ON MATCHING METHODS IN TURNING POINT PROBLEMS

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§ 1. Introduction.

We consider here the asymptotic nature of solutions of linear differential equations of the form

$$\varepsilon^h \frac{dy}{dx} = A(x, \varepsilon)y \tag{1. 1}$$

as the parameter ε tends to zero. Here $A(x, \varepsilon)$ is 2-by-2 matrix such that

$$\begin{pmatrix} 0 & 1 \\ x^q + \varepsilon\phi(x, \varepsilon) & 0 \end{pmatrix}, \tag{1. 2}$$

where the function $\phi(x, \varepsilon)$ is holomorphic in the complex variables x and ε in a domain of the x, ε-space defined by the inequalities

$$|x| \leqq x_0 < 1, \qquad 0 < |\varepsilon| \leqq \varepsilon_0, \qquad |\arg \varepsilon| \leqq \delta_0 \tag{1. 3}$$

and $\varepsilon\phi(x, \varepsilon)$ has a uniformly asymptotic expansion in powers of ε such that

$$\varepsilon\phi(x, \varepsilon) \simeq \sum_{\nu=1}^{\infty} \phi_\nu(x)\varepsilon^\nu \tag{1. 4}$$

as ε tends to zero in the domain (1. 3) with the coefficients $\phi_\nu(x)$ holomorphic for $|x| \leqq x_0$:

$$\phi_\nu(x) = \sum_{\mu=m_\nu}^{\infty} \phi_{\nu\mu} x^\mu, \qquad \phi_{\nu, m_\nu} \neq 0, \tag{1. 5}$$

where $m_\nu \geqq 0$.

The quantities h and q are positive integers. Thus the equation (1. 1) has a turning point at the origin. When $h=1$ and $q=1$ or $h=1$ and $q=2$, the asymptotic solutions of the equation (1. 1) were constructed by Langer [3] and Mckelvey [4]. Their methods are the reductions of the given equations to simpler related problems which can be solved by explicit technique. And for $h=1$ and any positive integers q, Sibuya [2] found some simpler related equations, but the analyses of them are seen not to be completed. On the other hand, Wasow [6], [7] claimed that the matching methods are also fruitful in fairly general cases. He treated the system (1. 1) with $h=1$ and an n-by-n matrix $A(x, \varepsilon)$. The matrix $A(x, \varepsilon)$ has an asymptotic expansion

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$$A(x, \varepsilon)\simeq \sum_{\nu=0}^{\infty} A_\nu(x)\varepsilon^\nu$$

where the coefficieuts $A_\nu(x)$ are holomorphic n-by-n matrices in the domain (1. 3) and

$$A_0(x)=\begin{pmatrix} 0, & 1 & & \\ & & \ddots & 0 \\ & 0 & & \\ & & & 1 \\ ax, & a_{n-1}x, & \cdots, & a_1x \end{pmatrix}+x^2A_0^{(2)}(x), \qquad a\neq 0.$$

Here the coefficient $A_0^{(2)}(x)$ is holomorphic for $|x|\leqq x_0$, and the quantities $a, a_{n-1}, \cdots, a_1$ are constants. At first he calculated the two types of formal solutions and then proved that there exist fundamental solutions whose asymptotic expansions coincide with the formal solutions in some neighborhoods of the turning point which overlap the full neighborhood of the turning point even for arbitrary small ε.

Now it will be proved here that the equation (1. 1) with (1. 2) can be treated by the matching methods employed by Wasow. The quantities h and q are arbitrary positive integers, but it requires the fundamental assumption which will be described below. We introduce the quantity

$$a=\frac{2h}{q+2}. \tag{1. 6}$$

The fundamental assumption is

$$m_\nu-q+\frac{\nu}{a}>0 \qquad (\nu\geqq 1). \tag{1. 7}$$

Under this condition, we can obtain the asymptotic representations of the fundamental solutions explicitly.

Sections 2 and 3 contain the calculations of the formal solutions and in Sections 4, 5 and 6, we prove that there exist fundamental solutions whose asymptotic expansions coincide with the formal solutions in several subdomains which overlap the full neighborhood of the turning point.

§ 2. Formal solutions for $x\neq 0$.

The linear transformation

$$y=\begin{pmatrix} 1 & 0 \\ 0 & x^{q/2} \end{pmatrix}u \tag{2. 1}$$

changes (1. 1) into

$$\varepsilon^h x^{-q/2}\frac{du}{dx}=\left\{\begin{pmatrix} 0 & 1 \\ 1+\varepsilon\phi(x, \varepsilon)x^{-q}, & 0 \end{pmatrix}-\begin{pmatrix} 0, & 0 \\ 0, & \dfrac{q}{2}\varepsilon^h x^{-q/2-1} \end{pmatrix}\right\}u. \tag{2. 2}$$

By (1. 4) and (1. 5) we have

$$
\begin{aligned}
1+\varepsilon\phi(x,\varepsilon)x^{-q} &\simeq 1+\sum_{\nu=1}^{\infty}\sum_{\mu=m_\nu}^{\infty}\phi_{\nu\mu}x^{\mu-q}\varepsilon^{\nu} \\
&\simeq 1+\sum_{\nu=1}^{\infty}\sum_{\mu=m_\nu}^{\infty}\phi_{\nu\mu}[x^{-1}\varepsilon^{a}]^{\nu/a}x^{\mu-q+\nu/a}.
\end{aligned} \tag{2.3}
$$

We remark here that by virtue of (1. 7), $\mu-q+\nu/a>0$ for $\nu\geqq 1$ and $\mu\geqq m_\nu$, and

$$
\frac{q}{2}\varepsilon^{h}x^{-q/2-1}=\frac{q}{2}[x^{-1}\varepsilon^{a}]^{h/a}, \qquad \varepsilon^{h}x^{-q/2}=[x^{-1}\varepsilon^{a}]^{h/a}x.
$$

Then (2. 2) can be written

$$
[x^{-1/a}\varepsilon]^{h}x\frac{du}{dx}=B(x,\varepsilon)u, \tag{2.4}
$$

where

$$
B(x,\varepsilon)\simeq\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}+\sum_{\nu=1}^{\infty}B_\nu(x)[x^{-1/a}\varepsilon]^{\nu}
$$

and $B_\nu(x)$ are holomorphic matrices functions of $x^{1/a}$. This means that for every m,

$$
B(x,\varepsilon)-\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}-\sum_{\nu=1}^{m}B_\nu(x)[x^{-1/a}\varepsilon]^{\nu}=E_m(x,\varepsilon)[x^{-1/a}\varepsilon]^{m+1},
$$

where $E_m(x,\varepsilon)$ is bounded in the domain (1. 3).

Here we put

$$
x=\tau^{a}, \tag{2.5}
$$

then (2. 4) becomes

$$
[\tau^{-1}\varepsilon]^{h}\tau\frac{du}{d\tau}=E(\tau,\varepsilon)u, \tag{2.6}
$$

where $E(\tau,\varepsilon)$ is holomorphic for τ and ε in the damain defined by the inequalities

$$
|\tau|\leqq\tau_0, \qquad 0<|\varepsilon|\leqq\varepsilon_0, \qquad |\arg\varepsilon|\leqq\delta_0, \tag{2.7}
$$

and has an asymptotic expansion when ε tends to zero:

$$
E(\tau,\varepsilon)\simeq\sum_{\nu=0}^{\infty}E_\nu(\tau)[\tau^{-1}\varepsilon]^{\nu}. \tag{2.8}
$$

The matrices functions $E_\nu(\tau)$ are holomorphic for $|\tau|\leqq\tau_0$, and

$$
E_0(\tau)=\begin{pmatrix}0 & a\\ a & 0\end{pmatrix}. \tag{2.9}
$$

Since the characteristic roots of $E_0(\tau)$ are distinct, there exists a linear transformation

$$
u=P(\tau,\varepsilon)z \tag{2.10}
$$

which changes the equation (2. 6) into

$$(2.11)\qquad [\tau^{-1}\varepsilon]^h \tau \frac{dz}{d\tau} = D(\tau, \varepsilon) z$$

with the following properties:

a) $D(\tau, \varepsilon)$ is holomorphic in both variables τ and ε for

$$(2.12)\qquad \begin{gathered} |\tau| \leqq \tau_0, \qquad |\arg \tau - \alpha_0| \leqq \frac{\pi}{2h}, \qquad 0 < |\varepsilon| \leqq \varepsilon_0, \\ |\arg \varepsilon| \leqq \delta_0, \qquad 0 < |\tau^{-1}\varepsilon| \leqq b_0, \end{gathered}$$

for sufficiently small b_0 and arbitrary α_0.

b) As $|\tau^{-1}\varepsilon|$ tends to zero, we have

$$(2.13)\qquad D(\tau, \varepsilon) \simeq \sum_{\nu=0}^{\infty} D_\nu(\tau) [\tau^{-1}\varepsilon]^\nu,$$

uniformly in (2.12).

c) The matrices $D_\nu(\tau)$ are diagonal and holomorphic for $|\tau| \leqq \tau_0$.

d) $D_0(\tau) = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$.

e) The matrix $P(\tau, \varepsilon)$ is holomorphic in the domain (2.12) and

$$(2.14)\qquad P(\tau, \varepsilon) \simeq \sum_{\nu=0}^{\infty} P_\nu(\tau) [\tau^{-1}\varepsilon]^\nu,$$

where $P_0(\tau)$ is nonsigular constant matrix and $P_\nu(\tau)$ holomorphic.

