a constant of modulus 1 and α is a real variable, and divide the integral into two parts of $|\xi-\zeta|\leqq R$ and $|\xi-\zeta|>R$. Then the above integral becomes

$$\xi^{(mb-q)/(m+q)}\int_0^R+\int_R^{\alpha_0}\{\exp[q_{jk}(\xi)-q_{jk}(\zeta)]\}\tilde{p}_{jk}(\sigma)\left[1+\delta_{jk}\frac{\alpha}{\xi}\right]^{\pi_k-\pi_j+(mb-q)/(m+q)}\delta_{jk}d\alpha,$$

where α_0 is a certain finite constant or ∞ depending on $\lambda_{jk}(\xi)$. For $j\neq k$, the inequality (3. 48) assures us that the above integral is a uniformly bounded function of ξ for $\xi\in\Sigma(\xi_1)$, and the integral of (3. 45) is of the order $O(s^{b-q/m})$ as $s\to\infty$ in $S(s_1)$. For $j=k$, $q_{jk}(s)=0$, $\pi_j=\pi_k$ in (3. 45), and the integral of (3. 45) along $\gamma_{jj}(s)$ is $O(s^{b+1})$. Thus Lemma 3. 3 follows at once from (3. 41).

Now using the above lemma, we get the asymptotic solution of the differential equation (3. 21) for each $\nu\geqq 1$ in the neighborhood of $s=\infty$, namely it will be proved the following lemma.

LEMMA 3. 4. *The differential equation* (3. 21) *possesses a particular solution of the form*

$$v^{(\nu)}(s)=s^{e\nu}\Omega(s)w^{(\nu)}(s)s^{\Pi}\exp[Q(s)], \tag{3. 50}$$

where $w^{(\nu)}(s)$ is bounded in the domain $S(s_0)$ and

$$e=\frac{1}{mh}+\frac{q}{m}+1. \tag{3. 51}$$

Proof. We prove this by induction. For $\nu=0$, the equation (3. 21) becomes the equation (3. 20) and the statements in Lemma 3. 4 is satisfied from Lemma 3. 2.

Assume it to be true for $\nu<r$. The μ-th term of the summation in (3. 21) has a form

$$H^{(\mu)}(s)v^{(r-\mu)}(s)=s^{f(r,\mu)}\Omega(s)H^{(\mu)}(s)^{*}w^{(r-\mu)}(s)s^{\Pi}\exp[Q(s)]$$

where $H^{(\nu)}(s)^{*}$ is defined in (3. 18) and

$$f(r,\mu)=\frac{\nu}{mh}+\frac{q}{m}+e(r-\mu).$$

The exponent $f(r,\mu)$ is the largest for $\mu=1$, and then for $\nu=r$ we can apply Lemma 3. 3 to the equation (3. 39) with $b=f(r,1)$.

Thus we get the following theorem.

THEOREM 3. 1. *Let $k(s)$ be a function such that*

$$k(s)=\begin{cases}0, & \text{if } |s|\leqq s_0.\\ 1, & \text{if } |s|>s_0.\end{cases} \tag{3. 52}$$

Then the differential equation (3. 10) *has a formal solution v of the form*

$$v\sim\Omega(s^{k(s)})\left\{\sum_{\nu=0}^{\infty}w^{(\nu)}(s)[s^{k(s)e}\rho]^{\nu}\right\}s^{k(s)\Pi}\exp[Q(s)], \tag{3. 53}$$

where $w^{(\nu)}(s)$ are bounded in the domain (1. 2) *and $|s|\leqq s_0$ if $k(s)=0$, and in the domain $s\in S(s_0)$, $w^{(\nu)}(s)$ is bounded if $k(s)=1$.*

REMARK. In the previous papers [3], [4], the connection formula between the solution of the equation (3. 20) in the neighborhood of $s=0$ and that in the neighborhood of $s=\infty$ can be obtained from a theorem of Okubo [5] or a theorem of Turrittin [6]. But in this case they are no longer applicable, then we must calculate the connection formula by the method of asymptotic matching.

§ 4. Existence theorem (1).

Here we prove the following existence theorem.

THEOREM 4. 1. *Let T be any sector of τ-plane with vertex at the origin and central angle less than $\pi/(m+q)h$, and let*

$$u\sim\sum_{\nu=0}^{\infty}\varepsilon^{\nu}u^{(\nu)}(\tau)\exp\left[\sum_{\nu=0}^{h}\varepsilon^{\nu-h}F^{(\nu)}(\tau)\right] \tag{4. 1}$$

be a formal solution of (2. 11) *which is defined in Theorem* 2. 1. *Then there exists an actual solution of* (2. 11)

$$u(\tau,\varepsilon)=\hat{u}(\tau,\varepsilon)\exp\sum_{\nu=0}^{h}\varepsilon^{\nu-h}F^{(\nu)}(\tau),$$

and for every integer r, there exists a domain D_1 of ε,τ-plane defined by

$$\tau\in T,\qquad 0<|\varepsilon|\leqq\varepsilon_1,\qquad |\arg\varepsilon|\leqq\delta_1,\qquad c_1|\varepsilon|^{1/(m+q)}\leqq|\tau|\leqq c_2 \tag{4. 2}$$

(ε_1, δ_1, c_1 and c_2 are certain constants independent of ε), in which it holds that

$$\hat{u}(\tau, \varepsilon)-\sum_{\nu=0}^{r} \varepsilon^{\nu} u^{(\nu)}(\tau)=E_r(\tau, \varepsilon)[\tau^{-(m+q)}\varepsilon]^{r+1}, \tag{4.3}$$

where $E_r(\tau, \varepsilon)$ is a bounded matrix function.

