MINIMAL SLIT DOMAINS AND MINIMAL SETS

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§ 1. Introduction.

1. The problems of mapping a plane domain onto a certain canonical domain such as a circular, radial and parallel slit domain have been discussed by many mathematicians. Grötzsch [3, 4] specified a number of extremum problems which provided the uniqueness of the mapping functions for the domains of infinite connectivity. His method, the method of strips, was succeeded by the refined method of the extremal metric which was usefully applied to the theory of univalent functions by Jenkins [6].

Recently Reich and Warschawski [20, 21] and Reich [19] dealt with these problems for the cases of both circular and radial slit disks and annuli. Their method was based on a sort of area theorem which was an extension of an inequality of Rengel. There are another approaches to the problems containing potential-theoretic techniques as in Ahlfors and Sario's book (Chap. III) [2] and others [7, 8, 9, 23, 24, 28, 30].

In the present note we first make a brief treatment of the problems except for the cases of general radial slit domains by means of the method of the extremal metric in § 3 as in Jenkins' book [6] in which the parallel slit theorem is obtained (pp. 81–85). Our principle, however, is an elementary equality in Hilbert space regarded as a precise evaluation of Rengel's inequality, instead of the geometrical observations. It leads to the specified extremum properties, and characterizations of the canonical domains in terms of the module or extremal length follow to it. These are essentially due to Grötzsch [3, 4]. Since we can show a localization principle which was given by Grötzsch [3] without proof, we shall define minimal sets of circular, radial and parallel slits for compact sets in § 4 which was first introduced by Koebe [9] for parallel slits.

We discuss several properties of the minimal sets and examine the inter-relations between the minimal sets of both circular and radial slits and the set of parallel slits. We show that the union of a finite number of disjoint minimal sets is minimal. The above results and the localization of the criterion for the minimality are proved in the stand-point of the method of the linear operator method [2] by Oikawa [15] in these Reports.

In § 5 we extend the minimality to non-compact sets and define quasi-minimal sets. The remained mapping problems are discussed and general radial slit mapping theorems due to Strebel [28] and Reich [19] are obtained, which may be another proof of the continuity of extemal lengths [28, 31] for an exhaustion in case of plane domains. Since a property of radial slit disks given by Strebel [27] contains an in-

Received March 22, 1965.

complete discussion, we give a correct one.

The author expresses his hertiest thanks to Professor K. Oikawa for his valuable advices in preparing this paper and to Professors Y. Komatu and M. Ozawa for their kind encouragements and valuable remarks.

§ 2. Preliminaries, modules and extremal lengths.

2. Let Ω be a domain and Γ be a family of locally rectifiable curves in Ω. If a non-negative measurable function $\rho(z)$ defined in Ω satisfies a condition

$$\int_{\gamma} \rho |dz| \geqq 1 \qquad \text{for all } \gamma \in \Gamma,$$

where the existence of the above integral (as a Lebesgue-Stieltjes integral) is assumed, we call ρ an *admissible metric* with respect to the L-normalization and denote by P the class of admissible metrics. Then we designate the quantity

$$\operatorname{mod} \Gamma = \inf_{\rho \in P} \iint_{\Omega} \rho^2 dxdy$$

as the *module* of Γ. The reciprocal of it is called *extremal length* $\lambda(\Gamma)$ of the family Γ [6]. If there exists an admissible metric ρ_0 whose square integral over Ω attains $\operatorname{mod} \Gamma$, it is called an extremal metric. The extremal metric for the module problem with finite module is unique if it exists [6].

The following lemma shows a property of continuous extremal metrics defined below.

We suppose that a curve family Γ has the following property:

(P) Let γ_1 and γ_2 be any two members of Γ (may coincide) and z_1, z_2 are contained in γ_1 and γ_2 respectively. γ_1' and γ_2' are denoted by the arcs before z_1 and after z_2 following suitable parametrizations of them respectively. If we join the two points by an arbitrary locally rectifiable curve α, the composed curve $\gamma_1' \cup \alpha \cup \gamma_2'$ belongs to Γ.

We remark that there exists a curve of the family Γ through any point z in Ω. Denoting by Γ_z the subfamily of Γ consisting of all curves through z, we state

LEMMA 1. *If $\rho_0(z)$ is a continuous extremal metric for a module problem of a curve family Γ with the property (P) and* $\operatorname{mod} \Gamma < \infty$, *then we have*

$$(1) \qquad \inf_{\gamma \in \Gamma_z} \int_{\gamma} \rho_0 |dz| = 1 \qquad \textit{for every point } z \textit{ in } \Omega.$$

The lemma is a generalization of the one in the previous paper of Oikawa and the author [17] and the proof is analogous to it. Similar results for the rectangle and circular ring are found as effectual lemmas in proving statements of uniqueness in Jenkins [6] but we use it in discussing certain properties of minimal slits domains. Jurchescu [7] used similar arguments to our proof for circular slit annuli.

PROOF. Contrary to the assertain, we suppose that

$$\inf_{\gamma\in\Gamma_z}\int_\gamma \rho_0|dz|=1+\delta, \qquad \delta>0. \tag{2}$$

We denote by $U(z,\varepsilon)$ the set of all points in the distance within ε from z with respct to ρ_0 metric. Since ρ_0 is continuous $U(z,\varepsilon)$ is open and connected and we have

$$\iint_{U(z,\varepsilon)}\rho_0^2 dxdy>0. \tag{3}$$

Indeed vanishing of the integral (3) implies that $\rho_0(z)\equiv 0$ in $U(z,\varepsilon)$ which contradicts the definition of $U(z,\varepsilon)$. Choosing ε smaller than $\delta/2$, we put

$$\rho_1=\begin{cases}0, & z\in U(z,\varepsilon),\\ \rho_0, & z\in\Omega-U(z,\varepsilon).\end{cases}$$

Then ρ_1 is also adminissible by (2) and we have $||\rho_1||_\Omega^2<||\rho_0||_\Omega^2$, using (3), which contradicts the extremality of ρ_0. Here and hereafter the norm stands for the square integral.