This will be proved as follows. At first, if we transform the equation (2.6) by

$$u = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} u^*$$

then (2.6) becomes

$$(2.6)^*\qquad [\tau^{-1}\varepsilon]^h \tau \frac{du^*}{d\tau} = E^*(\tau, \varepsilon) u^*,$$

where $E^*(\tau, \varepsilon)$ has the same properties as $E(\tau, \varepsilon)$:

$$E^*(\tau, \varepsilon) \simeq \sum_{k=0}^{\infty} E_k(\tau) [\tau^{-1}\varepsilon]^k,$$

$$E_0^*(\tau) = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}.$$

Now let us define the matrix $T_k(\tau, \varepsilon)$ by the formula

$$T_k(\tau, \varepsilon) = I + [\tau^{-1}\varepsilon] Q_k(\tau), \qquad k = 1, 2, \cdots,$$

where I is 2-by-2 unit matrix and $Q_k(\tau)$ is 2-by-2 matrix which is to be determined successively. The transformation

$$u^* = T_k(\tau, \varepsilon) u^k$$

changes the equation (2.6)* into

$$[\tau^{-1}\varepsilon]^{h}\tau \frac{du^{k}}{d\tau}=E^{k}(\tau, \varepsilon)u^{k}$$

where

$$E^{k}(\tau, \varepsilon)=\{I+[\tau^{-1}\varepsilon]^{k}Q_{k}(\tau)\}^{-1}E^{*}(\tau, \varepsilon)\{I+[\tau^{-1}\varepsilon]^{k}Q_{k}(\tau)\}$$
$$-[\tau^{-1}\varepsilon]^{h}\tau\{I+[\tau^{-1}\varepsilon]^{k}Q_{k}(\tau)\}^{-1}\frac{d}{d\tau}\{[\tau^{-1}\varepsilon]^{k}Q_{k}(\tau)\}.$$

If we replace $\{I+[\tau^{-1}\varepsilon]^{k}Q_{k}(\tau)\}^{-1}$ by its geometric series, we have

$$E^{k}(\tau, \varepsilon)\sim E_{0}^{*}(\tau)+\cdots+[\tau^{-1}\varepsilon]^{k-1}E_{k-1}^{*}(\tau)$$
$$+[E_{k}^{*}(\tau)+E_{0}^{*}(\tau)Q_{k}(\tau)-Q_{k}(\tau)E_{0}^{*}(\tau)][\tau^{-1}\varepsilon]^{k}+\cdots.$$

Since $E_{0}^{*}(\tau)$ is diagonal and has distinct characteristic values, $Q_{k}(\tau)$ can be chosen so as to make the coefficient of $[\tau^{-1}\varepsilon]^{k}$ diagonal. The infinite product $T_{1}(\tau, \varepsilon)\cdot T_{2}(\tau, \varepsilon)\cdots$ determines a formal series $\sum_{\nu=0}^{\infty}\hat{P}_{\nu}(\tau)[\tau^{-1}\varepsilon]^{\nu}$ with the holomorphic coefficients. By a Borel-Ritt theorem, we can construct a matrix function $\hat{P}(\tau, \mu)$, $\mu=\tau^{-1}\varepsilon$, such that

a) $\hat{P}(\tau, \mu)$ is holomorphic in τ and μ for
$|\tau|\leqq\tau_{0}$, $0<|\mu|\leqq b_{0}$, and arbitrary sector $\mathfrak{S}$ of μ.

b) As μ tends to zero

$$\hat{P}(\tau, \mu)\simeq\sum_{\nu=0}^{\infty}\hat{P}_{\nu}(\tau)\mu^{\nu}, \qquad \hat{P}_{0}(\tau)=I,$$

$$\frac{d\hat{P}(\tau, \mu)}{d\tau}\simeq\sum_{\nu=0}^{\infty}\frac{d\hat{P}_{\nu}(\tau)}{d\tau}\mu^{\nu},$$

$$\hat{P}(\tau, \mu)^{-1}\simeq\sum_{\nu=0}^{\infty}\hat{Q}_{\nu}(\tau)\mu^{\nu},$$

where $\hat{Q}_{\nu}(\tau)$ can be calculated from $P_{\mu}(\tau)$ $(\mu\leqq\nu)$ formally.

Thus the transformation

$$u=\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\hat{P}(\tau, \mu)z=P(\tau, \varepsilon)z$$

changes the differential equation (2. 6) into the equation (2. 11) with the desired properties.

Since all the matrices $D_{\nu}(\tau)$ of (2. 13) are diagonal, it is easy to calculate a formal series solution of the differential equation (2. 11).

THEOREM 1. *The differential equation* (2. 6) *possesses a formal matrix solution of the form*

$$u\sim\sum_{\nu=0}^{\infty}\varepsilon^{\nu}u_{\nu}(\tau)\exp\left[\sum_{r=0}^{h}\varepsilon^{r-h}F_{r}(\tau)\right] \tag{2. 15}$$

with the following properties:

$$u_{\nu}(\tau)=\tau^{-\nu}\hat{u}_{\nu}(\tau) \tag{2. 16}$$

where the $\hat{u}_\nu(\tau)$ *are polynomials of degree* ν, *at most, in* $\log \tau$ *whose coefficients are holomorphic in* $|\tau| \leqq \tau_0$, *and bounded in the domain* (2. 12).

$$F_\nu(\tau) = \tau^{h-\nu} \hat{F}_\nu(\tau) \qquad \textit{if } \nu \leqq h-1$$
$$F_h(\tau) = f_h \log \tau + \hat{F}_h(\tau), \tag{2. 17}$$

where $\hat{F}_\nu(\tau)$ ($\nu = 0, 1, \cdots, h$) *are holomorphic in* $|\tau| \leqq \tau_0$, *and* f_h *is a constant matrix.*

Proof. If the series (2. 13) were convergent,

$$z \sim \exp\left[\sum_{\nu=0}^{\infty} \varepsilon^{\nu-h} F_\nu(\tau)\right] \tag{2. 18}$$

with

$$F_\nu(\tau) = \int D_\nu(\tau) \tau^{h-\nu-1} d\tau \tag{2. 19}$$

would be an actual solution of (2. 12). If the determinations of the integrals (2. 19) are chosen whose series expasions have no constant terms, then $F_\nu(\tau)$ have the following properties:

$$F_\nu(\tau) = \tau^{h-\nu} \hat{F}_\nu(\tau) \qquad \text{if } \nu < h, \tag{2. 20}$$

$$F_\nu(\tau) = f_\nu \log \tau + \tau^{h-\nu} \hat{F}_\nu(\tau) \qquad \text{if } \nu \geqq h, \tag{2. 21}$$

where $\hat{F}_\nu(\tau)$ are holomorphic in $|\tau| \leqq \tau_0$ and f_ν are constant matrices. Then $F_\nu(\tau)$ have the properties (2. 17).

Also in the convergent case we may write

$$\begin{aligned} \exp\left[\varepsilon^{-h} \sum_{\nu=0}^{\infty} \varepsilon^\nu F_\nu(\tau)\right] &= \exp\left[\varepsilon^{-h} \sum_{\nu=h+1}^{\infty} \varepsilon^\nu F_\nu(\tau)\right] \exp\left[\varepsilon^{-h} \sum_{r=0}^{h} \varepsilon^r F_r(\tau)\right] \\ &= \sum_{\nu=0}^{\infty} \varepsilon^\nu G_\nu(\tau) \exp\left[\sum_{r=0}^{h} \varepsilon^{r-h} F_r(\tau)\right]. \end{aligned} \tag{2. 22}$$

Here

$$G_\nu(\tau) = \tau^{-\nu} \hat{G}_\nu(\tau), \tag{2. 23}$$

where $\hat{G}_\nu(\tau)$ are polynomials of degree ν at most in $\log \tau$ with holomorphic coefficients and $\hat{G}_\nu(\tau)$ are bounded at $\tau = 0$.

Clearly (2. 22) is a formal solution of (2. 11). The expression (2. 15) is obtained by multiplying the last member of (2. 22) to (2. 14) from the right and collecting the same powers of ε. Thus (2. 16) follows at once.

§ 3. Formal solutions in the neighborhoods of $x = 0$.

The transformation

$$x = \rho^{2h} s, \tag{3. 1}$$

$$y = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon^{h-a} \end{pmatrix} v \tag{3. 2}$$

with $\rho=\varepsilon^{1/(q+2)}$, $\rho>0$ for $\arg\varepsilon=0$, takes the differential equation (1. 1) into

$$\frac{dy}{ds}=\begin{pmatrix} 0 & 1 \\ s^q+\psi(s,\rho) & 0 \end{pmatrix} v, \tag{3. 3}$$

where

$$\psi(s,\rho)\simeq\sum_{\nu=1}^{\infty}\sum_{\mu=m_\nu}^{\infty}\psi_{\nu\mu}\rho^{\nu(q+2)+2h(\mu-q)}s^{\mu}.$$

We notice here that $\nu(q+2)+2h(\mu-q)=2h(\mu-q+\nu a^{-1})>0$. If we put

$$\psi(s,\rho)\simeq\sum_{\nu=1}^{\infty}d_\nu(s)\rho^\nu,$$

then

$$d_\nu(s)=s^{c(\nu)}d_\nu^*(s),$$

where $c(\nu)=\nu/2h+q-1/a$ and $d_\nu^*(s)$ are bounded at $s=\infty$.

Write the equation (3. 3) as

$$\frac{dv}{ds}=H(s,\rho)v \tag{3. 4}$$

where

$$H(s,\rho)\simeq\sum_{\nu=0}^{\infty}H_\nu(s)\rho^\nu$$

with

$$H_0(s)=\begin{pmatrix} 0 & 1 \\ s^q & 0 \end{pmatrix},\qquad H_\nu(s)=\begin{pmatrix} 0 & 0 \\ d_\nu(s) & 0 \end{pmatrix}=s^{c(\nu)}\begin{pmatrix} 0 & 0 \\ d_\nu^*(s) & 0 \end{pmatrix}.$$

If we introduce a matrix $\Omega(s)$ by

$$\Omega(s)=\begin{pmatrix} 1 & 0 \\ 0 & s^{q/2} \end{pmatrix},$$

then the matrix $H_\nu(s)$ can be written

$$H_\nu(s)=s^{c(\nu)-q/2}\Omega(s)H_\nu^*(s)\Omega(s^{-1}) \tag{3. 5}$$

where $H_\nu^*(s)$ is bounded at $s=\infty$.