Proof. This can be proved by the same method as in [4], but for the completeness we will repeat it. It is sufficient for us to prove the statements in Theorem 4.1 only for the equation (2.13).

Define the matrices $D^{(r)}(\tau, \varepsilon)$ and $z^{(r)}(\tau, \varepsilon)$ by

$$D^{(r)}(\tau, \varepsilon)=\sum_{\nu=0}^{r+h} D^{(\nu)}(\tau)[\tau^{-(m+q)}\varepsilon]^{\nu},$$

$$z^{(r)}(\tau, \varepsilon)=\exp\sum_{\nu=0}^{r+h} \varepsilon^{\nu-h} F^{(\nu)}(\tau).$$

Then $z^{(r)}(\tau, \varepsilon)$ is a fundamental solution of the equation

$$[\tau^{-(m+q)}\varepsilon]^{h}\tau\frac{dz}{d\tau}=D^{(r)}(\tau, \varepsilon)z.$$

By the transformation

$$z=z^{(r)}+w^{(r)}, \tag{4.4}$$

the equation (2.13) becomes

$$[\tau^{-(m+q)}\varepsilon]^{h}\tau\frac{dw^{(r)}}{d\tau}=D(\tau, \varepsilon)w^{(r)}+[D(\tau, \varepsilon)-D^{(r)}(\tau, \varepsilon)]z^{(r)}. \tag{4.5}$$

Define the matrices $K(\tau, \varepsilon)$, $D^{(h)}(\tau, \varepsilon)$, $\hat{w}^{(r)}(\tau, \varepsilon)$ and $\hat{z}^{(r)}(\tau, \varepsilon)$ by

$$K(\tau, \varepsilon)=\sum_{\nu=0}^{h} \varepsilon^{\nu} F^{(\nu)}(\tau),$$

$$D^{(h)}(\tau, \varepsilon)=\sum_{\nu=0}^{h} D^{(\nu)}(\tau)[\tau^{-(m+q)}\varepsilon]^{\nu},$$

$$\hat{w}^{(r)}(\tau, \varepsilon)=w^{(r)}(\tau, \varepsilon)\exp[-\varepsilon^{-h}K(\tau, \varepsilon)],$$

$$\hat{z}^{(r)}(\tau, \varepsilon)=z^{(r)}(\tau, \varepsilon)\exp[-\varepsilon^{-h}K(\tau, \varepsilon)],$$

then the equation (4.5) becomes

$$\begin{aligned}[\tau^{-(m+q)}\varepsilon]^{h}\tau\frac{d\hat{w}^{(r)}}{d\tau}&=D^{(h)}(\tau, \varepsilon)\hat{w}^{(r)}-\hat{w}^{(r)}D^{(h)}(\tau, \varepsilon)+[D(\tau, \varepsilon)-D^{(h)}(\tau, \varepsilon)]\hat{w}^{(r)}\\ &\quad+[D(\tau, \varepsilon)-D^{(r)}(\tau, \varepsilon)]\hat{z}^{(r)}.\end{aligned} \tag{4.6}$$

Here we have

$$\begin{aligned}D(\tau, \varepsilon)-D^{(h)}(\tau, \varepsilon)&=[\tau^{-(m+q)}\varepsilon]^{h+1}E^{(h)}(\tau, \varepsilon),\\ D(\tau, \varepsilon)-D^{(r)}(\tau, \varepsilon)&=[\tau^{-(m+q)}\varepsilon]^{r+h+1}E^{(r)}(\tau, \varepsilon),\end{aligned} \tag{4.7}$$

where $E^{(h)}(\tau, \varepsilon)$ and $E^{(r)}(\tau, \varepsilon)$ are bounded in (2. 14), and from (2. 22) we can write

$$
\begin{aligned}
\hat{z}^{(r)} &= \exp \sum_{\nu=h+1}^{r+h} \varepsilon^{\nu-h} F^{(\nu)}(\tau) \\
&= \exp \sum_{\nu=h+1}^{r+h} [\tau^{-(m+q)}\varepsilon]^{\nu+h} \hat{F}^{(\nu)}(\tau)
\end{aligned} \tag{4. 8}
$$

with bounded matrices $\hat{F}^{(\nu)}(\tau)$. Then $\hat{z}^{(r)}$ is bounded in (2. 14).

If we write the equation (4. 6) for each component of $\hat{w}^{(r)}(\tau, \varepsilon) = (\hat{w}_{jk}^{(r)}(\tau, \varepsilon))$, it becomes

$$
[\tau^{-(m+q)}\varepsilon]^h \tau \frac{dw_{jk}^{(r)}}{d\tau} = (d_j - d_k)\hat{w}_{jk}{}^{(r)} + [(D - D^{(h)})\hat{w}^{(r)} + (D - D^{(r)})\hat{z}^{(r)}]_{jk},
$$

where each of d_j $(j=1, \cdots, n)$ is the j-th diagonal element of $D^{(h)}(\tau, \varepsilon)$, and this equation can be converted by the method of variation of constants into the following integral equation:

$$
\begin{aligned}
\hat{w}_{jk}{}^{(r)}(\tau, \varepsilon) = \varepsilon^{-h} \int_{\gamma_{jk}} & [\exp \varepsilon^{-h}\{\mu_{jk}(\tau) - \mu_{jk}(\sigma)\}] [\{D(\sigma, \varepsilon) - D^{(h)}(\sigma, \varepsilon)\}\hat{w}^{(r)}(\sigma. \varepsilon) \\
& + \{D(\sigma, \varepsilon) - D^{(r)}(\sigma, \varepsilon)\}\hat{z}^{(r)}(\sigma, \varepsilon)]_{jk} \sigma^{(m+q)h-1} d\sigma \qquad (j, k = 1, \cdots, n),
\end{aligned} \tag{4. 9}
$$

where

$$
\mu_{jk}(\tau, \varepsilon) = K_j(\tau, \varepsilon) - K_k(\tau, \varepsilon)
$$

with diagonal elements $K_j(\tau, \varepsilon)$ of $K(\tau, \varepsilon)$, and γ_{jk} is an integral path which is described in later.

Now we prove the existence of the solution of (4. 9) by the fixed point theorem. Let $\mathfrak{F}$ be the set of all matrices $W(\tau, \varepsilon) = (w_{jk}(\tau, \varepsilon))$ whose components are holomorphic in (τ, ε) for (4. 2) and satisfy the inequality

$$
\| W(\tau, \varepsilon) \| \leqq M |\tau^{-(m+q)}\varepsilon|^{r+1}, \tag{4. 10}
$$

where

$$
\| W \| = \max_{1 \leqq j \leqq n} \left[\sum_{k=1}^{n} |w_{jk}| \right],
$$

and the constant M will be chosen appropriately. Clearly $\mathfrak{F}$ is closed, compact and convex with respect to the topology of uniform convergence on each compact subset of the domain (4. 2)· The mapping $\mathfrak{T}(W)$ is defined by right-hand term of (4. 9) with W in place of $\hat{w}^{(r)}(\tau, \varepsilon)$. To apply the fixed point theorem, we must only prove that the integral (4. 9) converges uniformly and the matrices $\mathfrak{T}(W)$ satisfy the inequality (4. 10) for all matrices W of $\mathfrak{F}$.

From (4. 7) and (4. 8), there exist constants M_h, M_r and B independent of ε, τ and $\tau^{-(m+q)}\varepsilon$ such that

$$
\begin{gathered}
\| D(\tau, \varepsilon) - D^{(h)}(\tau, \varepsilon) \| \leqq M_h |\tau^{-(m+q)}\varepsilon|^{h+1}, \\
\| D(\tau, \varepsilon) - D^{(r)}(\tau, \varepsilon) \| \leqq M_r |\tau^{-(m+q)}\varepsilon|^{r+h+1}, \\
\| \hat{z}^{(r)}(\tau, \varepsilon) \| \leqq B
\end{gathered}
$$

in the domain (2. 14). Then we have

$$|\mathfrak{T}(w_{jk})|$$

(4. 11) $$\leqq|\varepsilon|^{r+1}\int_{\gamma_{jk}}|\exp\varepsilon^{-h}\{\mu_{jk}(\tau)-\mu_{jk}(\sigma)\}|[M_h M|\sigma^{-(m+q)}\varepsilon|+M_r B]|\sigma|^{-(m+q)(r+1)-1}d\sigma.$$

Here we must determine the paths $\gamma_{jk}(\tau)$. In doing this remark that if we put

(4. 12) $$\mu_{jk}{}^{(0)}(\tau)=\frac{m}{m+q}\tau^{(m+q)h}(\lambda_j-\lambda_k),$$

then we have

(4. 13) $$\frac{d\mu_{jk}}{d\tau}(\tau,\varepsilon)=\frac{d\mu_{jk}{}^{(0)}(\tau)}{d\tau}[1+O(\tau)+O(\tau^{-(m+q)}\varepsilon)]$$

in the domain (2. 14) and for small $|\tau^{-(m+q)}\varepsilon|$. Let

(4. 14) $$|\tau^{-(m+q)}\varepsilon|\leqq\mu_1.$$

The conditions

(4. 15) $$\tau\in T,\qquad c_1|\varepsilon|^{1/(m+q)}\leqq|\tau|\leqq c_2$$

determine a region in the τ-plane which depends on ε. If we introduce the auxiliary variables

(4. 16) $$\zeta=\sigma^{(m+q)h},\qquad \xi=\tau^{(m+q)h},$$

then the image Σ of the region in the ξ-plane is a sector of annuli whose central angle at the origin is less than π. Let $\Sigma^*\supset\Sigma$ be an isosceles triangle with the same axis of symmetry as Σ, with its base tangent to the larger circular arc of the boundary of Σ and its sides passing through the endpoints of the smaller boundary of Σ. Without loss of generality we may assume that, for positive ε, the base of Σ^* is not parallel to any rays through the origin of the ξ-plane on which the quantity

(4. 17) $$\mathrm{Re}\left[\varepsilon^{-h}\frac{m}{m+q}\xi(\lambda_j-\lambda_k)\right]\quad(j\neq k)$$

equals to zero. δ_1 in (4. 2) is to be taken so small that (4. 17) remains different from zero for all ε with $|\arg\varepsilon|\leqq\delta_1\leqq\delta_0$. The size β of the two equal angles at the base of Σ^* is to be independent of ε and so small that any direction from an endpoint of the base of Σ^* into Σ^* is not parallel to a ray through the origin on which (4. 17) vanishes for some ε with $|\arg\varepsilon|\leqq\delta_1$.