3. Let Γ be a curve family with finite module. Obviously the admissible class P for Γ makes a convex set. By the fact Strebel [27] showed that there exists always a unique metric ρ_0 called a *generalized extremal* metric as a strong limit of minimal sequences. To explain the circumstances we add all strong limit functions of P to it and have a closed convex set. We call such a class P^* a *generalized admissible class* and its member a *generalized admissible metric.* We can easily see

$$\operatorname{mod}\Gamma=\inf_{\rho\in P}||\rho||^2=\min_{\rho\in P^*}||\rho||^2.$$

In fact obviously the infinima of norms of metrics of both classes P, P^* coincide and we take a minimal sequence $\{\rho_n\}$ of P^*. We have by an elementary equality

$$\left\|\frac{\rho_m+\rho_n}{2}\right\|^2+\left\|\frac{\rho_m-\rho_n}{2}\right\|^2=\frac{||\rho_m||^2+||\rho_n||^2}{2} \tag{4}$$

an inequality $||\rho_m-\rho_n||^2<4\varepsilon$ for so large m, n that $||\rho_m||^2-\operatorname{mod}\Gamma<\varepsilon$, $||\rho_n||^2-\operatorname{mod}\Gamma<\varepsilon$, since $\operatorname{mod}\Gamma\leqq||(\rho_m+\rho_n)/2||^2$. Then $\{\rho_n\}$ is a Cauchy sequence and we get a unique limit function ρ_0 satisfying $||\rho_0||^2=\operatorname{mod}\Gamma$ for all minimal sequences.

Inserting into (4) a metric ρ of P^* and a generalized extremal metric ρ_0, we have

$$\frac{1}{2}||\rho-\rho_0||^2\leqq||\rho||^2-||\rho_0||^2 \tag{5}$$

which is a fundamental inequality in the subsequent arguments.

§ 3. Mapping problems.

4. We now state several mapping theorems. Although Theorems 1-7 contained in Reich and Warschawski [19, 20], Jenkins [6] and essentially in Grötzsch [3, 4], or partly in many other articles [2, 7, 8, 9, 23, 24, 28, 30], we make simple proofs in order to make this note self-contained. The problems will be considered for the domains of infinite connectivity and the results for the domains of finite connectivity are supposed (Rengel [22], Komatu [10] and Nehari [13]).

We speak of a strict difinition of boundary components. Let Ω be a plane domain and $\{V_n\}_{n=1}^{\infty}$ be a sequence of its ends i.e. non-compact subregions with compact relative boundary satisfying the following conditions: $\{V_n\}$ is decreasing, the relative boundary of V_n consists of one Jordan closed curve and $\cap V_n=\phi$. Then we obtain a closed set $C=\cap \overline{V}_n$, where $\overline{V}_n$ is the closure of V_n taken in the complex sphere and call C and $\{V_n\}$ a *boundary component* of Ω and a *defining sequence* of C respectively. Two defining sequences $\{V_n\}$, $\{V'_n\}$ define the same boundary component if and only if there exist suitable m, m' such that $V'_n\supset V_m$ and $V_n\supset V'_{m'}$ for every n and they are said to be equivalent. We can take a equivalent class of defining sequences as a *boundary component* in the sense of Kerékjártó-Stoilow.

Let $T(z)$ be a topological mapping of Ω. Then the images $T(V_n)$ of a defining sequence $\{V_n\}$ make a defining sequence in $T(\Omega)$ and define a boundary component. We call it the *image* of a boundary component under a topological mapping T.

We mean by a (normal) exhaustion of Ω a sequence of relatively compact subregions $\{\Omega_n\}_{n=1}^{\infty}$ such that $\overline{\Omega}_n\subset\Omega_{n+1}$, $\cup\Omega_n=\Omega$ and the relative boundary of Ω_n consists of a finite number of analytic Jordan closed curves containing at least one boundary component in its exterior with respect to Ω_n.

5. Let Ω be a plane domain and a, b, C, C' be two points of Ω and two boundary components respectively. We denote by $\mathfrak{F}_{ab}(\Omega)$ the family of univalent functions $f(z)$ with the properties:

1. $f(a)=1-f'(a)=0$,
2. f has a simple pole at b with residue $c(f)$.

Then we have

THEOREM 1. *There exists a unique function $f_{ab}(z)$ within $\mathfrak{F}_{ab}$ with the following properties:*

1. $|c(f_{ab})|\leqq|c(f)|$ *for* $f\in\mathfrak{F}_{ab}$.
2. *The images of the boundary components of Ω under f_{ab} are circular slits (possibly points) of total area zero.*
3. *If an annulus $A(q, Q)=\{q<|w|<Q\}$ contains the image of the boundary under f_{ab}, the module of the family of locally rectifiable closed curves separating the two boundaries of $A(q, Q)$ in the image domain Δ is equal to* $(2\pi)^{-1}\log(Q/q)$.

Conversely the property 1 *or* 3 *characterizes* f_{ab}.

Proof. Let $\{\Omega_n\}$ be a normal exhaustion of Ω. Then there exist extremal functions f_n within $\mathfrak{F}_{ab}(\Omega_n)$ with the properties in Theorem 1. By the normalization

1 we can extract a subsequence $\{f_\nu\}$ with a limit f_0 from $\{f_n\}$. Let Δ' be the intersection of Δ with $A(q, Q)$ containing the image of the boundary of Ω under f_0. Putting $\rho_0=(2\pi)^{-1}|f_0'/f_0|$ and $\rho_\nu=(2\pi)^{-1}|f_\nu'/f_\nu|$, we have from the weak convergence of ρ_ν which is the result of its uniform convergence on any compact subset of Ω

$$\underline{\lim}\,||\rho_\nu||^2_{\Omega'}\geqq||\rho_0||^2_{\Omega'} \tag{6}$$

where Ω' is the inverse image of Δ' and ρ_ν is defined zero outside of Ω_ν in the sequal.

On the other hand, the uniform convergence of f_ν on the inverse image of the boundary of $A(q, Q)$ shows that ρ_ν is an extremal metric for the module problem in a subdomain of Ω' arbitrarily close to it. We have

$$\overline{\lim}\,||\rho_\nu||^2=\frac{1}{2\pi}\log\frac{Q}{q}\leqq||\rho_0||^2$$

by the admissibility of ρ_0, which implies

$$\frac{1}{2\pi}\log\frac{Q}{q}=\lim||\rho_\nu||^2=||\rho_0||^2=\operatorname{mod}\Delta', \tag{7}$$

where we use the notation "mod" as a functional of a domain. Since $\rho_0(\rho_0(w)=1/|w|$ in $\Delta')$ is a continuous extremal metric, the infinimum of the logarithmic lengths of closed curves through a fixed point w in Δ' and separating the origin from the point at infinity is 2π by Lemma 1 and hence the boundary components of Δ are circular slits. By the equality (7) the logarithmic area of slits and also the area vanish. We can deduce from the weak convergence of f_ν'/f_ν and the convergence of norms $||\rho_\nu||$ that it converges to f_0'/f_0 strongly. By the uniqueness of extremal metrics, the limits are same for all subsequences of $\{f_n\}$ and hence it converges to f_0.