Let us consider the asymptotic properties of $H(s,\rho)$, and we put

$$H(s,\rho)-\sum_{\nu=0}^{m}H_\nu(s)\rho^\nu=E_m(s,\rho).$$

Clearly in the domain (1. 3) and $|s|\leqq s_0$ for arbitrary s_0,

$$E_m(s,\rho)=\rho^{m+1}E_{1,m}(s,\rho) \tag{3. 6}$$

where $E_{1,m}(s,\rho)$ is bounded there. Now if we denote $m/(q+2)+aq$ by $d(m)$,

$$\begin{aligned}\psi(s,\rho)-\sum_{\nu=1}^{m}d_\nu(s)\rho^\nu\simeq &\sum_{\nu\leqq d(m)}{}^{*}\ \sum_{\mu\leqq c(m)}{}^{*}\ \phi_{\nu\mu}\rho^{\nu(q+2)+2h(\mu-q)}s^{\mu}\\ &+\sum_{\nu>d(m)}\ \sum_{\mu\leqq c(m)}\phi_{\nu\mu}\rho^{\nu(q+2)+2h(\mu-q)}s^{\mu}\\ &+\sum_{\nu\leqq d(m)}\ \sum_{\mu>c(m)}\phi_{\nu\mu}\rho^{(q+2)\{\nu+2(a-h)\}}x^{\mu}\\ &+\sum_{\nu>d(m)}\ \sum_{\mu>c(m)}\phi_{\nu\mu}\rho^{(q+2)\{\nu+2(a-h)\}}x^{\mu},\end{aligned}\tag{3. 7}$$

where $\sum^*\sum^*$ involves the summation of the terms of powers of ρ higher than ρ^m. The first and second sums of the right members of (3. 7) can be written $\rho^{m+1}s^{c(m)}E^*_{1,m}$ in the domain (1. 3) and $|s|>s_0$, where $E^*_{1,m}$ is bounded there, and the remainder terms can be written $\rho^{m+1}s^{c(m+1)}E^*_{2,m}$ in the domain (1. 3), where $E^*_{2,m}$ is bounded there. Then in the domain (1. 3) and $|s|>s_0$, we have

$$E_m(s,\rho)=s^{q/2-1/a}\Omega(s)E_{2m}(s,\rho)\Omega(s^{-1})[s^{1/2h}\rho]^{m+1}\tag{3. 8}$$

where $E_{2m}(s,\rho)$ is bounded.

Let

$$v\sim\sum_{\nu=0}^{\infty}v_\nu(s)\rho^\nu\tag{3. 9}$$

be a formal solution of (3. 4). Then $v_\nu(s)$ must satisfy the following equations:

$$\frac{dv_0}{ds}=\begin{pmatrix}0 & 1\\ s^q & 0\end{pmatrix}v_0=H_0(s)v_0,\tag{3. 10}$$

$$\frac{dv_\nu}{ds}=H_0(s)v_\nu+\sum_{\mu=1}^{\nu}H_\mu(s)v_{\nu-\mu}.\tag{3. 11}$$

The asymptotic solution of (3. 10) can be obtained from a result of Turrittin [5]. Let S be the sector defined by

$$S:\quad \frac{-3\pi}{2(2+q)}\leqq\arg s\leqq\frac{\pi}{2(2+q)}.\tag{3. 12}$$

Then the differential equation (3. 10) possesses a fundamental matrix solution $v_0(s)$ of the form

$$v_0(s)=s^{q/4}\Omega(s)w_0(s)\exp[Q(s)],\tag{3. 13}$$

where

$$Q(s)=\frac{2}{2+q}s^{(2+q)/2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\equiv\begin{pmatrix}q_1(s) & 0\\ 0 & q_2(s)\end{pmatrix},\tag{3. 14}$$

$$w_0(s)\simeq\sum_{\nu=0}^{\infty}w_{0\nu}s^{-(2+q)\nu/2},\qquad \text{as } s \text{ tends to } \infty \text{ in } S,\tag{3. 15}$$

where $w_{0\nu}$ are constant matrices and w_{00} is nonsingular. If $\omega^{2+q}=1$, the matrix $\begin{pmatrix}1 & 0\\ 0 & \omega\end{pmatrix}v_0(\omega s)$ is a fundamental solution of (3. 10) whenever $v_0(s)$ is one. Hence there exist $2+q$ fundamental matrices solutions whose asymptotic properties as well as their stokes multipliers are known in $2+q$ sectors respectively.

Next we calculate the solution of the differential equation (3. 11), which is of the form

$$\frac{dt}{ds}=H_0(s)t+F(s) \tag{3. 16}$$

with entire coefficients. The integral

$$t(s)=\int_{\Gamma(s)} v_0(s)v_0(\sigma)^{-1}F(\sigma)d\sigma \tag{3. 17}$$

is a solution of (3. 16) if $\Gamma(s)$ designates a set of paths $\gamma_{jk}(s)$ in σ-plane ending at $\sigma=s$ for every scalar integral contained in (3. 17). The paths will be specified later.

Let us define $\hat{\mathfrak{t}}(s)$, $\hat{\mathfrak{v}}_0(s)$ and $\hat{\mathfrak{F}}(s)$ by the relations

$$t(s)=\Omega(s)\hat{\mathfrak{t}}(s)\exp[Q(s)], \tag{3. 18}$$

$$v_0(s)=\Omega(s)\hat{\mathfrak{v}}_0(s)\exp[Q(s)], \tag{3. 19}$$

$$F(s)=\Omega(s)\hat{\mathfrak{F}}(s)\exp[Q(s)]. \tag{3. 20}$$

Then (3. 17) becomes

$$\hat{\mathfrak{t}}(s)=\hat{\mathfrak{v}}_0(s)\int_{\Gamma(s)}\exp[Q(s)-Q(\sigma)]\hat{\mathfrak{v}}_0(\sigma)^{-1}\hat{\mathfrak{F}}(\sigma)\exp[Q(\sigma)-Q(s)]d\sigma. \tag{3. 21}$$

Let $S(s_0)$ be the domain defined by

$$S(s_0):\quad s\in S,\quad \text{and}\quad |s|>s_0>0. \tag{3. 22}$$

For the application we have in mind that $\hat{\mathfrak{F}}(s)$ satisfies the condition

$$\hat{\mathfrak{F}}(s)s^{-b}\quad \text{is bounded in } S(s_0). \tag{3. 23}$$

It follows then that

$$\hat{\mathfrak{v}}_0(s)^{-1}\hat{\mathfrak{F}}(s)s^{-b+q/4}\quad \text{is bounded in } S(s_0). \tag{3. 24}$$

With the abbreviations

$$q_{jk}(s)=q_j(s)-q_k(s)=2\operatorname{sign}(k-j)\frac{2}{2+q}s^{(2+q)/2}=\beta_{jk}s^{(2+q)/2}. \tag{3. 25}$$

From (3. 21), (3. 24) and (3. 25) every element of the matrix in the integrand of (3. 21) has the form

$$p_{jk}(\sigma)\sigma^{b-q/4}\exp[\beta_{jk}(s^{(2+q)/2}-\sigma^{(2+q)/2})], \tag{3. 26}$$

where $p_{jk}(s)$ is bounded in $S(s_0)$.

We introduce here the auxiliary variables

$$\zeta=\sigma^{(2+q)/2},\qquad \xi=s^{(2+q)/2}. \tag{3. 27}$$

The sector S in the σ-plane corresponds to a half plane Σ in the ζ-plane.

Clearly every line

$$\mathrm{Re}\,[q_{jk}(\zeta^{2/(q+2)})]=\mathrm{Re}\,\beta_{jk}\zeta=0 \qquad (j\neq k),$$

has one half line inside Σ. Hence we can draw rays l_{12} and l_{21} through the origin into the interior of Σ such that $\mathrm{Re}\,(\beta_{jk}\zeta)$ increases monotonically to ∞ along $l_{jk}(j\neq k)$. Denote by $\lambda_{jk}(\xi)(j\neq k)$ the straight half line in Σ which is parallel to l_{jk} and has one end point at a point ξ of Σ. Then $\mathrm{Re}\,(\beta_{jk}\zeta)$ also increases monotonically to ∞ along $\lambda_{jk}(\xi)$. We define $\gamma_{jk}(s)$ for $j\neq k$ as the curve in the sector S of the σ-plane whose image under (3. 27) is $\lambda_{jk}(\xi)$. Then $\mathrm{Re}\,[q_{jk}(\sigma)]$ grows monotonically to ∞ as σ recedes to ∞ along $\gamma_{jk}(s)$.

In order to make sure that all points of $\gamma_{jk}(s)$ lie in the domain $S(s_0)$ of the σ-plane, we must limit s to a domain $S(s_1)$, where s_1 is sufficiently large. As to the paths $\gamma_{jj}(s)$, it suffices to take them as segments from some point s_2 in $S(s_1)$ to $s\in S(s_1)$, where s_2 is so large that these segments lie in $S(s_0)$ for all $s\in S(s_1)$.

LEMMA 1. *If the differential equation* (3. 16) *satisfies the condition* (3. 23), *then it possesses a solution of the form*

$$t(s)=s^{b+1}\Omega(s)t^*(s)\exp\,[Q(s)], \tag{3. 28}$$

where $t^*(s)$ is bounded as $s\to\infty$ in S.