Let ξ_1, ξ_2, ξ_3 be the vertices of Σ^* as in Figure 1. The radii r_1, r_2 of the circular arcs that bound Σ are

(4. 18) $$r_1=c_1{}^{(m+q)h}|\varepsilon|^h\leqq r_2=c_2{}^{(m+q)h}.$$

Since the shape of Σ^* is independent of ε, there exist positive constants k_1 and k_2 depending only on β such that

(4. 19) $$|\xi_1|=k_1 r_1,\qquad |\xi_2|\leqq k_2 r_2.$$

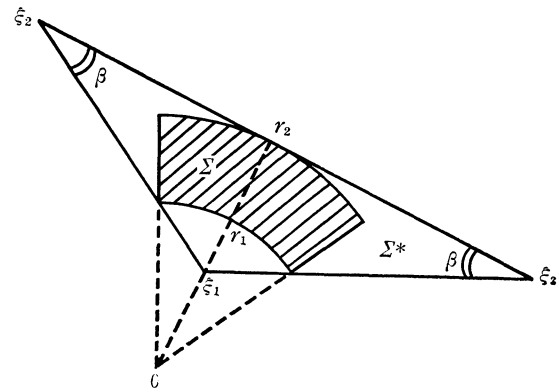

Fig. 1

Now c_1 and c_2 in (4. 15) must be chosen so that the inverse image H^* of Σ^* in the τ-plane lies in the domain where (2. 14) is satisfied. Σ^* lies in the ring

(4. 20) $$|\xi_1| \leqq |\xi| \leqq |\xi_2|.$$

Hence, by (4. 18), (4. 19) and (4. 20), H^* lies in

$$k_1^{1/(m+q)h} c_1 |\varepsilon|^{1/(m+q)} \leqq |\tau| \leqq k_2^{1/(m+q)h} c_2.$$

The first of these inequalities implies that

$$|\tau^{-(m+q)}\varepsilon| \leqq k_1^{-1/h} c_1^{-(m+q)}.$$

Therefore, (4. 14) can be satisfied by taking c_1 large enough. The condition $|\tau| \leqq \tau_0$ is satisfied if c_2 is taken sufficiently small. In order to be sure $|\xi_1| < |\xi_2|$, it may be necessary to take ε_1 in (4. 2) smaller than ε_0 in (1. 2). Now consider the region Σ^* in the ζ-plane and let $\zeta = \xi$ be some point in Σ^*. From the method of construction of Σ^*, the quantity (4. 17) with ζ in place of ξ, changes monotonically if ζ moves from ξ_2 along a straight segment to ξ and then to ξ_3. Hence this quantity increases along one of the two paths $\xi_2\xi$ or $\xi_3\xi$. For $j \neq k$, let $\lambda_{jk}(\xi)$ be the one of these two segments along which (4. 17), with ζ for ξ, increases. The inverse image of $\lambda_{jk}(\xi)$ under (4. 16) will be our path $\gamma_{jk}(\tau)$. For $j=k$, we may take either of these paths as $\gamma_{jj}(\tau)$. Finally from (4. 13), we can choose τ_0 and μ_1 so small that $\mathrm{Re}\,[\varepsilon^{-h}\mu_{jk}(\sigma, \varepsilon)]$ also increases along $\gamma_{jk}(\tau)$. Now let us consider the integral (4. 11). From the way $\gamma_{jk}(\tau)$ was constructed, we have

$$|\quad (w_{jk})| \leqq |\varepsilon|^{r+2} M_h M \int |\sigma|^{-(m+q)(r+2)-1} |d\sigma|$$

$$+ |\varepsilon|^{r+1} M_r B \int |\sigma|^{-(m+q)(r+1)-1} |d\sigma|.$$

About the integral appearing in above inequality, we prove the following lemma.

LEMMA 4. 1. *There exists a constant M_1, depending on β and γ but not on*

ε, μ_1 and τ_0 such that

$$\int_{r_{jk}(\tau)} |\sigma|^{-(m+q)r-1}|d\sigma| \leqq M_1|\tau|^{-(m+q)r} \qquad (r>0). \tag{4.21}$$

Proof. If we write this integral in term of ζ, (4. 21) becomes

$$\frac{1}{(m+q)h}\int_{\lambda_{jk}(\xi)} |\zeta|^{-r/h-1}|d\zeta|.$$

To fix the idea assume that $\lambda_{ij}(\xi)$ starts at ξ_2. Let θ denote the polar angle in the ζ-plane. Designate by p, θ_p the polar coordinates of the end point of the perpendicular from $\zeta=0$ onto the straight line on which $\lambda_{ij}(\xi)$ is situated, and denote by θ_ξ the polar angle of ξ. Then along $\lambda_{jk}(\xi)$ we have

$$-\frac{\pi}{2}+\beta \leqq \theta_p-\theta < \frac{\pi}{2}.$$

Let $\lambda_{jk}^{(1)}(\xi)$ be the part of $\lambda_{jk}(\xi)$ where

$$|\theta_p-\theta| \leqq \frac{\pi}{2}-\beta.$$

If $\lambda_{jk}^{(1)}(\xi)$ is not empty,

$$\int_{\lambda_{jk}^{(1)}(\xi)} |\zeta|^{-r/h-1}|\,d\zeta\,| = p^{-r/h}\int_{\theta_p-\pi/2-\beta}^{\theta_\xi} \cos{(\theta-\theta_p)}^{-r/h+1}d\theta$$

and we have $|\theta_\xi-\theta_p| \leqq \pi/2-\beta$ so that

$$p=|\xi|\cos{(\theta_\xi-\theta_p)} \geqq |\xi|\sin\beta.$$

Hence

$$\int_{\lambda_{jk}^{(1)}(\xi)} |\zeta|^{-r/h-1}|d\zeta| = (\sin\beta)^{-r/h}|\xi|^{-r/h}\int_{-\pi/2+\beta}^{\pi/2-\beta} \cos^{r/h-1}\theta d\theta. \tag{4.22}$$