To show the extremal property 1, for $f(z)\in\Omega$ we put $\rho=(2\pi)^{-1}|f'/f|$ which is admissible in Ω'. Denoting by $M_Q(f)$, $m_q(f)$ the maximum and minimum moduli of f on the inverse images of circles $|w|=Q$, $|w|=q$ under f respectively, we have by the fundamental inequality (5) in § 2

$$\frac{1}{2\pi}\left(\log\frac{M_Q(f)}{Q}-\log\frac{m_q(f)}{q}\right)\geqq\frac{1}{2}||\rho-\rho_0||^2_{\Omega'},$$

since the logarithmic area of an annulus $A(m_q(f), M_Q(f))$ is larger than $||\rho||^2_{\Omega'}$ and making q and Q tend to zero and infinity respectively

$$(2\pi)^{-1}\log\left|\frac{c(f)}{c(f_0)}\right|\geqq\frac{1}{2}||\rho-\rho_0||^2_{\Omega}. \tag{8}$$

The characterizations by the property 1 or 3 are obvious. In fact, if $|c(f)|=|c(f_0)|$, we have $|f'/f|\equiv|f_0'/f_0|$ in Ω and the extremal property of f_0 is deduced from the property 3 and (8). We designate f_0 as f_{ab}.

REMARK. If an annulus $A(q, Q)$ containing the boundary of Δ has the property 3, it holds for all $A(q, Q)$ for $q'<q, Q'>Q$ by the superadditivity of modules [6]. Such improvements of the characterizations in terms of modules will be discussed in § 4.

6. To transfer to slit disks, we denote by $\mathfrak{F}_{aC}(\Omega)$ the family of univalent functions $f(z)$ with the properties:

1. $f(a)=1-f'(a)=0$,
2. the image $f(C)$ separates the origin from the point at infinity.

We introduce a notation

$$M_C(f)=\max_{w\in f(C)}|w|$$

and state

THEOREM 2. *If Ω is mapped onto a bounded domain by a univalent function of $\mathfrak{F}_{aC}$, there exists a unique function $f_{aC}(z)$ within $\mathfrak{F}_{aC}$ with the following properties:*

1. *$M_C(f_{aC})\leqq M_C(f)$ for $f\in\mathfrak{F}_{aC}$.*
2. *The images of boundary components other than C are circular slits (possibly points) of total area zero.*
3. *$f_{aC}(C)$ is a circle with radius Q $(Q=M_C(f_{aC}))$ and if a circle with radius q $(q<Q)$ separates the origin from the boundary of the image domain Δ, the module of locally rectifiable closed curves separating two boundaries of the annulus $A(q, Q)$ in Δ is equal to $(2\pi)^{-1}\log(Q/q)$.*

Conversely the property 1 *or* 3 *characterizes f_{aC}* [21].

Proof. The proof is similar to that of Theorem 1 and omitted.

REMARK. The assumption in Theorem 2 is the fact that C is not weak [25].

Let $\mathfrak{F}_{CC'}(\Omega)$ be a family of univalent functions $f(z)$ with the property: $f(C')$ and $f(C)$ divide the other from the origin and the point at infinity respectively. Denoting by $m_C(f)$ the quantity

$$m_C(f)=\min_{w\in f(C)}|w|$$

we have

THEOREM 3. *If Ω is mapped by a function of $\mathfrak{F}_{CC'}$ onto a relatively compact domain in the finite punctured plane delating the origin, there exists a unique function $f_{CC'}(z)$ within $\mathfrak{F}_{CC'}$ save linear transformations with fixed points at zero and infinity having the following properties:*

1. *$M_C(f_{CC'})/m_{C'}(f_{CC'})\leqq M_C(f)/m_{C'}(f)$ for $f\in\mathfrak{F}_{CC'}$.*
2. *The images of boundary components other than C, C' are circular slits (possibly points) of total area zero.*
3. *$f_{CC'}(C')$, $f_{CC'}(C)$ are two circles with radii q, Q and the module of locally rectifiable closed curves separating the two circles in the image domain is equal to $(2\pi)^{-1}\log(Q/q)$.*

Conversely the property 1 *or* 3 *characterizes* $f_{CC'}$ [20].

Proof. We point out only differences from that of Theorem 1. Let $\{\Omega_n\}$ be a normal exhaustion of Ω and C_n, C'_n the boundary curves of Ω_n containing C, C' respectively. Then there exists a function f_n within $\mathfrak{F}_{C_nC'_n}(\Omega_n)$ with the properties in Theorem 3 and an additional normalization that $f_n(C'_n)$ coincides with the unit circle. The quantity $Q_n = M_{C_n}(f_n)$ is increasing and bounded by the assumption. Denoting by Q the limiting value of Q_n, we select a subsequence $\{f_\nu\}$ with a limit f_0, under which the image domain is contained by an annulus $A(1, Q)$. Then under the notations in Theorem 1, we have

$$||\rho_0||^2 \leqq \underline{\lim}||\rho_\nu||^2 \leqq \overline{\lim}||\rho_\nu||^2 \leqq \mathrm{mod}\, \Delta \leqq ||\rho_0||^2 \tag{9}$$

by the weak convergence of f'_ν/f_ν, the monotonity of modules and the admissibility of ρ_0, which implies the properties 2 and 3. As to the extremality, put $\rho = (2\pi)^{-1}|f'/f|$ for $f \in \mathfrak{F}_{CC'}$ and insert ρ_0, ρ into (5). We have

$$\frac{1}{2}||\rho - \rho_0||^2 \leqq (2\pi)^{-1}\left(\log \frac{M_C(f)}{m_{C'}(f)} - \log M_C(f_0)\right).$$

In the above inequality implies that $\rho \equiv \rho_0$ and $f = af_0$.

REMARK. If Ω is regarded as a bordered surface not necessarily compact with compact borders C, C', we take such a special exhaustion $\{\Omega_n\}$ that $(\bar{\Omega}_n^* \cap \Omega) \subset \Omega_{n+1}$, $\cup \Omega_n^* = \Omega$ and the boundary of Ω_n^* consists of the borders C, C' and a finite number of analytic Jordan closed curves containing at least one boundary component of Ω, instead of the normal exhaustion $\{\Omega_n\}$ in the proof and the sequence $\{f_n^*\}$ of the normalized extremal functions contains convergent subsequences to f_0, whose convergences are uniform on any compact subset in $\Omega \cup C \cup C'$. In fact the strong convergences of the logarithmic derivatives of subsequences are valid as before from (9) and the uniform convergence on borders are deduced by the usual inversion method.