Proof. If (3. 26) is integrated along $\gamma_{jk}(s)$, it becomes in terms of ζ and ξ

$$\frac{2}{2+q}\int_{\lambda_{jk}(\xi)}\exp\,[\beta_{jk}(\xi-\zeta)]p_{jk}(\sigma)\zeta^{2(b-3q/4)/(q+2)}d\zeta. \tag{3. 29}$$

Let ζ on $\lambda_{jk}(\xi)$ be expressed in the from

$$\zeta=\xi+\delta_{jk}r$$

where δ_{jk} is a constant of modulus 1 and r is the arc length on $\lambda_{jk}(\xi)$. Then (3. 29) can be written

$$\frac{2}{2+q}\,\xi^{2(b-3q/4)/(q+2)}\int_0^\infty \exp\,[-\beta_{jk}\delta_{jk}r]\,p_{jk}(\sigma)\left[1+\delta_{jk}\frac{r}{\xi}\right]^{2(b-3q/4)/(q+2)}dr. \tag{3. 30}$$

If $j\neq k$, then $\mathrm{Re}\,(\beta_{jk}\delta_{jk})>0$ on $\lambda_{jk}(\xi)$ and thus the integral in (3. 30) is a uniformly bounded function of ξ for $s\in S(s_1)$. Hence (3. 30) is of the order $O(s^{b-3q/4})$ as $s\to\infty$ in $S(s_1)$. If $i=k$, $\beta_{jj}=0$ in (3. 26), and then the integral of (3. 26) along $\gamma_{jj}(s)$ is $O(s^{b-q/4+1})$. It follows then that

$$\begin{aligned}\hat{\boldsymbol{t}}(s)&=\hat{\boldsymbol{v}}_0(s)s^{b-q/4+1}t^{**}(s)\\&=s^{b+1}t^*(s)\end{aligned}$$

where $t^*(s)$ is uniformly bounded as $s\to\infty$ in S, and so Lemma 1 follows from (3. 18).

LEMMA 2. *The differential equation* (3. 11) *possesses a particular solution of the form*

$$v_\nu(s)=s^{q/4+\nu e}\Omega(s)w_\nu(s)\exp\,[Q(s)], \tag{3. 31}$$

where $w_\nu(s)$ is bounded as $s\to\infty$ in the sector S, and

(3. 32)
$$e=1+\frac{1}{2h}+\frac{q}{2}-\frac{1}{a}.$$

Proof. We prove this by induction. For $\nu=0$, the statement in Lemma 2 is contained in (3. 13), (3. 14) and (3. 15). Assume it to be true for $\nu<m$. For $\nu=m$ the μ-th term of the summation in (3. 13) has the form

$$H_\mu(s)v_{m-\mu}(s)=s^{f(m,\mu)}\Omega(s)H_\mu^* w_{m-\mu}^* \exp[Q(s)],$$

where

$$f(m,\mu)=\frac{\mu}{2h}+\frac{q}{2}-\frac{1}{a}+(m-\mu)e+\frac{q}{4}.$$

The exponent $f(\nu,\mu)$ in this expression is largest for $\mu=1$, and then for $\nu=m$ we can apply Lemma 1 to the equation (3. 11) with

$$b=f(m,1).$$

This leads us to the formula (3.31) for $\nu=m$, and Lemma 2 is proved.

Then we get the following theorem.

THEOREM 2. *Let $k(s)$ be defined by*

(3. 33)
$$k(s)=\begin{cases} 0 & \textit{if } |s|\leqq s_0 \\ 1 & \textit{if } |s|>s_0, \end{cases}$$

then the differential equation (3. 4) *has a formal matrix solution v of the form*

(3. 34)
$$v\sim\sum_{\nu=0}^{\infty}\Omega(s^{k(s)})w_\nu(s)s^{k(s)q/4}[s^{k(s)e}\rho]^\nu \exp[Q(s)],$$

where $w_\nu(s)$ are bounded in the domain (1. 3) *and $|s|\leqq s_0$ or $|s|>s_0$ according to $k(s)=0$ or* 1.

§ 4. Existence Theorem (1).

In this section we prove the following theorem.

THEOREM 3. *For every sector T of the τ-plane with vertex at the origin and central angle less than π/h, and for every positive integer m, there exists a domain of ε, τ-plane defined by*

(4. 1)
$$\tau\in T,\qquad 0<|\varepsilon|\leqq\varepsilon_1,\qquad |\arg\varepsilon|\leqq\delta_1,\qquad c_1|\varepsilon|\leqq|\tau|\leqq c_2,$$

(ε_1, δ_1, c_1 and c_2 are certain constants independent of ε) and an actual solution $u(\tau,\varepsilon)$ of the differential equation (2.6) *of the form*

(4. 2)
$$u(\tau,\varepsilon)=\hat{u}(\tau,\varepsilon)\exp\left[\sum_{\nu=0}^{h}\varepsilon^{\nu-h}F_\nu(\tau)\right],$$

which is related to the formal solution (2. 15) *as*

$$\hat{u}(\tau, \varepsilon)-\sum_{\nu=0}^{m} u_\nu(\tau)\varepsilon^\nu = E_m(\tau, \varepsilon)[\tau^{-1}\varepsilon]^{m+1}, \tag{4. 3}$$

where $E_m(\tau, \varepsilon)$ *is a matrix function that is bounded in the domain* (4. 1).

Proof. At first we analyze the equation (2. 11).
Let

$$D_m(\tau, \varepsilon)=\sum_{\nu=0}^{m+h} D_\nu(\tau)[\tau^{-1}\varepsilon]^\nu, \tag{4. 4}$$

$$z_m(\tau, \varepsilon)=\exp\left[\sum_{\nu=0}^{m+h} \varepsilon^{\nu-h}F_\nu(\tau)\right]. \tag{4. 5}$$

Then $z_m(\tau, \varepsilon)$ is a fundamental matrix solution of a differential equation

$$[\tau^{-1}\varepsilon]^h \tau \frac{dz}{d\tau}=D_m(\tau, \varepsilon)z. \tag{4. 6}$$

We write (2. 11) in the form

$$[\tau^{-1}\varepsilon]^h \tau \frac{dz}{d\tau}=[D_m+(D-D_m])z, \tag{4. 7}$$

$$(D-D_m)=[\tau^{-1}\varepsilon]^{m+h+1}E_m(\tau, \varepsilon), \tag{4. 8}$$

where $E_m(\tau, \varepsilon)$ is bounded in (2. 12) provided ε is taken small enough.

By the method of variation of constants, any solution of the integral equation

$$z(\tau, \varepsilon)=z_m(\tau, \varepsilon)+\varepsilon^{-h}\int_{\Gamma(\tau)} z_m(\tau, \varepsilon)z_m(\sigma, \varepsilon)^{-1}[D(\sigma, \varepsilon)-D_m(\sigma, \varepsilon)]z(\sigma, \varepsilon)\sigma^{h-1}d\sigma \tag{4. 9}$$

satisfies the given differential equation (2. 11). Here $\Gamma(\tau)$ designates a set of paths of integration $\gamma_{jk}(\tau)$, $(j, k=1, 2)$ in the σ-plane which are described later.
Let

$$K(\tau, \varepsilon)=\sum_{\nu=0}^{h} \varepsilon^\nu F_\nu(\tau), \tag{4. 10}$$

$$\hat{z}_m(\tau, \varepsilon)=z_m(\tau, \varepsilon)\exp[-\varepsilon^{-h}K(\tau, \varepsilon)], \tag{4. 11}$$

$$\hat{z}(\tau, \varepsilon)=z(\tau, \varepsilon)\exp[-\varepsilon^{-h}K(\tau, \varepsilon)]. \tag{4. 12}$$

Then (4.6) becomes

$$\hat{z}(\tau, \varepsilon)=\hat{z}_m(\tau, \varepsilon)+L[\hat{z}(\tau, \varepsilon)], \tag{4. 13}$$

where

$$\begin{aligned} L[\hat{z}(\tau, \varepsilon)]=\varepsilon^{-h}\int_{\Gamma(\tau)} &\hat{z}_m(\tau, \varepsilon)\exp\{\varepsilon^{-h}[K(\tau, \varepsilon)-K(\sigma, \varepsilon)]\}\hat{z}_m(\sigma, \varepsilon)^{-1} \\ &\cdot[D(\sigma, \varepsilon)-D_m(\sigma, \varepsilon)]\hat{z}(\sigma, \varepsilon)\exp\{\varepsilon^{-h}[K(\sigma, \varepsilon)-K(\tau, \varepsilon)]\}\sigma^{h-1}d\sigma. \end{aligned} \tag{4. 14}$$

From (4. 10), (4. 11) and Theorem 1 (2. 17), we have

$$\hat{z}_m(\tau, \varepsilon) = \exp\left[\sum_{\nu=h+1}^{m+h} \varepsilon^{\nu-h} F_\nu(\tau)\right]$$

$$= \exp\left[\sum_{\nu=h+1}^{m+h} (\tau^{-1}\varepsilon)^{\nu-h} \hat{F}_\nu(\tau)\right],$$

where $\hat{F}_\nu(\tau)$ are bounded for $|\tau| \leqq \tau_0$. Hence if, in addition, $|\tau^{-1}\varepsilon|$ is small enough, for instance,

$$(4.15) \qquad |\tau^{-1}\varepsilon| \leqq b,$$

then $\hat{z}_m(\tau, \varepsilon)$ as well as $\hat{z}_m(\tau, \varepsilon)^{-1}$ are bounded. (4. 15) is satisfied if c_1 in (4. 1) is sufficiently large.

Let $K_j(\tau, \varepsilon)$, $j=1, 2$ be the diagonal elements of $K(\tau, \varepsilon)$ and set

$$(4.16) \qquad \mu_{jk}(\tau, \varepsilon) = K_j(\tau, \varepsilon) - K_k(\tau, \varepsilon).$$

Then the $j-k$ element of $L[\hat{z}(\tau, \varepsilon)]$ has the form

$$(4.17) \qquad L[\hat{z}(\tau, \varepsilon)]_{jk} = \int_{\gamma_{jk}} \exp \varepsilon^{-h}[\mu_{jk}(\tau) - \mu_{jk}(\sigma)] L_{jk}[\hat{z}(\sigma, \varepsilon)] \varepsilon^{m+1} \sigma^{-(m+2)} d\sigma,$$

where $L_{jk}[\hat{z}(\tau, \varepsilon)]$ is a linear form of the two components in the k-th column of $\hat{z}(\tau, \varepsilon)$. By (4. 8) and boundedness of $\hat{z}_m$, $\hat{z}_m^{-1}$, the coefficients of this linear form are bounded if (2. 12) and (4. 15) are satisfied.