Let $\lambda_{jk}^{(2)}(\xi)$ be the part of $\lambda_{ik}(\xi)$ not in $\lambda_{jk}^{(1)}(\xi)$ and assume that it is not empty. On this segment, $|d\zeta| < |\,d|\zeta|\,|\sec\beta$, and therefore

$$\int_{\lambda_{jk}^{(2)}(\xi)} |\zeta|^{-r/h-1}|\,d\zeta\,| \leqq \frac{h\sec\beta}{r}|\xi^*|^{-r/h}, \tag{4.23}$$

where ξ^* is the left end point of $\lambda_{jk}^{(2)}(\xi)$. If $\lambda_{jk}^{(1)}(\xi)$ is not empty $|\xi| \leqq |\xi^*|$, and if $\lambda_{jk}^{(1)}(\xi)$ is empty ξ^* coincide with ξ. Then ξ^* can be replaced by ξ in (4. 23), and Lemma 4. 1 follows at once by adding (4. 22) and (4. 23).

From Lemma 4. 1 we have

$$|\mathfrak{T}(w_{jk})| \leqq [M_h M\mu_1 + M_r B]M_1|\tau^{-(m+q)}\varepsilon|^{r+1},$$

hence if μ_1 in (4. 14) is taken so small and M so large that

$$n[M_h M\mu_1 + M_r B]M_1 \leqq M,$$

then we can conclude that

$$|\mathfrak{T}(W)|\leqq M|\tau^{-(m+q)}\varepsilon|^{r+1},$$

and also from uniform convergence of the integral (4. 11) we have

$$\mathfrak{T}(W)\subset W.$$

If we apply the fixed point theorem, we can prove the existence of the solution of the equation (4. 9) and hence (4. 6) of the form

$$\hat{w}^{(r)}(\tau, \varepsilon)=[\tau^{-(m+q)}\varepsilon]^{r+1}E_r(\tau, \varepsilon),$$

where $E_r(\tau, \varepsilon)$ is bounded in (4. 2). Then the function

$$w(\tau, \varepsilon)=\hat{w}^{(r)}(\sigma, \varepsilon)+\hat{z}^{(r)}(\tau, \varepsilon)\exp[\varepsilon^{-h}K(\tau, \varepsilon)]$$

is a fundamental matrix solution of the differential equation (2. 13). By a usual method, we can prove that the solution $w(\tau, \varepsilon)$ is independent of r. This completes the proof of Theorem 4. 1.

§ 5. Existence theorem (2).

In this Section we prove the existence theorem corresponding to the formal solution (3. 53) in Theorem 3. 1.

THEOREM 5. 1. *Let S be the sector in the s-plane defined in Section* 3, *and let*

$$v\sim\Omega(s^{k(s)})\left\{\sum_{\nu=0}^{\infty}w^{(\nu)}(s)\,[s^{k(s)e}\rho]^{\nu}\right\}s^{k(s)\Pi}\exp[Q(s)] \tag{5. 1}$$

be a formal solution of (3. 10) *whose existence was proved in Theorem* 3. 1. *Then, there exists an actual solution $v(s, \rho)$ of* (3. 10) *of the form*

$$v(s, \rho)=\Omega(s^{k(s)})\hat{v}(s, \rho)s^{k(s)\Pi}\exp[Q(s)],$$

and for every integer r, there exists a domain D_2 of s, ρ-plane defined by

$$s\in S,\qquad 0<|\rho|\leqq\rho_2,\qquad |\arg\rho|\leqq\delta_2,\qquad |s^e\rho|\leqq c_3 \tag{5. 2}$$

(ρ_2, δ_2 *and* c_3 *are some constant independent of* ρ), *in which it holds that*

$$\hat{v}(s, \rho)-\sum_{\nu=0}^{r}w^{(\nu)}(s)[s^{k(s)e}\rho]^{\nu}=E_r(s, \rho)[s^{k(s)e}\rho]^{r+1},$$

where $E_r(s, \rho)$ is bounded.