7. We next mention mappings onto radial slit domains. Under the notations in Nos. 5 and 6 we state three theorems.

THEOREM 4. *There exists a unique function $g_{ab}(z)$ within $\mathfrak{F}_{ab}(\Omega)$ with the properties:*

1. $|c(g_{ab})| \geqq |c(f)|$ *for* $f \in \mathfrak{F}_{ab}$.
2. *The images of the boundary components of Ω under g_{ab} are radial slits (possibly points) of total area zero.*
3. *If an annulus $A(q, Q)$ contains the image of the boundary of Ω under g_{ab}, the module of the family of locally rectifiable curves joining the two boundaries of $A(q, Q)$ in Δ $(= g_{ab}(\Omega))$ is equal to $2\pi/\log(Q/q)$.*

Conversely the properties 1 *or* 3 *characterizes* g_{ab}.

THEOREM 5. *If C is not a point and isolated from other boundary components, there exists a unique function $g_{aC}(z)$ within $\mathfrak{F}_{aC}(\Omega)$ with the properties:*

1. $m_C(g_{aC}) \geqq m_C(f)$ *for* $f \in \mathfrak{F}_{aC}$.

2. *The images of boundary components other than C are radial slits (possibly points) of total area zero.*

3. $g_{aC}(C)$ *is a circle with radius* Q ($Q = m(g_{aC})$) *and if a circle with radius* q ($q < Q$) *separates the origin from the boundary of the image domain, the module of locally rectifiable curves joining the two boundaries of the annulus* $A(q, Q)$ *in* $\varDelta$ *is equal to* $2\pi/\log(Q/q)$.

Conversely the properties 1 *or* 3 *characterizes* g_{aC} [21].

THEOREM 6. *If neither C nor C′ is a point and if they are isolated from other boundary components, there exists a unique function* $g_{CC'}(z)$ *within* $\mathfrak{F}_{CC'}(\varOmega)$ *save linear transformations with fixed points at zero and infinity with the properties:*

1. $m_C(g_{CC'})/M_{C'}(g_{CC'}) \geqq m_C(f)/M_{C'}(f)$ *for* $f \in \mathfrak{F}_{CC'}$.

2. *The images of boundary components other than C, C′ are radial slits (possibly points) of total area zero.*

3. $g_{CC'}(C')$, $g_{CC'}(C)$ *are two circles with radii* q, Q *and the module of locally rectifiable curves joining the two circles in the image domain is equal to* $2\pi/\log(Q/q)$ [28].

Proof. We give a proof of Theorem 5 which is applied to the other two with a little modifications. We may suppose that C is an analytic curve. Let $\{\varOmega_n\}$ be an exhaustion of $\varOmega$ in the sense of Remark in No. 6 and g_n the extremal functions of $\varOmega_n$ in Theorem 5 [10, 22]. As before for a subsequence $\lim g_\nu = g_0$. Let Q_ν be the radius of the outer boundary of $g_\nu(\varOmega_\nu)$. Q_ν is decreasing and $\lim Q_\nu = Q$, which is the radius of that of $g_0(\varOmega)$, from the uniform convergence of g_ν on C by usual inversions. Put $g_0(\varOmega) = \varDelta$ and $\varDelta' = \varDelta \cap A(q, Q)$ for so small q that the circle $|w| = q$ separates the origin from the boundary of $\varDelta$. Then the metrics $\rho_0 = |g_0'/(g_0 \log(Q/q))|$, $\rho_\nu = g_\nu/(g_\nu \log(Q_\nu/M_q(g_\nu))$ are admissible and since $\lim M_q(g_\nu) = q$, we have

$$\|\rho_0\|^2 \leqq \underline{\lim}\|\rho_\nu\|^2 \leqq \overline{\lim}\|\rho_\nu\|^2 \leqq \operatorname{mod}\varDelta' \leqq \|\rho_0\|^2$$

and $\|\rho_0\|^2 = 2\pi/\log(Q/q) = \operatorname{mod}\varDelta'$. From the convergence of $\log(Q_\nu/M_q(g_\nu))$ we note that $\|g_\nu'/g_\nu - g_0'/g_0\|^2 \to 0$.

To show the property 1, we set $\rho = |f'/f \log(m_C(f)/M_q(f))|$ in the intersection of $f(\varOmega)$ with $A(M_q(f), m_C(f))$ and zero elsewhere for $f \in \mathfrak{F}_{aC}$.

$$\frac{1}{2}\|\rho - \rho_0\|^2 \leqq 2\pi \Big/ \log\frac{m_C(f)}{M_q(f)} - 2\pi \Big/ \log\frac{Q}{q}.$$

Multiplying it by $\log(Q/q)\log(m_C(f)/M_q(f))$ and tending q to zero, we have for the inverse image $\varOmega^*$ of $\{|w| < m_C(f)\}$ under f

$$\frac{1}{2}\left\|\left|\frac{f'}{f}\right| - \left|\frac{g_0'}{g_0}\right|\right\|^2 \leqq 2\pi \log\frac{Q}{m_C(f)}$$

which implies the property 1 and the converse.

8. Let $\mathfrak{F}_a(\varOmega)$ be the family of meromorphic univalent functions $f(z)$ with the

expansion at the point a: $f(z)=1/(z-a)+c_1(f)(z-a)+\cdots$. Then we state

THEOREM 7. *There exists a unique function $p_a^\theta(z)$ within $\mathfrak{F}_a(\Omega)$ with the properties:*

1. $\mathrm{Re}(e^{-2i\theta}c_1(p_a^\theta))\geqq\mathrm{Re}(e^{-2i\theta}c_1(f))$ *for* $f\in\mathfrak{F}_a$.
2. *The images of the boundary components of Ω under p_a^θ are parallel slits (possibly points) with inclination θ.*
3. *Let $S(L,\theta)$ be a square defined by $|\mathrm{Re}(e^{-i\theta}w)|<L$, $|\mathrm{Im}(e^{-i\theta}w)|<L$. If $S(L,\theta)$ contains the boundary of the image domain Δ, the module of the family of locally rectifiable curves joining the sides with inclination $\theta+\pi/2$ in Δ is equal to one.*

Conversely the property 1 *or* 3 *characterizes* p_a^θ.

Proof. The result is essentially due to Grötzsch [3]. de Possel [18] showed the property 1 and the characterization by it. Jenkins [6] proved the theorem by means of the method of extremal metrics and our proof is similar to his except the inequality (5) and omitted here.

The following facts are evident from the proofs of the preceding theorems.