Next we choose the path $\gamma_{jk}(\tau)$ in such a way that the exponential function in (4. 17) remains bounded as ε tends to zero.

Let

$$(4.18) \qquad \mu_{jk}^{(0)}(\tau) = \frac{1}{h} \tau^h a[\exp(\pi ik) - \exp(\pi ij)] \qquad (i = \sqrt{-1}),$$

then after a short calculation we get

$$(4.19) \qquad \frac{d\mu_{jk}}{d\tau}(\tau) = \frac{d\mu_{jk}^{(0)}}{d\tau}(\tau)[1 + O(\tau) + O(\tau^{-1}\varepsilon)].$$

The condition

$$(4.20) \qquad \tau \in T, \qquad c_1|\varepsilon| \leqq |\tau| \leqq c_2$$

determines a damain H in the τ-plane which depends on ε. For convenience we introduce auxiliary variables ξ and ζ by

$$(4.21) \qquad \xi = \tau^h,$$

$$(4.22) \qquad \zeta = \sigma^h.$$

Let Σ be the image of H in the ξ-plane under 4. 21) and in the ζ-plane under (4. 22). Σ and H are sectors of annuli. Their central angles at the origin are less than π in the ξ-plane and less than π/h in the τ-plane.

Let $\Sigma^* \supset \Sigma$ be an isosceles triangle with the same axis of symmetry as Σ, and its sides passing through the endpoints of the smaller circlar arc of boundary of Σ, and its base tangent to the larger circular arc of boundary of Σ. (see Figure 1.)

Without loss of generality we can assume that the base of Σ^* is not parallel to the imaginary axis of ξ-plane. Here we choose δ_1 in (4. 1) so small that the base of Σ^* is not parallel to any ray through the origin of ξ-plane on which

$$\mathrm{Re}\,[\varepsilon^{-h}\xi]=0 \tag{4. 23}$$

for all ε such that $|\arg \varepsilon| \leqq \delta_1 \leqq \delta_0$.

The size β is to be independent of ε and so small that any direction from the point ξ_2 or ξ_3 into Σ^* is not parallel to a ray through the origin on which (4.23) hold for some ε with $|\arg \varepsilon| \leqq \delta_1$. (see Figure 1)

If r_1 and r_2 are the radii of the circular arcs that bound Σ, then we have

$$r_1=|c_1\varepsilon|^h<r_2=c_2{}^h. \tag{4. 24}$$

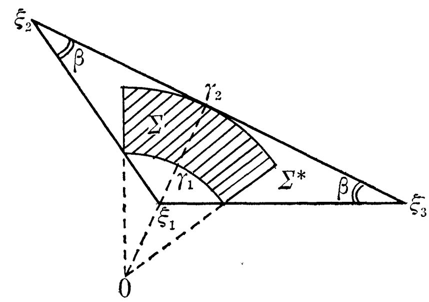

Fig. 1.

Since the shape of Σ^* is independent of ε, there exist positive constants k_1 and k_2 depending only on β such that

$$|\xi_1|=k_1r_1, \qquad |\xi_2| \leqq k_2r_2. \tag{4. 25}$$

Now c_1 and c_2 in (4.1) must be chosen such that the inverse image H^* of Σ^* in the τ-plane lies in the domain where (2. 12) and (4. 15) are satisfied. Σ^* lies in the ring

$$|\xi_1| \leqq |\xi| \leqq |\xi_2|. \tag{4. 26}$$

Then we have

$$k_1{}^{1/h}c_1|\varepsilon| \leqq |\tau| \leqq k_2{}^{1/h}c_2. \tag{4. 27}$$

The first of these inequalities implies that

$$|\tau^{-1}\varepsilon| \leqq k_1{}^{-1/h}c_1{}^{-1}. \tag{4. 28}$$

Then (4. 15) can be satisfied by taking c_1 large enough. The condition $|\tau| \leqq \tau_0$ is satisfied if c_2 is taken sufficiently small. In order to hold $|\xi_1|<|\xi_2|$, it may be necessary to take ε_1 in (4. 1) smaller than ε_0.

Now let us consider the domain Σ^* in the ζ-plane and let $\zeta=\xi$ be some point in Σ^*. From the methods that Σ^* was constructed, the quantity

$$\mathrm{Re}\,[\varepsilon^{-h}\mu_{jk}^{(0)}(\xi^{1/h})]=\mathrm{Re}\left[\varepsilon^{-h}\frac{a}{h}(\exp \pi ik-\exp \pi ij)\right] \tag{4. 29}$$

with ζ in place of ξ, changes monotonically if ζ moves from ξ_2 along a straight segment to ξ and to ξ_3. Therefore (4. 29) with ζ for ξ decreases along one of the straight paths $\xi_2\xi$ or $\xi_3\xi$. For $j \neq k$, let $\lambda_{jk}(\xi)$ be one of the two segments along which (4. 29) decreases. The inverse image in the τ-plane of $\lambda_{jk}(\xi)$ will be our path $\gamma_{jk}(\tau)$. For $j=k$, we may take either of these paths as $\gamma_{jj}(\tau)$.

Finally we choose τ_0 and b so small that $\mathrm{Re}\,[\varepsilon^{-h}\mu_{jk}(\sigma, \varepsilon)]$ also decreases along $\gamma_{jk}(\tau)$. If b is decreased, it may be necessary to increase c_1 and to decrease ε_1, but this reduction does not destroy any of the inequalities already established.

We prove the existence of solutions of the integral equation (4. 13) by the methods of successive approximations. Define the norm of a 2-by-2 matrix $M=(m_{jk})$ by

$$||M||=\max_{j}\sum_{k=1}^{2}|m_{jk}|.$$

Since $\hat{z}_m(\tau, \varepsilon)$ is bounded in the domain (2. 12) and (4. 15), we can assume

$$||\hat{z}_m(\tau, \varepsilon)||\leqq B$$

where B is some positive constant.

The successive approximations for (4. 13) are the matrices functions given recursively by the formulas

$$\hat{z}^{(0)}(\tau, \varepsilon)=\hat{z}_m(\tau, \varepsilon), \tag{4. 30}$$

$$\hat{z}^{(n+1)}(\tau, \varepsilon)=\hat{z}_m(\tau, \varepsilon)+L[\hat{z}^{(n)}(\tau, \varepsilon)], \qquad n=0, 1, 2, \cdots. \tag{4. 31}$$

In order to state the existence of solution, it is sufficient to prove that

a) All $\hat{z}^{(n)}(\tau, \varepsilon)$ are bounded and holomorphic.

b) The sequence $\{\hat{z}^{(n)}(\tau, \varepsilon)\}$ is uniformly convergent to a bounded and holomorphic matrix function $\hat{z}(\tau, \varepsilon)$.

c) $\hat{z}(\tau, \varepsilon)$ is a solution of integral equation (4. 13).

In proving these statements, we need some estimates of the integral (4. 17). The $\mathrm{Re}\,[\varepsilon^{-h}\mu_{jk}(\tau, \varepsilon)]$ is monotonic decreasing along $\gamma_{jk}(\tau)$ and the coefficients of $L_{jk}(\hat{z})$ are bounded, then $L[\hat{z}(\tau, \varepsilon)]$, as defined in (4. 17), satisfies an inequality of the form

$$|[L(\hat{z})]_{jk}|\leqq M_1|\varepsilon|^{m+1}\sup_{\tau\in H^*}||z||\int_{\gamma_{jk}(\tau)}\sigma^{-(m+2)}d\sigma \tag{4. 32}$$

for $\tau\in H^*$, $|\varepsilon|\leqq\varepsilon_1$, and $|\arg\varepsilon|\leqq\delta_1$.

The constant M_1 and other constants M_r to be introduced below may depend on m, ε_1, δ_1, c_1 and c_2, but not on ε and τ.

Using (4. 22) we have

$$\int_{\gamma_{jk}(\tau)}\sigma^{-(m+2)}d\sigma=\frac{1}{h}\int_{\lambda_{jk}(\xi)}\zeta^{-(m+h+1)/h}d\zeta.$$

and prove the following lemma concerning the last integral.

LEMMA 3. *There exists a constant M_2, depending on β and m but not on ε, b and τ_0 such that*

$$\int_{\lambda_{jk}(\xi)}|\zeta|^{-(m+h+1)/h}d\zeta\leqq M_2|\xi|^{-(m+1)/h}. \tag{4. 33}$$

Proof. To fix the ideas, assume that $\lambda_{jk}(\xi)$ starts at ξ_2. Let θ be the polar angle in the ζ-plane. Designate by p and θ the polar coordinates of the end point of the perpendicular from $\zeta=0$ onto the straight line $\lambda_{jk}(\xi)$ and denote by θ_ξ the polar angle of ξ. Then we have

$$-\frac{\pi}{2}+\beta \leqq \theta_p-\theta<\frac{\pi}{2} \tag{4.34}$$

along $\lambda_{jk}(\xi)$. We note here that the following relations hold on the integral path $\lambda_{jk}(\xi)$:

$$|\zeta|=p\cos^{-1}(\theta_p-\theta),$$

$$d|\zeta|=-p\frac{\sin(\theta_p-\theta)d\theta}{\cos^2(\theta_p-\theta)},$$

$$|d\zeta|=p\frac{|d\theta|}{\cos^2(\theta_p-\theta)}.$$

Now let $\lambda_{jk}^{(1)}(\xi)$ be the part of $\lambda_{jk}(\xi)$ where

$$|\theta_p-\theta|\leqq\frac{\pi}{2}-\beta.$$

If $\lambda_{jk}^{(1)}(\xi)$ is not empty,

$$\int_{\lambda_{jk}^{(1)}(\xi)}|\zeta|^{-(m+h+1)/h}|d\zeta|=p^{-(m+1)/h}\int_{\theta_p-\pi/2+\beta}^{\theta_\xi}[\cos(\theta-\theta_p)]^{(m-h+1)/h}d\theta,$$

and $|\theta_\xi-\theta_p|\leqq\pi/2-\beta$ so that

$$p=|\xi|\cos(\theta_\xi-\theta_p)\geqq|\xi|\sin\beta.$$

Hence we have

$$\int_{\lambda_{jk}^{(1)}(\xi)}|\zeta|^{-(m+h+1)/h}|d\zeta|\leqq(\sin\beta)^{-(m+1)/h}|\xi|^{-(m+1)/h}\int_{-\pi/2+\beta}^{\pi/2-\beta}[\cos\theta]^{-(m-h+1)/h}d\theta. \tag{4.35}$$

Denote by $\lambda_{jk}^{(2)}(\xi)$ the complement of $\lambda_{jk}^{(1)}(\xi)$ in $\lambda_{jk}(\xi)$ and assume it not empty, then on this segment we have, $|d\zeta|<|d|\zeta||\sec\beta$, and therefore

$$\int_{\lambda_{jk}^{(2)}(\xi)}|\zeta|^{-(m+h+1)/h}|d\zeta|<\frac{h\cdot\sec\beta}{m+1}|\xi^*|^{-(n+1)/h} \tag{4.36}$$

where ξ^* is the left end point of $\lambda_{jk}^{(2)}(\xi)$. Now when $\lambda_{jk}^{(1)}(\xi)$ is empty, $\xi^*=\xi$, and when $\lambda_{jk}^{(1)}(\xi)$ is not empty, then $|\xi^*|>|\xi|$, so we can replace ξ^* in (4.36) by ξ. Adding (4.35) to (4.36) we get Lemma 3.

