ON BIRECURRENT TENSORS

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It has been recently proved[1] that if, in a compact Riemannian manifold of class C^∞, a tensor field T satisfies the covariant differential equation $\nabla_j \nabla_i T = \beta_j \nabla_i T$, then the covariant derivative of the tensor T vanishes identically. On the other hand, it is known [2][2] that if, in a complete irreducible Riemannian manifold of class C^∞, the m-th covariant derivative of any tensor field for some $m \geqq 1$ vanishes identically, then so does the covariant derivative of the tensor field.

In connection with these results, Prof. Yano suggests to study the tensor field T satisfying the covariant differential equation

$$(1) \qquad \nabla_m \nabla_l T + \alpha_{ml} T = 0,$$

α_{ml} being a tensor field of type (0, 2). The main purpose of this note is to get a sufficient condition for such a tensor field to vanish identically.

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§ 1. Let M be an n-dimensional Riemannian manifold of class C^∞ covered by a system of local coordinates $\{x^h\}$[3] and g_{ji} the components of the metric tensor. Let all vector fields and tensor fields be of class C^∞. If a tensor T, say T_{kji}, satisfies the equation (1), we call T a recurrent tensor of second order (or briefly a birecurrent tensor) and α_{ml} the associated tensor of the birecurrent tensor T.

For any birecurrent tensor the associated tensor α_{ml} is symmetric. In fact, transvecting T^{kji} to $\nabla_m \nabla_l T_{kji} + \alpha_{ml} T_{kji} = 0$, we get

$$\nabla_m \nabla_l (T_{kji} T^{kji})/2 - \nabla_m T^{kji} \cdot \nabla_l T_{kji} + \alpha_{ml} T_{kji} T^{kji} = 0.$$

Since the first and the second terms in the first member are symmetric in m and l, so is the last term, that is, α_{ml} too.

First we consider the case in which the tensor field S, say S_{kji}, satisfies covariant differential equations of slightly general form

$$(2) \qquad \nabla_m \nabla_l S_{kji} + \beta_m \nabla_l S_{kji} + \alpha_{ml} S_{kji} = 0,$$

β_m being a vector field. Transvecting $g^{ml} S^{kji}$ to (2) and putting $S^2 = S_{kji} S^{kji}$, we get

$$\Delta S^2 - 2 \nabla_m S_{kji} \cdot \nabla^m S^{kji} + \beta^m \nabla_m S^2 + 2\alpha_m{}^m S^2 = 0,$$

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1) Nomizu and Yano, personal communication.

2) Numbers in brackets refer to the bibliography at the end of the note.

3) Throughout this note, indices run over the range 1, 2, ⋯, n.

where Δ denotes the Laplacian operator. It follows from the equation above that

$$L(S^2)\overset{\text{def}}{=}g^{ji}(\partial^2/\partial x^j\partial x^i)S^2+(\beta^h-g^{ji}\{{}_j{}^h{}_i\})\partial S^2/\partial x^h \tag{3}$$
$$=2(\nabla_h S_{kji}\cdot\nabla^h S^{kji}-\alpha_h{}^h S^2).$$

Accordingly, if $\alpha_h{}^h$ is negative then we have $L(S^2)\geqq 0$ everywhere on the manifold. Consequently, by virtue of E. Hopf's theorem [3], $L(S^2)$ vanishes identically in a compact Riemannian manifold and therefore so does S^2. Thus we have

THEOREM 1. *If, in a compact Riemannian manifold, a tensor field S satisfies* (2), *and $g^{ji}\alpha_{ji}$ is negative, then the tensor field S vanishes identically.*

Making use of this theorem, we can easily prove the following result concerning the birecurrent tensor.