COROLLARY 1. *The extremal functions of the above problems are the limit functions of those of exhaustions, where suitable normalizations are added if necessary: e.g. the inner radius and the image of a point on C' are fixed in the case of annuli with isolated distinguished boundaries. The convergences are uniform on any compact subsets of Ω and $\|f_n'/f_n-f_0'/f_0\|_{\Omega_n}^2\to0$, $\|p_n'-p_0'\|_{\Omega_n}^2\to0$ in the cases of circular or radial slits and parallel slits respectively.*

9. For the later use, we prove a special mapping theorem as a lemma. Let Ω be such a domain that its outer boundary is a closed Jordan curve with distinguished four points ζ_1, ζ_2, ζ_3, ζ_4 and its inner boundary is a compact set E contained in the interior of the curve. We suppose that the four points lie in the natural order. Then we state

LEMMA 2. *There exists a unique mapping function $\varphi(z)$ such that*

1. *φ maps Ω onto a rectangle R defined by $0<\mathrm{Re}\,w<L$, $0<\mathrm{Im}\,w<L'$, minus a compact set in such a way that the four points correspond to vertices of R and especially ζ_1, ζ_2 to 0, L respectively,*
2. *φ maps E onto a horizontal slits (possibly points) of total area zero,*
3. *the module of the family of locally rectifiable curves joining two vertical sides is equal to L'/L.*

Conversely if a slit rectangle R with the normalization in 1 *satisfies the condition* 3, *then $\varphi(z)=z$. Furthermore let $\{R_n\}$ be an exhaustion of R fixing the outer boundary in the sense of Remark in No.* 6 *and φ_n the mapping functions of R_n in this lemma with the same normalization. φ_n converges to z uniformly on any compact subsets of the union of the domain and the outer boundry.*

Proof. We owe the proof to Grötzsch [3] who proved it in the case of finite connectivity. We regard Ω as a non-compact bordered surface with a compact border and make a double of Ω denoted by $\hat{\Omega}$, identifying the arcs $\widehat{\zeta_2\zeta_3}$, $\widehat{\zeta_4\zeta_1}$ with their counterparts respectively. $\hat{\Omega}$ has two closed boundary curves C, C' consisting of two arcs $\widehat{\zeta_1\zeta_2}$, $\widehat{\zeta_3\zeta_4}$ respectively. Then by Theorem 3 a function $f_{CC'}$ maps $\hat{\Omega}$ onto a circular slit annulus with inner radius one having the properties in Theorem 3. Since $\hat{\Omega}$ has an anti-conformal mapping onto itself fixing the arcs $\widehat{\zeta_2\zeta_3}$ $\widehat{\zeta_4\zeta_1}$, they correspond by $f_{CC'}$ to two segments on a diameter of the outer circle. We put $\varphi=A\log f_{CC'}+B$ for suitable imaginary constants A, B and then the restriction of it in Ω is the desired function. Indeed the properties 1, 2 are obvious. Let Γ_1, Γ_2 be the family of locally rectifiable curves joining the segments $f_{CC'}(\widehat{\zeta_4\zeta_1})$ and $f_{CC'}(\widehat{\zeta_2\zeta_3})$ in the upper and lower halves respectively. Then the module of the family of curves separating two boundaries of $A(1, Q)$ in the image domain is equal to $(2\pi)^{-1}\log Q$ where Q is the radius of the outer circle, and since each member of it contains both members of Γ_1 and Γ_2, a well-known inequality $(\operatorname{mod}\Gamma_1)^{-1}+(\operatorname{mod}\Gamma_2)^{-1}\leqq 2\pi/\log Q$ [2] shows that $\operatorname{mod}\Gamma_1=\operatorname{mod}\Gamma_2=(4\pi)^{-1}\log Q$, which implies the property 3. The later half is obvious from Theorem 3 and Corollary 1.

We remark that Lemma 2 is also valid in case that the arcs $\widehat{\zeta_2\zeta_3}$, $\widehat{\zeta_4\zeta_1}$ are isolated from E under the exhaustions in which the isolated arcs are fixed.

§ 4. Minimal slit domains and minimal sets.

10. We call the image domains in Theorems 1, 2, 3 (4, 5, 6) a *minimal* circular (radial) slit plane, disk, annulus respectively. The image domains p_a^0, $p_a^{\pi/2}$ are called a minimal horizontal and vertical slit plane. By the preceding theorems they possess the extremal properties and characterizations in terms of modules. The extremal function of each minimal domain is the function $w=z$ and the extremal functions in the exhaustion converge to it in the respective sense adding suitable normalizations, if necessary. As in Remark after Theorem 1, the criteria for minimal circular slit domains comply localization principles. Analogous remarks are valid for other minimal domains, but they are not trivial since the superadditivity of modulus is not effective. We show them for horizontal slit domains, although they were already stated by Grötzsch [3] without proof.

On the other hand Professor Oikawa [15] will give interesting and effective characterizations by means of linear operator methods in the subsequent paper in these Reports, which may be regarded as a generalization of Koebe's [9] and these facts will be clarified by them.

To this end we need the following

LEMMA 3. *Let $\{V_j\}_{j=1}^N$ be a finite number of ends of a domain Ω and f_n univalent functions defined in an exhaustion $\{V_{jn}\}$ of V_j. If $\lim f_n=z$ uniformly on any compact subsets of V_j, then for sufficiently large n there exist quasi-conformal mappings F_n in subdomains of Ω such that F_n is equal to f_n in the intersection of V_{jn} with subends V_j' of V_j for all j and to z in $\Omega-\cup V_j$, and the maximal dilata-*

tions K_n of F_n tend to one with n increasing.

Proof. Put $f_n(z)=z+\zeta_n(z)$. Let V_j^*, V_j' be subends of V_j and of V_j^* with closed analytic Jordan curves C_j^*, C_j' as relative boundaries respectively. We denote by S_j the ring domain bounded by C_j^*, C_j' and by $\omega_j(z)$ the harmonic measure of C_j' in S_j. We set

$$F_n(z)=\begin{cases} f_n(z), & \text{in } V_j'\cap V_{jn}, \\ z+\omega_j(z)\zeta_n(z), & \text{in } S_j, \\ z, & \text{in } \Omega-\cup V_j^*. \end{cases}$$

The restriction of F_n on S_j maps C_j^*, C_j' onto simple closed curves C_j^*, $f_n(C_j')$ and the jacobian of F_n does not vanish for sufficiently large n. Hence F_n maps S_j onto a ring domain bouned by C_j^*, $f_n(C_j')$ quasi-conformally and by the continuations across C_j^* and C_j' [12], F_n is quasi-conformal in $\cup_j((\Omega-V_j^*)\cup V_{jn})$. Simple calculations show that $\lim K_n=1$.