With this prepapations we prove the properties a), b) and c).

a) From (4.32) and Lemma 3, we have

$$||\hat{z}^{(n+1)}(\tau,\varepsilon)||\leqq B+M_3|\tau^{-1}\varepsilon|^{m+1}\sup_{H^*}||\hat{z}^{(n)}(\tau,\varepsilon)||, \tag{4.37}$$

so that we can conclude that all $\hat{z}^{(n)}(\tau,\varepsilon)$ are bounded and holomorphic in the domain H^*.

b) Let Δ_k be defined by

$$\Delta_k(\tau,\varepsilon)=||\hat{z}^{(k+1)}(\tau,\varepsilon)-\hat{z}^{(k)}(\tau,\varepsilon)||.$$

Then we have

$$\Delta_k(\tau, \varepsilon) \leqq M_3|\tau^{-1}\varepsilon|^{m+1}\Delta_{k-1}(\tau, \varepsilon)$$
$$\leqq [M_3|\tau^{-1}\varepsilon|^{m+1}]^k B.$$

If the constant b in (4. 15) is taken so small that

(4. 38) $$M_3 b^{m+1} < 1,$$

then the series $\sum_{k=0}^{\infty} \Delta_k(\tau, \varepsilon)$ is uniformly convergent on H^*. Thus the series

$$\hat{z}_0^{(0)}(\tau, \varepsilon) + \sum_{k=0}^{\infty} [\hat{z}^{(k+1)}(\tau, \varepsilon) - \hat{z}^{(k)}(\tau, \varepsilon)]$$

is absolutely and uniformly convergent on H^*, and consequently the partial sum

$$\hat{z}_0^{(0)}(\tau, \varepsilon) + \sum_{k=0}^{n-1} [\hat{z}^{(k+1)}(\tau, \varepsilon) - \hat{z}^{(k)}(\tau, \varepsilon)] = \hat{z}^{(n)}(\tau, \varepsilon)$$

tends uniformly on H^* to a bounded and holomorphic limit matrix function $\hat{z}(\tau, \varepsilon)$.

c) Since $\hat{z}^{(n)}(\tau, \varepsilon)$ converges uniformly to $\hat{z}(\tau, \varepsilon)$, then from (4. 31), $\hat{z}(\tau, \varepsilon)$ is clearly a solution of the integral equation (4. 13).

Thus we get a bounded solution of the integral equation (4. 13), and this implies that the matrix function

$$z(\tau, \varepsilon) = \hat{z}(\tau, \varepsilon) \exp [\varepsilon^{-h} K(\tau, \varepsilon)]$$

satisfies the differential equation (2. 11). Moreover wet get an asymptotic property:

(4. 39) $$||\hat{z}(\tau, \varepsilon) - \hat{z}_m(\tau, \varepsilon)|| < M_4|\tau^{-1}\varepsilon|^{m+1}.$$

To complete the proof of Theorem 3 we have to get a similar inequality as (4. 39) for a solution of the equation (2. 6). Let us put

(4. 40) $$\hat{u}(\tau, \varepsilon) = P(\tau, \varepsilon)\hat{z}(\tau, \varepsilon),$$

then the matrix function

(4. 41) $$u(\tau, \varepsilon) = \hat{u}(\tau, \varepsilon) \exp\left[\varepsilon^{-h} \sum_{\nu=0}^{h} \varepsilon^{\nu} F_{\nu}(\tau)\right]$$

is a fundamental solution of (2.6). The matrix function $P(\tau, \varepsilon)$ is bounded in the domain (2. 12) which contains the domain (4. 1). Hence we have

(4. 42) $$||\hat{u}(\tau, \varepsilon) - P(\tau, \varepsilon)\hat{z}_m(\tau, \varepsilon)|| \leqq M_5|\tau^{-1}\varepsilon|^{m+1},$$

where

$$\hat{z}_m(\tau, \varepsilon) = \exp\left[\sum_{\nu=h+1}^{m+h} \varepsilon^{\nu-h} F_{\nu}(\tau)\right].$$

If this quantity is expanded in powers of ε, it coincides with the formal power series $\sum_{\nu=0}^{\infty} \varepsilon^{\nu} G_{\nu}(\tau)$ of (2. 22) up to the term $\varepsilon^m G_m(\tau)$. In order to calculate the difference

$$\exp\left[\sum_{\nu=h+1}^{m+h} \varepsilon^{\nu-h} F_{\nu}(\tau)\right] - \sum_{\nu=0}^{m} \varepsilon^{\nu} G_{\nu}(\tau),$$

we write for abbreviation,

$$\chi(\tau, \varepsilon)=\sum_{\nu=h+1}^{m+h} \varepsilon^{\nu-h}F_{\nu}(\tau)=\sum_{\nu=1}^{m} \varepsilon^{\nu}F_{\nu+h}(\tau).$$

Remembering (2. 21), we have

$$||\varepsilon^{\nu}F_{\nu+h}(\tau)|| \leqq q_{\nu}|\tau^{-1}\varepsilon|^{\nu},$$

where q_{ν} is a constant, and then $\chi(\tau, \varepsilon)$ has the order of magnitude $O(\tau^{-1}\varepsilon)$, and hence $\exp\chi$ differs from the partial sum $\sum_{\nu=0}^{m} \chi(\tau, \varepsilon)^{\nu}/\nu!$ by an expression of order of magnitude $O[(\tau^{-1}\varepsilon)^{m+1}]$. Since $\sum_{\nu=0}^{m} \varepsilon^{\nu}G_{\nu}(\tau)$ is obtained from $\sum_{\nu=0}^{m} \chi(\tau, \varepsilon)^{\nu}/\nu!$ by discarding a finite number of terms of order of magnitude $O[(\tau^{-1}\varepsilon)^{m+1}]$, we get in the domain (4. 1),

$$\left\|\hat{z}_m(\tau, \varepsilon)-\sum_{\nu=0}^{m} \varepsilon^{\nu}G_{\nu}(\tau)\right\| \leqq M_6|\tau^{-1}\varepsilon|^{m+1}.$$

If we multiply the matrix $P(\tau, \varepsilon)$ from the left and replace $P(\tau, \varepsilon)$ by its asymptotic series (2. 14), we have

$$\left\|P(\tau, \varepsilon)\hat{z}_m(\tau, \varepsilon)-\sum_{\nu=0}^{m} \varepsilon^{\nu}u_{\nu}(\tau)\right\| \leqq M_7|\tau^{-1}\varepsilon|^{m+1}. \tag{4. 43}$$

Thus we conbine the inequality (4. 42) with (4. 43) and get

$$\left\|\hat{u}(\tau, \varepsilon)-\sum_{\nu=0}^{m} \varepsilon^{\nu}u_{\nu}(\tau)\right\| \leqq M_8|\tau^{-1}\varepsilon|^{m+1}.$$

This completes the proof of Theorem 3.

§ 5. Existence Theorem (2).

Corresponding to the formal series solution in Theorem 2, we prove the following existence theorem.

THEOREM 4. *For every positive integer m, there exists a domain of the s, ρ-plane defined by*

$$s \in S, \qquad 0<|\rho| \leqq \rho_1, \qquad |\arg \rho| \leqq \delta_2, \qquad |s^e\rho| \leqq c_3, \tag{5. 1}$$

(ρ_1, δ_2 *and* c_3 *are certain constants and* e *is a number defined by* (3. 32)) *and a fundamental matrix solution* $v(s, \rho)$ *of the differential equation* (3. 4) *which is related to the formal series solution* (3. 34) *by the formula*

$$v(s, \rho)-\sum_{\nu=0}^{m} u_{\nu}(s)\rho^{\nu}=s^{k(s)q/4}\Omega(s^{k(s)})E_n(s, \rho)[s^{k(s)e}\rho]^{m+1} \exp[Q(s)], \tag{5. 2}$$

where $E_m(s, \rho)$ *is a matrix function bounded in the domain* (5. 1).

Proof. Let

$$v_m(s, \rho)=\sum_{\nu=0}^{m} v_{\nu}(s)\rho^{\nu} \tag{5. 3}$$

be a finite sum of the series (3. 34). This satisfies a differential equation

$$\frac{dv}{ds}=v'=H_m v, \qquad H_m=v_m'\cdot v_m^{-1}, \tag{5. 4}$$

where v_m' denotes the derivative of v_m with respect to s.