Proof. Let

$$v^{(r)}(s, \rho)=\sum_{\nu=0}^{r}v^{(\nu)}(s)\rho^{\nu} \tag{5. 3}$$

be a finite sum of the series (3. 53). This satisfies a differential equation

$$\frac{dv}{ds}=H_r(s, \rho)v,\qquad H_r(s, \rho)=v^{(r)}(s, \rho)'v^{(r)}(s, \rho)^{-1},$$

where $v^{(r)\prime}$ denote the derivative of $v^{(r)}$ with respect to s. Clearly $v^{(0)}(s)$ is a nonsingular matrix and all $v^{(\nu)}(s)$ are entire matrices. Hence if $s_0>0$ is chosen arbitrarily, $v^{(r)}(s,\rho)^{-1}$ exists for

$$(5.4)\qquad |\rho|\leqq\rho_3,\qquad |s|\leqq s_0,$$

where ρ_3 is a sufficiently small positive number depending on s_0 and r. On the other hand, from Lemma 3.4 we have

$$(5.5)\qquad v^{(r)}(s,\rho)=\Omega(s)\left\{\sum_{\nu=0}^{r} w^{(\nu)}(s)[s^e\rho]^\nu\right\}s^{\Pi}\exp[Q(s)],$$

where $w^{(\nu)}(s)$ are bounded for $|s|>s_0$ and $s\in S$, and $w^{(0)}(s)$ is nonsingular for $s\neq 0$. Then it follows from (5.5) that $v^{(m)}(s,\rho)^{-1}$ exists for

$$(5.6)\qquad s\in S,\qquad |s^e\rho|\leqq\mu_2,\qquad |s|>s_0,$$

where μ_2 is a sufficiently small positive number depending on s_0 and r.

Define a function $\hat{\boldsymbol{v}}^{(r)}(s,\rho)$ by

$$(5.7)\qquad v^{(r)}(s,\rho)=\Omega(s^{k(s)})\hat{\boldsymbol{v}}^{(r)}(s,\rho)s^{k(s)\Pi}\exp[Q(s)].$$

Then, from the above discussions, $\hat{\boldsymbol{v}}^{(r)}(s,\rho)$ is bounded and nonsingular if s and ρ satisfy the condition (5.4) or (5.6).

If we put

$$(5.8)\qquad v(s,\rho)=v^{(r)}(s,\rho)+z^{(r)},$$

then the equation (3.10) becomes

$$(5.9)\qquad \frac{dz^{(r)}}{ds}=H(s,\rho)z^{(r)}+H(s,\rho)v^{(r)}(s,\rho)-v^{(r)}(s,\rho)'.$$

Then any solution of the integral equation

$$z^{(r)}=\int_{\Gamma(s)} v^{(0)}(s)v^{(0)}(\sigma)^{-1}[\{H(\sigma,\rho)-H^{(0)}(\sigma)\}z^{(r)}+\{H(\sigma,\rho)v^{(r)}(\sigma,\rho)-v^{(r)}(\sigma,\rho)'\}]d\sigma$$

satisfies the differential equation (5.9). Here $\Gamma(s)$ denotes a set of n^2 paths of integration in the σ-plane ending at s.

If $\hat{\boldsymbol{v}}^{(0)}(s)$ and $\hat{z}^{(r)}(s,\rho)$ are defined by

$$(5.10)\qquad \begin{aligned} v^{(0)}(s)&=\Omega(s^{k(s)})\hat{\boldsymbol{v}}^{(0)}(s)s^{k(s)\Pi}\exp[Q(s)],\\ z^{(r)}(s,\rho)&=\Omega(s^{k(s)})\hat{z}^{(r)}(s,\rho)s^{k(s)\Pi}\exp[Q(s)],\end{aligned}$$

then the above integral equation becomes

$$(5.11)\qquad \begin{aligned} \hat{z}^{(r)}=\hat{\boldsymbol{v}}^{(0)}(s)\int_{\Gamma(s)} & s^{k(s)\Pi}[\exp\{Q(s)-Q(\sigma)\}]\sigma^{-k(\sigma)\Pi}\hat{\boldsymbol{v}}^{(0)}(\sigma)^{-1}Q(\sigma^{-k(\sigma)})\\ &\cdot[\{H(\sigma,\rho)-H^{(0)}(\sigma)\}\Omega(\sigma^{k(\sigma)}\hat{z}^{(r)}\sigma^{k(\sigma)\Pi}\exp[Q(\sigma)]\\ &+\{H(\sigma,\rho)v^{(r)}(\sigma,\rho)-v^{(r)}(\sigma,\rho)'\}]s^{-k(s)\Pi}\exp[-Q(s)]d\sigma.\end{aligned}$$

After a short calculation, using (3.17), (3.18), (3.19), (3.50) and (5.7) we get

$$H(\sigma,\rho)v^{(r)}-v^{(r)}(\sigma,\rho)'=\Omega(\sigma^{k(\sigma)})E(\sigma,\rho)[\sigma^{k(\sigma)e}\rho]^{r+1}\sigma^{-k(\sigma)}\sigma^{k(\sigma)\Pi}\exp[Q(\sigma)],$$

and

$$H(\sigma,\rho)-H^{(0)}(\sigma)=\Omega(\sigma^{k(\sigma)})E(\sigma,\rho)[\sigma^{k(\sigma)e}\rho]\sigma^{-k(\sigma)}\Omega(\sigma^{-k(\sigma)}),$$

where $E(\sigma,\rho)$ is a bounded matrix in (5. 4) or (5. 6). If these relations are inserted into (5. 11), we have

$$\begin{aligned}\hat{z}^{(r)}=\hat{v}^{(0)}(s)\int_{\Gamma(s)}& s^{k(s)\Pi}[\exp\{Q(s)-Q(\sigma)\}]\sigma^{-k(\sigma)\Pi}\hat{v}^{(0)}(\sigma)^{-1}E(\sigma,\rho)\\ &\cdot\{\sigma^{k(\sigma)(e-1)}\rho\cdot\hat{z}^{(r)}+[\sigma^{k(\sigma)e}\rho]^{r+1}\sigma^{-k(\sigma)}\}\sigma^{k(\sigma)\Pi}s^{-k(s)\Pi}\exp\{Q(\sigma)-Q(s)\}d\sigma,\end{aligned}\tag{5. 12}$$

where $E(\sigma,\rho)$ is some bounded matrix.