Then we have

THEOREM 8. *Let Ω be a plane domain. If there exists such a rectangle R with horizontal and vertical sides of the lengths L, L' containing the boundary of Ω in its interior that the module of the family of curves joining the vertical sides in Ω is equal to L'/L, then Ω is a minimal horizontal slit plane.*

Proof. Let $\{R_n\}$ be an exhaustion of $\{R\cap\Omega\}$ in Lemma 2. The functions $\varphi_n(z)$ mapping R_n onto slit rectangles with the same lower horizontal sides tend to z. Let S be a square containing R. Then the function F_n in Lemma 3 maps $R_n\cup(S-R)$ onto S minus a finite number of horizontal slits. We have $\mathrm{mod}(S\cap\Omega)\geqq 1/K_n$, and $\mathrm{mod}(S\cap\Omega)=1$, tending n to infinity which implies the minimality of Ω by Theorem 7.

11. Now we define minimal sets which were first introduced by Koebe [9]. Let E be a compact set in the z-plane. If E^c is a minimal horizontal (vertical) slit plane, then E is called a *minimal set of horizontal* (*vertical*) *slits*. Similarly if E^c contains the origin and is a minimal circular (radial) slit plane, E is called a *minimal set of circular* (*radial*) *slits*. For simplicity's sake we remark that those minimal sets lie on trajectories of suitable quadratic digerentials. Indeed minimal sets of horizontal, vertical, circular and radial slits lie on the trajectories of the quadratic differentials[1)] dz^2, $-dz^2$, $-dz^2/z^2$, dz^2/z^2 respectively. Generalizations of minimal sets for other quadratic differentials are possible but not done here.

Summing up the results obtained, we can deduce the following properties of minimal sets:

i) Any compact subset of a minimal set is minimal.

ii) Linear transformations $z+c$ for parallel slits, cz for other preserve the mini-

1) For the definitions of quadratic differentials, related notions and their properties see Jenkins [6].

mality.

iii) The area of a minimal set vanishes.

iv) Two points in E^c lying on a trajectory can be joined in it by an arc arbitrarily close to the distance of them with respect to extrmal metrics (euclidean or logarithmic), where E is a minimal set.

In fact these properties follow from tne monotonity, conformal invariance of modules, Theorems in § 3 and Lemma 1.

A compact set E with the projection of the linear measure zero into the imaginary axis is a minimal set of horizontal slits, and similar statements are valid for others. Koebe [9] conjectured the necessity of the condition, but Grötzsch [3] showed an example of a minimal set of horizontal slits such that the projection into the imaginary axis becomes an interval.

Professors J. Tamura and K. Oikawa informed us two simple examples which are a compact set of class $N_{\mathfrak{D}}$ [1] with an interval as the projection into a line in any direction and a totally disconnected compact set without the property iv) with respect to the euclidean metric [29].

12. We prove

LEMMA 4. *Let Ω be a domain with a set of conformal or anti-conformal mappings $\{\phi\}$ onto itself and $\tilde{\Gamma}$ a family of locally rectifiable curves with finite module and invariant under each ϕ. Let $\tilde{P}^*$ be the subclass of all generalized admissible metrics satisfying the following condition: $\tilde{\rho}(z)=\tilde{\rho}\circ\phi(z)|\phi'(z)|$ for all ϕ.* [2] *Then we have*

$$\operatorname{mod}\tilde{\Gamma}=\min_{\tilde{\rho}\in\tilde{P}^*}\|\tilde{\rho}\|^2$$

Proof. We put $\rho_\phi=\rho\circ\phi|\phi'|$ for any generalized admissible metric ρ and

$$\tilde{\rho}=\frac{\rho+\rho_\phi}{2}$$

which is invariant under ϕ_j. Then we have

$$\|\tilde{\rho}\|^2+\left\|\frac{\rho-\rho_\phi}{2}\right\|^2=\frac{\|\rho\|^2+\|\rho_\phi\|^2}{2},$$

which implies the assertion since $\|\rho\|^2=\|\rho_\phi\|^2$.

Now we state an improvement of the property 3 in § 3 and Theorem 8, which is valid for other minimal sets:

THEOREM 9. *Let E be a minimal set of horizontal slits and R a rectangle with sides parallel to the axes. Let Γ^* be the family of locally rectifiable curves in $R-E$ with end points on two vertical sides which have neighborhoods contained in E^c. Then the module of Γ^* is equal to the ratio of the sides.*

2) ϕ' is read as $d\phi/d\bar{z}$ in anti-conformal cases.

Conversely if a closed rectangle $\bar{R}$ contains a compact sets E and the module of the above family Γ^* is equal to the ratio of sides, then E is a minimal set of horizontal slits.

Proof. We may assume that R is defined by $|\mathrm{Re}\, z|<L$, $|\mathrm{Im}\, z|<L'$ and $R\cap E \neq \phi$. If a large square $S(L^*, 0)$ contains E, we take a rectangle R^* defined by $|\mathrm{Re}\, z|<L^*$, $|\mathrm{Im}\, z|<L'$. By the property iv) there exist two Jardan arcs C, C' joining two vertical sides of R^* in the intersections of R^*-E with the half planes $\mathrm{Im}\, z<-(L'-\varepsilon)$ and $\mathrm{Im}\, z>L'-\varepsilon$ respectively. We denote by T the quadrangle bounded by C, C' and two vertical segments joining them. Let $\{\Omega_n\}$ be an exhaustion of E^c. Then the parallel slit mapping functions p_n, the function p_∞^0 of Ω_n, converge uniformly to z on the periphery of T for sufficiently large n and hence the rectangle defined by $|\mathrm{Re}\, z|<L^*+\varepsilon$, $|\mathrm{Im}\, z|<L'-2\varepsilon$ lies through the image of T under p_n for sufficiently large n. We have from the same reason as in [6] p. 82 (or [1])

$$\mathrm{mod}\,(R^*-E)\geqq \mathrm{mod}\,(R^*\cap\Omega_n)\geqq \frac{L'-4\varepsilon}{L^*+2\varepsilon}$$

and

$$\mathrm{mod}\,(R^*-E)=\frac{L'}{L^*}.$$

Next R^*-R consists of two rectangles denoted by R_1, R_2 from the left. Let Γ_1, Γ_2 be the families of locally rectifiable curves joining the vertical sides in R_1-E, R_2-E respectively. Since any locally rectifiable curve joining the vertical sides of R^* in R^*-E contains each members of the three families, we have by the inequality of the extremal length used in Lemma 2 [2]

$$(\mathrm{mod}\,\Gamma_1)^{-1}+(\mathrm{mod}\,\Gamma^*)^{-1}+(\mathrm{mod}\,\Gamma_2)^{-1}\leqq L^*/L.$$

On the other hand we note that $\mathrm{mod}\,\Gamma_1\leqq L'/(L^*-L)$, $\mathrm{mod}\,\Gamma^*\leqq L'/L$, $\mathrm{mod}\,\Gamma_2 \leqq L'(L^*-L)$, which implies that $\mathrm{mod}\,\Gamma^*=L'/L$.