Clearly $v_0(s)$ is a nonsingular matrix and all $v_\nu(s)$ are entire functions. Hence if $s_0>0$ is chosen arbitrarily, $v_m(s, \rho)^{-1}$ exists for

$$|\rho|\leqq\rho_0, \qquad |s|\leqq s_0, \tag{5. 5}$$

where ρ_0 depends on s_0 and m. By Lemma 2, for $|s|>s_0$ and $s\in S$, we have

$$v_m(s, \rho)=\sum_{\nu=0}^{m} s^{q/4}\Omega(s)w_\nu(s)[s^e\rho]^\nu \exp[Q(s)] \tag{5. 6}$$

with bounded matrices functions $w_\nu(s)$. The matrix function $w_0(s)$ is nonsingular for $s\neq 0$ from (3. 13) and nonsingularity of $v_0(s)$. Then it follows from (5. 6) that $v_m(s, \rho)^{-1}$ exists for $s\in S$ and

$$|s^e\rho|\leqq\eta_0, \qquad |s|>s_0, \tag{5. 7}$$

where η_0 is sufficiently small positive number depending on s_0.

Define a function $w_m(s, \rho)$ by the equation

$$v_m(s, \rho)=s^{k(s)q/4}\Omega(s^{k(s)})w_m(s, \rho)\exp[Q(s)], \tag{5. 8}$$

where $k(s)$ is defined in (3. 33). Then from the above arguments the matrix functions $w_m(s, \rho)$ is bounded and nonsingular if s and ρ satisfy the condition (5. 5) or (5. 7). We write the given equation (3. 4) in the form

$$\begin{aligned}\frac{dv}{ds}&=H(s, \rho)v=H_m(s, \rho)v+[H(s, \rho)-H_m(s, \rho)]v\\ &=H_m(s, \rho)v+[H(s, \rho)v_m(s, \rho)-v_m'(s, \rho)]v_m(s, \rho)^{-1}v.\end{aligned}$$

Then any solution of the integral equation

$$v(s, \rho)=v_m(s, \rho)+\int_{\Gamma(s)} v_m(s, \rho)v_m(\sigma, \rho)^{-1}[Hv_m-v_m']v_m(\sigma, \rho)^{-1}v(\sigma, \rho)d\sigma \tag{5. 9}$$

satisfies the differential equation (3. 4). As in the proof of Theorem 3, $\Gamma(s)$ is a set of paths $\gamma_{jk}(s)$ ending at s.

If $w(s, \rho)$ is defined by

$$v(s, \rho)=s^{k(s)q/4}\Omega(s^{k(s)})w(s, \rho)\exp[Q(s)], \tag{5. 10}$$

then the equation (5. 9) becomes

$$\begin{aligned}w(s, \rho)=w_m(s, \rho)+\int_{\Gamma(s)} & w_m(s, \rho)\exp[Q(s)-Q(\sigma)]w_m(\sigma, \rho)^{-1}\Omega(\sigma^{-k(\sigma)})\sigma^{-k(\sigma)q/4}\\ &\cdot(Hv_m-v_m')\exp[-Q(\sigma)]w_m(\sigma, \rho)^{-1}w(\sigma, \rho)\exp[Q(\sigma)-Q(s)]d\sigma.\end{aligned} \tag{5. 11}$$

In order to solve this equation, we need an estimate of the quantity Hv_m-v_m'.

$$
\begin{aligned}
Hv_m - v'_m &= \left[\sum_{\nu=0}^{m} H_\nu \rho^\nu + R_m\right] \sum_{\nu=0}^{m} v_\nu \rho^\nu - \sum_{\nu=0}^{m} v'_\nu \rho^\nu \\
&= \sum_{\nu=0}^{m} \rho^\nu \left[\sum_{\mu=0}^{\nu} H_\mu v_{\nu-\mu} - v'_\nu\right] + \sum{}^{*} H_\mu v_{\nu-\mu} \rho^\nu + R_m v_m,
\end{aligned} \tag{5.12}
$$

where the summation $\sum^*$ is for

$$
\mu \leqq m, \qquad \nu-\mu \leqq m, \qquad \nu > m,
$$

and R_m represents the remainder terms in the series expansion of $H(s, \rho)$. The first of the three right hand terms in (5. 12) is zero, because v_ν is a solution of (3. 11). The second summation can be written

$$
s^{k(s)q/4}\Omega(s^{k(s)}) \sum{}^{*} \rho^\nu s^{k(s)g(\nu,\mu)} H^*_\mu w_{\nu-\mu} \exp[Q(s)] \tag{5.13}
$$

by virtue of (3. 5) and (3. 31), where $g(\nu, \mu)=\mu/2h+q/2-1/a+(\nu-\mu)e$. For each ν, the maximum power of s is attained by $\mu=1$, that is

$$
\frac{\mu}{2h}+\frac{q}{2}-\frac{1}{a}+(\nu-\mu)e \leqq \nu e - 1,
$$

then the summation (5. 13) can be written

$$
s^{k(s)q/4}\Omega(s^{k(s)}) E(s, \rho)[s^{k(s)e}\rho]^{m+1} s^{-k(s)} \exp[Q(s)], \tag{5.14}
$$

where $E(s, \rho)$ is bounded in (5. 5) or (5. 7) respectively, and e is a number given in Theorem 4. At last, $R_m v_m$ has the same form as in (5. 14) with a different bounded matrix function $E(s, \rho)$ in the domain (1. 3) and (5. 5) or (5. 7), because we have from (3. 6), (3. 8) and (5. 8),

$$
R_m v_m = s^{k(s)q/4}\Omega(s^{k(s)}) \tilde{E}(s, \rho) s^{k(s)i(m)} \rho^{m+1} \exp[Q(s)],
$$

where $i(m)=q/2-1/a+(m+1)/2h$, and $\tilde{E}(s, \rho)$ is a bounded function. If we notice here that

$$
\frac{q}{2}-\frac{1}{a}+(m+1)\frac{1}{2h} \leqq (m+1)e-1 \qquad (h \geqq 1)
$$

then $R_m v_m$ is seen to have the form (5.14).

Inserting these results into (5. 12), we have

$$
H_m v_m - v'_m = s^{k(s)q/4}\Omega(s^{k(s)}) E(s, \rho)[s^{k(s)e}\rho]^{m+1} s^{-k(s)} \exp[Q(s)] \tag{5.15}
$$

with another bounded function $E(s, \rho)$.

Thus the integral equation (5. 11) becomes

$$
\begin{aligned}
w(s, \rho) = w_m(s, \rho) + w_m(s, \rho) \int_{\Gamma(s)} &\exp[Q(s)-Q(\sigma)] w_m(\sigma, \rho)^{-1} E(\sigma, \rho) \\
&\cdot w_m(\sigma, \rho)^{-1} w(\sigma, \rho) \exp[Q(\sigma)-Q(s)][\sigma^{k(\sigma)e}\rho]^{m+1} \sigma^{-k(\sigma)} d\sigma.
\end{aligned} \tag{5.16}
$$

Since the matrices functions w_m and w_m^{-1} are bounded in the domain considered, each element of the matrix which forms the integrand in (5. 16) has the form

$$
\exp[q_{jk}(s)-q_{jk}(\sigma)] L_{jk}[w(\sigma, \rho)][\sigma^{k(\sigma)e}\rho]^{m+1} \sigma^{-k(\sigma)}, \tag{5.17}
$$

where $q_{jk}(s)$ is defined in (3. 25) and $L_{jk}[w]$ is a linear conbination of the components in 1-st or 2-nd column of w. The coefficients of this linear conbination are functions of σ and ρ bounded in (1. 3) and (5. 5) or (5. 7).

Next let us choose the paths of integrations for each element (5. 17).

As in Section 3, we map the sector S of the σ-plane and s-plane into the ζ-plane and ξ-plane respectively by

(5. 18) $$\zeta=\sigma^{(2+q)/2}, \qquad \xi=s^{(2+q)/2}.$$

Let a half plane Σ in the ζ-plane or ξ-plane be the image of the sector S and let $\mathcal{H}$ be a closed half disk in Σ which satisfies

(5. 19) $$|\zeta^{2e/(q+2)}\rho|\leqq\eta_0$$

for each ρ. On the circular arc of the boundary of $\mathcal{H}$, there exists for every pair j, k $(j\neq k)$ a unique point ζ_{jk} at which $\mathrm{Re}\,[\beta_{jk}\zeta]$ attains its maximum in $\mathcal{H}$. (see Figure 2). The number β_{jk} is defined in (3. 25).

Then the quantity

$$\mathrm{Re}\,[q_{jk}(s)-q_{jk}(\sigma)]=\mathrm{Re}\,[\beta_{jk}(\xi-\zeta)]$$

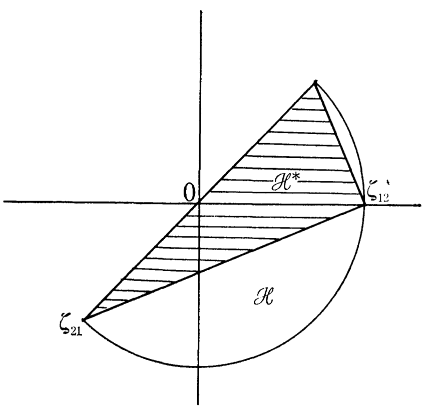

Fig. 2.

increases monotonically when ζ moves from ζ_{jk} to ξ along a straight segment. Here we limit ξ to the triangular domain $\mathcal{H}^*$ whose vertices are ζ_{jk} and two end points of the diameter of $\mathcal{H}$. If ξ is any point of $\mathcal{H}^*$, then the integral path $\lambda_{jk}(\xi)$ is defined by the segment joining ξ and ζ_{jk}. Thus for ζ on $\lambda_{jk}(\xi)$, there exists a positive constant p, independent of j, k and ρ such that

(5. 20) $$\mathrm{Re}\,[\beta_{jk}(\xi-\zeta)]<-p|\zeta-\xi|.$$

We take here the inverse image of $\lambda_{jk}(\xi)$ in the σ-plane as the integral path $\gamma_{jk}(s)$ for $j\neq k$, and the path $\gamma_{jj}(s)$ is to be the ray from the origin to s.