We prove the existence of the solution of this equation by the fixed point theorem. Let $\mathfrak{F}$ be the set of all matrices $\hat{Z}=(\hat{z}_{jk}(s,\rho))$ whose elements are holomorphic in the domain D_2 for $|s|<s_0$ and $|s|>s_0$ $(s\neq\infty)$ respectively and satisfy the inequality such that

$$\begin{aligned}&\|\hat{Z}\|=\max_{1\leqq j\leqq n}\sum_{k=1}^{n}|\hat{z}_{jk}(s,\rho)|\leqq M|\rho|^{r+1}, && \text{if } |s|<s_0,\\ &\|\hat{Z}\|\leqq M|s^e\rho|^{r+1}, && \text{if } |s|>s_0.\end{aligned}\tag{5. 13}$$

The constant M and others in (5. 2) are defined later. The mapping $\mathfrak{T}(\hat{Z})$ is given by the right term in (5. 12) with $\hat{Z}$ for $\hat{z}^{(r)}$, and with dividing the integral into $|s|<s_0$ and $|s|>s_0$. If we write $\mathfrak{T}(\hat{Z})$ for each component,

$$\begin{aligned}\mathfrak{T}(\hat{z}_{ij})=s^{k(s)(\pi_j-\pi_k)}\int_{\gamma jk(s)}&\sigma^{-k(\sigma)(\pi_j-\pi_k)}[\exp\{q_{jk}(s)-q_{jk}(\sigma)\}]\\ &\cdot\{L_{jk}(\hat{Z})\sigma^{k(\sigma)(e-1)}\rho+p_{jk}(\sigma,\rho)\sigma^{k(\sigma)e(m+1)}\rho^{r+1}\sigma^{-k(\sigma)}\}d\sigma,\end{aligned}\tag{5. 14}$$

where π_j is the diagonal element of Π, $L_{jk}(\hat{Z})$ is a linear form of k-th column of $\hat{Z}$ with bounded coefficients, $p_{jk}(\sigma,\rho)$ is a bounded matrix and $q_{jk}(s)$ is from (3. 38) and (3. 44),

$$q_{jk}(s)=\frac{m}{m+q}(\lambda_j-\lambda_k)s^{(m+q)/m}+\cdots+(q_{jm+q}-q_{km+q})s^{1/m}.\tag{5. 15}$$

If we transform σ and s into ζ and ξ by the relations (3. 46), the sector S corresponds to the sector Σ, and if we insert (5. 15) into (5. 14), we have

$$\begin{aligned}|\mathfrak{T}(\hat{z}_{jk})|\leqq|\rho|^{r+1}\cdot|\xi^{k(s)(\pi_j-\pi_k)m/(m+q)}|&\int_{\lambda jk(\xi)}\zeta^{-k(\sigma)(\pi_j-\pi_k)m/(m+q)}\\ &\cdot|\exp\{q_{jk}(s)-q_{jk}(\sigma)\}|\cdot B\{M|\zeta|^{k(\sigma)em/(m+q)}|\rho|+1\}\\ &\cdot|\zeta|^{k(\sigma)\{e(r+1)m/(m+q)-m/(m+q)\}}\cdot|\zeta|^{-q/(m+q)}|d\zeta|,\end{aligned}\tag{5. 16}$$

where B is some constant. Let $\mathcal{K}$ be the closed disk in Σ in which

$$|\zeta^{em/(m+q)}\rho|\leqq\mu_3\tag{5. 17}$$

for eace ρ. From the way that Σ was constructed, the quantity

$$\mathrm{Re}\,[(\lambda_k-\lambda_k)\zeta] \tag{5.18}$$

does not vanish on both boundary lines of Σ. and the cental angle of Σ does not exceed π. For each pair (j, k) $j\neq k$, draw a line in the ζ-plane on which (5.18) vanishes, and then the ζ-plane is divided into two half planes. If Σ is contained in the negative half plane, the path of integration $\lambda_{jk}(\xi)$ is to be the segment from the origin to ξ. On the other hand, if Σ and the positive half plane have common part, there exists one and only one point ζ_{jk} on the circular arc of $\mathcal{H}$ such that the quantity (5.18) takes its maximum at the point ζ_{jk}. Then in this case we take the segment from ζ_{jk} to ξ in Σ as the path of integration $\lambda_{jk}(\xi)$. For $j=k$, $\lambda_{jj}(\xi)$ is the ray from origin to ξ. $\gamma_{jk}(s)$ is to be the inverse image in the σ-plane of $\lambda_{jk}(\xi)$ under the transformation (3.46). Here we limit ξ to the convex polygon $\mathcal{H}^*$ in $\mathcal{H}$ whose vertices are ζ_{jk} and two end points of the boundary lines. By this choice of paths of integration, there exists a positive constant p independent of j, k and ρ for $\xi\in\mathcal{H}^*$ and ζ on $\lambda_{jk}(\xi)$ $(j\neq k)$, such that

$$\mathrm{Re}\,[(\lambda_j-\lambda_k)(\xi-\zeta)]\leqq -p|\xi-\zeta|. \tag{5.19}$$