We show the converse. Add to the rectangle suitable rectangles, if necessary, and the horizontal sides are isolated from E. The condition of the theorem remains valid because of the superadditivity of modules. We give the sets R, E an inversion with respect to the right hand vertical sides and denote by R', E' their images. The union of $R-E$, $R'-E'$ and the open segments comlementary to E of the side is a slit rectangle (not necessarily connected) and the module of the family Γ of locally rectifiable curves joining the vertical sides in the slit rectangle is $L'/2L$. Indeed for a invariant subfamily $\tilde{\Gamma}$ of Γ, we can restrict the metrics to invariant metrics $\tilde{\rho}$ by Lemma 4. Such metrics $\tilde{\rho}$ are symmetric with respect to the side and $2\tilde{\rho}$ is admissible for the module problem of Γ^*, and hence $\mathrm{mod}\,\tilde{\Gamma}=\mathrm{mod}\,\Gamma=L'/2L$.

Repeat the same process for the left vertical side and vanish the slits other than E. We see that the rectangle containing E complies the assumption of Theorem 8 and E is minimal.

REMARK. In order to select such a curve as C in the proof which lies in a narrow strip, it is sufficient that the compact set E consists of horizontal slits without the minimality.

COROLLARY 2. *If E is a minimal set of circular slits, then any disk (annulus) with center at the origin minus E is a minimal slit disk (annulus).*

13. We state the interrelations of minimal sets:

THEOREM 10. *A minimal set of circular (radial) slits is mapped by the function $w=\log z$ (not single-valued) onto a non-compact set of vertical (horizontal) slits and any compact subsets of it is minimal.*

Proof. We may assume that the minimal set E is contained in an annulus $A(q, Q)$ and the compact subset E^* of the image E' of E in a rectangle R defined by $\log q<\operatorname{Re} w<\log Q$, $0<\operatorname{Im} w<2n\pi$. For the case of radial slits, let Γ be the family of locally rectifiable curves joining the vertical sides in $R-E'$. Then we can deduce from Theorem 9 and the superadditivity of modules that $\operatorname{mod}\Gamma=2n\pi/\log(Q/q)$, hence E^* is a minimal set by Theorem 8.

For the case of circular slits, we identify the horizontal sides of R as usual, delete the vertical slit from it and have a planar surface $\mathfrak{R}$. The function $z=e^w$ maps $\mathfrak{R}$ onto an n-sheeted smooth covering surface of $A(q, Q)-E$ denoted by $\mathfrak{A}$. Obviously $\mathfrak{A}$ has n cover transformations regarded as a cyclic group of order n. Let Γ_1 be the family of locally rectifiable curves separating the boundaries over those of $A(q, Q)$. Then any admissible $\tilde{\rho}$ for invariant subfamily $\tilde{\Gamma}_1$ in the sense of Lemma 4 is an metric in $A(q, Q)$ using z as a local parameter and $n\tilde{\rho}$ is admissible for the module problem in Theorem 3. Then by Lemma 4 and the monotonity of modules we have $\operatorname{mod}\Gamma_1=(2n\pi)^{-1}\log(Q/q)$ and hence conclude that the module of curves joining the horizontal sides of R in $R-E^*$ is equal to the same value, which implies the minimality of E^*, removing superfluous parts of E'.

Although it is a restrictive case, we have as a converse of Theorem 10:

THEOREM 11. *A minimal set of vertical (horizontal) slits contained in a rectangle with vertical sides 2π is mapped by e^{iz} onto a minimal set of circular (radial) slits.*

Proof. For the case of vertical slits, let the set E be contained in a rectangle R defined by $\log q<\operatorname{Re} z<\log Q$, $0<\operatorname{Im} z<2\pi$. Then the function $w=e^{iz}$ maps the identified surface $R-E$ as in Theorem 10 onto a circular slit annulus and the curve family Γ_1 in Theorem 10 corresponds to the family of the curves joining two points on the sides with the same real parts. Let $\{R_n\}$ be the exhaustion of $R-E$ in the sense of Lemma 2. Then we can make functions F_n which maps R_n onto the same rectangle with a finite number of slits quasi-conformally by Lemma 2 and 3. Since the images of the curves belonging to Γ and contained in R_n enjoy the same property, we have

$$\operatorname{mod} \Gamma \geqq \frac{\log (Q/q)}{2\pi K_n} \qquad (\lim K_n = 1)$$

which implies the minimality of the circular slits. For the horizontal slits the assertion is obvious from the relation of the curve families.

We remark that for the latter case the condition that the set is isolated from the horizontal sides is removable as it is seen from the proof.

14. It is known that the union of a finite number of sets of class $N_{\mathfrak{B}}$ ($N_{\mathfrak{D}}$) is also null (Kuroda [11]). We show a similar result for disjoint minimal sets.

THEOREM 12. *The union of a finite number of disjoint minimal sets of the same type is minimal.*

Proof. It is sufficient to prove for horizontal slits. Let $\{E_j\}_{j=1}^N$ be the sets and E is the union. We take a large square $S(L, 0)$ containing E and disjoint ends $\{V_j\}_{j=1}^N$ of $S-E$ with E as its ideal boundary. Since the complement of E is a minimal horizontal slit domain, the slit mapping functions of the exhaustion of it tend to z in each end V_j. Hence we can make quasi-conformal mappings F_n mapping subdomains of the complement of E onto domains with a finite number of slits and with the maximal dilatation arbitrarily close to one by Lemma 3. $F_n = z$ outside $\cup V_j$ and hence we have $\operatorname{mod}(S(L, 0)-E)=1$ from the same reason as Theorem 8, which implies the minimality of E.

For the general case the problem is still open.

§ 5. Quasi-minimal sets.

15. A set E is called *quasi-minimal set of horizontal slits etc.*, if any compact subset of E is minimal. Theorem 10, 11 and the properties except iv) of minimal sets are valid for quasi-minimal sets using the term "quasi-minimal" instead of "minimal". We show additional properties of quasi-minimal sets:

i) If an annulus contains a quasi-minimal set E of circular slits and $A(q, Q)-E$ is a domain, then it is a minimal slit annulus.

ii) Let a set E be contained in $A(q, Q)$ and $A(q, Q)-E$ be a domain. If both intersections of E with annuli $A(q, r)$, $A(r, Q)$, $q<r<Q$, are quasi-minimal, it is a minimal slit annulus.