Now, we choose the positive constants ρ_1 and δ_2 in (5. 1) so small that the domain

(5. 21) $$0<|\rho|\leqq\rho_1\leqq\rho_0, \qquad |\arg\rho|\leqq\delta_2,$$

satisfies the condition (1. 3). Let $M(s, \rho)$ be some matrix function which is uniformly bounded in the domain (5. 21) and $\xi=s^{(2+q)/2}\in\mathcal{H}^*$, and let M_0 be the least upper bound of $||M(s, \rho)||$ in this domain.

Consider the integral of (5. 17), and write

(5. 22) $$I(M)=\int_{\gamma_{jk}(s)}\exp\,[q_{jk}(s)-q_{jk}(\sigma)]L_{jk}[M(\sigma)][\sigma^{k(\sigma)}\rho]^{m+1}\sigma^{-k(\sigma)}d\sigma.$$

If we change the variables s and σ into ξ and ζ, this integral becomes

(5. 23) $$I(M)=\int_{\lambda_{jk}(\xi)} \exp[\beta_{jk}(\xi-\zeta)]L_{jk}[M]\zeta^{k(\sigma)2\{(m+1)e-1\}/(q+2)}\zeta^{-q/(q+2)}\rho^{m+1}d\zeta.$$

Let us calculate this integral for each case of $j\neq k$, and $j=k$.

(1) $j\neq k$. First, when all the points of $\gamma_{jk}(s)$ are in the domain $|\sigma|>s_0$, then (5. 23) becomes by virtue of $k(\sigma)=1$

(5. 24) $$I(M)=\int_{\lambda_{jk}(\xi)} \exp[\beta_{jk}(\xi-\zeta)]L_{jk}(M)\zeta^{2(m+1)e/(q+2)-1}\rho^{m+1}d\zeta.$$

Then, for $\xi\epsilon\mathcal{H}^*$,

(5. 25) $$|I(M)|\leqq|\rho|^{m+1}C_1M_0\int_{\lambda_{jk}(\xi)} \exp[-p|\xi-\zeta|]|\zeta|^{2(m+1)e/(q+2)-1}|d\zeta|,$$

where C_1 and C_ν introduced below are some constants which depend on s_0, m, η_0, ρ_1 and δ_2, but are independent of ρ. If the path of integration is extended beyond ζ_{jk} to infinity along a straight ray and if we put

$$\xi-\zeta=r\cdot\exp[i\theta] \qquad (i=\sqrt{-1})$$

then we have

(5. 26) $$\begin{aligned}|I(M)|\leqq&|\rho|^{m+1}C_1M_0|\xi|^{2(m+1)e/(q+2)-1}\\ &\cdot\int_0^\infty \exp[-pr]\left[1+\frac{2r}{\xi}\cos\theta+\frac{r^2}{\xi^2}\right]^{(m+1)e/)q+2)-1/2}dr.\end{aligned}$$

Hence, if $|s|>s_0$, that is $|\xi|>\xi_0=s_0^{(q+2)/2}$, then we have from (5.26),

$$|I(M)|\leqq C_2M_0[\xi^{2e/(q+2)}\rho]^{m+1}.$$

Next if $|s|>s_0$ and some parts of $\gamma_{jk}(s)$ are contained in $|\sigma|\leqq s_0$, then in this parts

$$\begin{aligned}|I(M)|&\leqq|\rho|^{m+1}C_3M_0\int\exp[-p|\xi-\zeta|]|\zeta|^{-q/(q+2)}|d\zeta|\\ &\leqq C_4M_0|\rho|^{m+1}.\end{aligned}$$

In the same way, if $|s|\leqq s_0$ and if $\gamma_{jk}(s)$ is contained in $|\sigma|\leqq s_0$, then $I(M)=O(|\rho|^{m+1})$. And if $|s|\leqq s_0$ and $\gamma_{jk}(s)$ has the parts on which $|\sigma|>s_0$, then the contributions of this parts are $O(|\rho|^{m+1})$.

(2) $j=k$. In this case we have

(5. 27) $$|I(M)|\leqq\rho^{m+1}C_5M_0\int_{\lambda_{jj}(\xi)}|\zeta|^{k(\sigma)2\{(m+1)e-1\}/(q+2)}|\zeta|^{-q/(q+2)}|d\zeta|.$$

If $|\xi|\leqq\xi_0$, the integral (5. 27) has the order of magnitude $O(|\rho|^{m+1})$, and if $|\xi|>\xi_0$, it is $O[\{|\xi|^{2e/(q+2)}\rho\}^{m+1}]$.

Then we get in each case

(5. 28) $$|I(M)|\leqq C_6M_0[|s|^{k(s)e}\rho]^{m+1}.$$

Now we solve the integral equation (5. 16) which can be written

(5. 29) $$w(s,\rho)=w_m(s,\rho)\{1+L[w(\sigma,\rho)]\}$$

where 1 is the unit matrix and

$$L[w]=\int_{\Gamma(s)}\exp[Q(s)-Q(\sigma)]w_m(\sigma,\rho)^{-1}E(\sigma,\rho)w_m(\sigma,\rho)^{-1} \cdot w\cdot\exp[Q(\sigma)-Q(s)][\sigma^{k(\sigma)e}\rho]^{m+1}\exp[-k(\sigma)]d\sigma. \tag{5.30}$$

As in Section 4, we do this by the methods of successive approximations. Let H^* be the inverse image in the s-plane of $\mathfrak{H}^*$. Since $w_m(s,\rho)$ is bounded, we can suppose $||w_m(s,\rho)||\leqq B$ for some constant B.

The successive approximations for (5.29) are

$$w^{(0)}(s,\rho)=w_m(s,\rho), \tag{5.31}$$

$$w^{(k+1)}(s,\rho)=w_m(s,\rho)\{1+L[w^{(k)}(s,\rho)]\}. \tag{5.32}$$

Then from (5.28), we have

$$||w^{(k+1)}(s,\rho)||<B\left\{1+C_7\sup_{H^*}||w^{(k)}(s,\rho)||\cdot|s^{k(s)e}\rho|^{m+1}\right\}.$$

for some constant C_7. Then all $w^{(k)}(s,\rho)$ are bounded in (5.21) and H^*, and are holomorphic in s and ρ in (5.12), H^* and $s<\infty$.

Let $\Delta_k(s,\rho)$ be defined by

$$\Delta_k(s,\rho)=\sup||w^{(k+1)}(s,\rho)-w^{(k)}(s,\rho)||.$$

Then

$$\begin{aligned}\Delta_k(s,\rho)&\leqq BC_8\Delta_{k-1}(s,\rho)|s^{k(s)e}\rho|^{m+1}\\&\leqq(BC_8|s^{k(s)e}\rho|)^kB.\end{aligned}$$

If ρ_1 and η_0 are taken so small that we have in the domain (5.21) and H^*

$$BC_8|s^{k(s)e}\rho|^{m+1}<1, \tag{5.33}$$

then the series $\sum_{k=0}^{\infty}\Delta_k(s,\rho)$ is uniformly convergent and this implies that the sequence $\{w^{(k)}(s,\rho)\}$ converges absolutely and uniformly to a bounded matrix function $w(s,\rho)$ which is clearly a solution of the integral equation (5.29) in the domain (5.21) and H^*.

Now if c_3 in (5.1) is chosen small enough, then the domain (5.1) is contained in H^*, so that the inequalities already established hold in the domain (5.1). Thus we have a solution matrix $w(s,\rho)$ of the differential equation (3.4) in the domain (5.1) and at the same time we get an asymptotic property

$$||w(s,\rho)-w_m(s,\rho)||\leqq C_9|s^{k(s)e}\rho|^{m+1}, \tag{5.34}$$

which proves Theorem 4 from (5.8) and (5.10).

§ 6. Conclusions and Remarks.

CONCLUSIONS. From Theorem 3, (2.1) and (2.5), it follows that the differential equation (1.1) has a fundamental matrix solution in the domain (4.1) of the form

$$y=\begin{pmatrix}1 & 0\\ 0 & x^{q/2}\end{pmatrix}u(\tau,\varepsilon), \tag{6.1}$$

where $u(\tau, \varepsilon)$ is defined in (4. 2).

On the other hand, from Theorem 4, (4. 1) and (3. 2), there exists another fundamental matrix solution in the domain (5. 1) of the form

$$y=\begin{pmatrix} 1 & 0 \\ 0 & \varepsilon^{h-a} \end{pmatrix} v(s, \rho), \tag{6. 2}$$

where $v(s, \rho)$ is defined in Theorem 4. Now in order to state that these two solutions can be patched together, it is sufficient to prove that the domains (4. 1) and (5. 1) overlap for all sufficiently small ε.

The inequality $c_1|\varepsilon| \leqq |\tau| \leqq c_2$ becomes in terms of x and ε,

$$c_1{}^a|\varepsilon|^a \leqq |x| \leqq c_2{}^a, \tag{6. 3}$$

and the inequality $|s^e\rho| \leqq c_3$ becomes

$$|x| \leqq c_3{}^{1/a}|\varepsilon|^{a-1/e(q+2)}, \tag{6. 4}$$

where

$$e=1+\frac{1}{2h}+\frac{q}{2}-\frac{1}{a}=\frac{h(q+2)-(q+1)}{2h}>0.$$

We remark also that $a-1/e(q+2)>0$ if $h>1$, and $a-1/e(q+2)=0$ if $h=1$ for any positive integer q. The fact that $e>0$ assures us that the two domains (6. 3) and (6. 4) overlap for arbitrarily small ε. Thus for suitable point belonging to both domains, we can determine the matching matrix of the two solutions (6. 2) and (6.3).

REMARK. If the fundamental assumption (1. 7) is removed, it is more difficult to analyze the equation (1. 1). According to the results of Iwano and Sibuya [1], the assumption (1. 7) means that the domain (1. 3) is divided into only two subdomains in each of which the solution of the equation (1. 1) moves quite differently as ε tends to zero. In near future, it will be treated more general equations.

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