COROLLARY. *If, in a compact Riemannian manifold, $g^{ji}\alpha_{ji}$ is negative for the associated tensor α_{ji} of a birecurrent tensor T, then T vanishes identically.*

§**2.** Next we shall prove the following

THEOREM 2. *If, for any birecurrent tensor field T with associated tensor α_{ji} in a complete Riemannian manifold, α_{ji} is negative definite, $\nabla_k\alpha_{ji}+\nabla_j\alpha_{ik}+\nabla_i\alpha_{kj}=0$ and either the length of T or that of $\nabla_j T$ is bounded, then T vanishes identically.*

Proof. Let p be any fixed point in M and γ: $t\to\gamma(t)$ a geodesic passing through $p=\gamma(0)$ and parametrized by arc length t. We put $x^h(t)=x^h(\gamma(t))$ and we choose a system of orthonormal parallel vector fields $\{e_1{}^h(t), e_2{}^h(t), \cdots, e_n{}^h(t)\}$ along γ. For any vector fields e_l, e_m and e_n in the frame and a birecurrent tensor T, say T_{kji}, with associated tensor α_{ml}, we put

$$f(t)=(T_{kji}e_l{}^k e_m{}^j e_n{}^i)(t). \tag{4}$$

If follows from the definition that

$$d^2f(t)/dt^2+\alpha_{ml}(dx^m/dt)(dx^l/dt)(f(t))=0.$$

Since the associated tensor α_{ji} is negative definite and satisfies the condition $\nabla_k\alpha_{ji}+\nabla_j\alpha_{ik}+\nabla_i\alpha_{kj}=0$, it is seen that $\alpha_{ml}(dx^m/dt)(dx^l/dt)$ is a negative constant. Thus we denote it by $-c$. This means that a solution of the last equation is

$$f(t)=A\exp(\sqrt{c}\,t)+B\exp(-\sqrt{c}\,t),$$

where A and B are constants. Consequently, if either A or B is non-zero constant, then $f(t)$ diverges as t tends to infinity.

On the other hand, making use of the property that $T_{kji}T^{kji}$ is bounded and taking account of the definition (4) of $f(t)$, we see that $f(t)$ is bounded. Thus, if we assume that either A or B is non-zero constant, the bounded $f(t)$ diverges to infinity. This is a contradiction. Hence we get $A=B=0$, that is, $f(0)=0$. This implies that

$$(T_{kji}e_l{}^k e_m{}^j e_n{}^i)(0)=0$$

for any vector fields e_l, e_m and e_n in the given frame. Since the last equation is valid for all points in M and for all frames along any geodesic passing through the given point, we conclude that T_{kji} vanishes identically.

In the following, we consider the case in which $\nabla_m T_{kji} \cdot \nabla^m T^{kji}$ is bounded. If we put

$$\tilde{f}(t) = (\nabla_h T_{kji}(dx^h/dt) e_l{}^k e_m{}^j e_n{}^i)(t) \tag{5}$$

for arbitrary vector fields e_l, e_m and e_n in the given frame, then it follows that $\tilde{f}(t)$ is bounded and that

$$d^2\tilde{f}(t)/dt^2 - c\tilde{f}(t) = 0.$$

Therefore, by the same process as above, we have

$$\nabla_h T_{kji}(dx^h/dt) = 0.$$

Differentiating this covariantly along the given geodesic γ, we obtain

$$\nabla_m \nabla_h T_{kji}(dx^h/dt)(dx^m/dt) = 0,$$

from which we get

$$\alpha_{mh}(dx^m/dt)(dx^h/dt) T_{kji} = 0.$$

From the property of α_{mh} to be negative definite it follows that T_{kji} vanishes identically. Thus Theorem 2 is proved.

In particular, if, for any birecurrent tensor field T with associated tensor α_{ji}, we have $\alpha_{ji} = kg_{ji}$, then we call T a restricted birecurrent tensor. If k is a negative constant, then the associated tensor kg_{ji} satisfies the assumption of Theorem 2. Thus we find

COROLLARY. *In a complete Riemannian manifold, let T be a restricted birecurrent tensor with associated tensor kg_{ji}. If k is a negative constant and either the length of T or that of $\nabla_j T$ is bounded, then T vanishes identically.*

It is known [1] that, in a complete Riemannian manifold M with constant scalar curvature which can be mapped, by a concircular transformation, onto another Riemannian manifold with constant scalar curvature, a scalar field τ defining the transformation is a restricted birecurrent scalar on M. According to the corollary above, if M is with negative scalar curvature and τ is bounded, then τ vanishes identically. This means that there exists no concircular transformation of such a type in a complete Riemannian manifold with negative constant scalar curvature.

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