Here we choose constant numbers R and K independent of $\xi\in\Sigma$, j, k, ρ such such

$$\begin{aligned} &\mathrm{Re}\,[q_{jk}(s)-q_{jk}(\sigma)]\leqq K && \text{if } |\xi-\zeta|\leqq R, \\ &\mathrm{Re}\,[q_{jk}(s)-q_{jk}(\sigma)]\leqq -p|\xi-\zeta| && \text{if } |\xi-\zeta|>R, \end{aligned} \tag{5.20}$$

for ζ on $\lambda_{jk}(\xi)$. Next we estimate the integral

$$J\equiv\int_{\lambda_{jk}(\xi)} |\zeta^{-k(\sigma)(\pi_j-\pi_k)m/(m+q)}|\,|\exp[q_{jk}(s)-q_{jk}(\sigma)]|\,|\zeta|^{k(\sigma)h(r)-q/(m+q)}|d\zeta|, \tag{5.21}$$

where $h(r)=e(r+1)m/(m+q)-m/(m+q)$.

LEMMA 5.1. *There exists a constant C independent of ξ and ρ such that*

$$J\leqq C|\xi^{-k(\sigma)(\pi_j-\pi_k)m/(m+q)}|\,|\xi|^{k(\sigma)e(r+1)m/(m+q)}, \tag{5.22}$$

for $\xi\in\mathcal{H}^*$.

Proof. For $j=k$, Lemma is obvious, and then we prove only for $j\neq k$. The contribution of the integral J on the path of $\lambda_{jk}(\xi)$ on which the inequality $|\zeta|\leqq\xi_2$ is satisfied, is by virtue of (5.20)

$$J_1\leqq C_1\xi_1{}^{m/(m+q)},$$

where C_1 and C_r introduced below are some constants independent of ξ and ρ. If $|\xi|>\xi_1$ and on the part of $\lambda_{jk}(\xi)$ on which $|\zeta|>\xi_1$, we have

$$\begin{aligned} J_2&\leqq C_2|\xi^{-(\pi_j-\pi_k)m/(m+q)}|\,|\xi|^{e(r+1)m/(m+q)-1} \\ &\leqq C_2|\xi^{-(\pi_j-\pi_k)m/(m+q}|\,|\xi|^{e(r+1)m/(m+q)}, \end{aligned}$$

and when $|\xi|\leqq\xi_1$, the contribution of the part of $\lambda_{jk}(\xi)$ on which $|\zeta|>\xi_1$ is

$$J_3\leqq C_3|\xi_1{}^{-(\pi_j-\pi_k)m/(m+q)}|\,|\xi_1|^{e(r+1)m/(m+q)}.$$

Thus if we add J_1 and J_2, or J_1 and J_3 we get Lemma 5. 1 for $j \neq k$.

From (5. 16) and (5. 22). we have

$$|\mathfrak{T}(\hat{z}_{jk})| \leqq BC\{M|s^{k(s)e}\rho|+1\}|s^{k(s)e}\rho|^{r+1},$$

and then

$$\|\mathfrak{T}(\hat{Z})\| \leqq nBC\{M|s^{k(s)e}\rho|+1\}|s^{k(s)e}\rho|^{r+1}. \tag{5. 23}$$

Here we choose μ_3 in (5. 17), ρ_2 in (5. 2) so small, and M in (5. 13) so large that

$$nBC\{M\mu_3+1\} \leqq M,$$

$$nBC\{M\rho_2+1\} \leqq M.$$

Let s_0 in (5. 4) or (5. 6) be chosen such that $s_0 > s_1 = [(m+q/m)\xi_1]^{m/(m+q)}$, and let $\rho_2 \leqq \rho_3$, δ_2 in (5. 2) be chosen so small that corresponding ε satisfies the condition (1. 2). The constant c_3 in (5. 2) must be chosen so small that $c_3 \leqq \mu_2$ and the inverse image H^* in the s, ρ-plane of $\mathcal{H}^*$ in the ξ, ρ-plane under the transformation (3. 46) contains the domain (5. 2). From this choice of constants in (5. 2), we can apply the fixed point theorem to $\mathfrak{F}$ and $\mathfrak{T}$, and there exists a solution of the equation (5. 11) and hence of the differential equation (5. 9) of the form

$$z^{(r)}(s, \rho) = \Omega(s^{k(s)})\hat{z}^{(r)}(s, \rho)s^{k(s)\Pi} \exp[Q(s)]$$

with

$$|\hat{z}^{(r)}(s, \rho)| \leqq M|s^{k(s)e}\rho|^{r+1}.$$

Hence, $v(s, \rho) = v^{(r)}(s, \rho) + z^{(r)}(s, \rho)$ is a fundamental solution of (3. 10). The fact that $v(s, \sigma)$ is independent of r can be proved easily by an usual method. This completes the proof of Theorem 5. 1.

As a conclusion, we remark that the two domains D_1 and D_2 are overlapped with each other for arbitrarily small ε, and this fact makes it legitimate to identify the two types of solutions:

$$y = \Omega(x)u(\tau, \varepsilon),$$

$$y = \Omega(\varepsilon^a)v(s, \rho),$$

where the functions $u(\tau, \varepsilon)$ and $v(s, \rho)$ are defined in Theorem 4. 1 and in Theorem 5. 1 respectively.

When the fundamental assumption (1. 11) is not satisfied, that is, when the characteristic polygon Π consists of several segments, the asymptotic nature of the solution is quite complicated.

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