In fact, i) is deduced from Corollary 2, and ii) from i) and the superadditivity of modules. We remark that ii) is valid for a countable number of annuli if the set of radii of the circles complementary to the union of the annuli is of linear measure zero, and shall use it for radial slits and radial divisions. The property ii) does not hold in general for radial slits and circular divisions. Concerning this direction we show

THEOREM 13. *Let E be a compact set intersecting the imaginary axis whose* *intersections with the right and left half planes are both quasi-minimal sets of horizontal slits. If for a rectangle containing its intersection with the imaginary axis, the module of the family of the curves joining its vertical sides in E^c is equal to the ratio of sides, then E is minimal.*

Proof. We may assume that the rectangle R is defined by $|\mathrm{Re}\, z|<L_1$, $|\mathrm{Im}\, z|<L_2$ and that it satisfies the condition in Theorem 9 by the same reason as its proof. Then $E\cap\bar{R}$ is minimal. It is sufficient to show the minimality of its intersection with the closed parallel strip $|\mathrm{Im}\, z|\leqq L_2$, denoted by E_1, from the above property ii). Let $S(L,0)$ contain E_1. Since the set of all slits of E_1 with lengths not less than $L_1/4$ has the projection into imaginary axis which is closed and of linear measure zero, we can take a finite number of strips such that their two boundaries join the vertical sides of $S(L,0)$ in E_1^c, they contain all the above slits and the amount of the oscillations of $\mathrm{Im}\, z$ in each strip is less than ε. They devide $S(L,0)$ into a finite number of quadrangles, say S_ν. Since the set of slits contained in S_ν is compact, it can be covered by the union of a finite number of Jordan domains whose boundary lies in $S_\nu-E_1$ with its diameter less than $2L_1/5$. Then we can devide the slits in S_ν into three minimal sets by means of two cross-cuts consisting of segments on $\mathrm{Re}\, z=\pm L_1$ and a part of boundaries of the Jordan domains contained in right and left halves of R. Hence by Theorem 12 the slits in S_ν are minimal and the minimality of E_1 is easily deduced from the superadditivity of modules.

16. We last give some remarks to the remained mapping problems especially with respect to radial slit domains. The results are essencially due to Strebel [27, 28] and Reich [19]. We note

LEMMA 5. *Let f_n be univalent functions in a domain Ω and E a part of its boundary components. If $\lim f_n=f$ uniformly on any compact subset in Ω and if $f_n(E)$ is quasi-minimal, then $f(E)$ is so.*

Proof. It is sufficient to prove for horizontal slits. Put $\Delta=f(\Omega)$ and take an end V of Δ containing the given compact subset E^* of E. Let $g_n(w)$ be $f_n\circ f^{-1}(w)$ and $S(L,0)$ contain E^*. Then, since $\lim g_n=w$, we have $\mathrm{mod}\,(S(L,0)-E^*)=1$ by Lemma 4.

We remark that the lemma is valid for the sequence $\{f_n\}$ defined in an exhaustion $\{\Omega_n\}$ of Ω. The following remark deduced from it is suggested by Professor Oikawa:

REMARK. If the boundary component C is weak, the problem in Theorem 2 loses its meaning. The extremal functions $f_{aC_n}(z)$ for an exhaustion $\{\Omega_n\}$ of Ω, however, contain convergent subsequences and the image of its boundary under each limit function is a quasi-minimal set. By Corollary 2 the intersection of the image domain with any disk with radius Q is a minimal slit disk with radius Q. We may call the domain a minimal slit disk with infinite radius, but the uniqueness of the limit functions is not known.

We define for a domain Ω and a boundary component C an exhaustion $\{\Omega_n\}$ in the direction of C according to Oikawa [6] such that $a\in\Omega_n\subset\Omega$, $(\bar{\Omega}_n\cap\Omega)\subset\Omega_{n+1}$, $\cup\Omega_n=\Omega$, the relative boundary C_n of Ω_n is an analytic Jordan closed curve separating a from C. The exhaustion in the direction of C, C' is defined similarly.

We treat radial slit disks representatively. For the above exhaustion Ω_n is mapped onto a radial slit disk with radius Q_n by the function $g_n=g_{aC_n}$ in Theorem 5. Since Q_n is increasing, we have a limit value Q (possibly infinite) which is called the *extremal radius* of C by Strebel [28] and denoted by $R(a, c)$. Let Γ_q be the family of locally rectifiable curves joining the circle $|z-a|=q$ and C. Then using the continuity of extremal lengths with respect to the above exhaustions, the following equality is shown by him [28] (see also [15, 16]).

$$R(a, C)=\lim_{q\to 0}(2\pi\lambda(\Gamma_q)+\log q), \tag{10}$$

where λ is the extremal length.

17. We now state the results of Strebel [27, 28] and Reich [19] and give another proof of them which is also a proof of the continuity of extremal lengths (distances) for such a planar surface [28, 31]. Since annulus cases are similar, we show for disk cases the following:

THEOREM 14. *If $Q=R(a, C)<\infty$, the radial slit mapping functions g_n of an exhaustion $\{\Omega_n\}$ in the direction of C converge to a function g_{aC} in the sense that $||g_n'/g_n-g_{aC}'/g_{aC}||^2_{\Omega_n}\to 0$. The function g_{aC} has the following properties:*

1. *The images of the boundary components other than C under g_{aC} are a quasi-minimal set of radial slits.*
2. *The image of C is a circle with radius Q having possible incisions of linear measure zero.*
3. *The module of the family of curves joining a small circle with radius q and $g_{aC}(C)$ is equal to $2\pi/\log(Q/q)$* [27, 28, 19].

REMARK. The image domain stated by Reich [19] is different from it, but the property ii) in No. 15 (the remaked from) introduces his statements [27].

Proof. We have $\lim g_\nu=g_0$ for a subsequence. Put $\bar{\rho}_\nu=|g_\nu'/g_\nu\log(Q_\nu/M_q(g_\nu))|$, $\underline{\rho}_\nu=|g_\nu'/g_\nu\log(Q_\nu/m_q(g_\nu))|$ which are the extremal metrics for the module problems of locally rectifiable curves joining two boundaries of annuli $A(M_q(g_\nu), Q_\nu)$, $A(m_q(g_\nu), Q_\nu)$ in $g_\nu(\Omega_\nu)$ respectively from Theorem 5. We have by (5) for

$$\frac{1}{2}||\bar{\rho}_\nu-\underline{\rho}_\mu||_{A(M_q(g_\nu), Q)}\leqq||\bar{\rho}_\nu||^2-||\underline{\rho}_\mu||^2=2\pi\Big/\log\frac{Q_\nu}{M_q(g_\nu)}-2\pi\Big/\log\frac{Q_\nu}{m_q(g_\nu)}.$$

Since $\lim Q_\nu/M_q(g_\nu)=\lim Q_\nu/m_q(g_\nu)=Q/q$, noting that $\lim||\bar{\rho}_\nu-\underline{\rho}_\nu||^2=0$, we see that $||g_\nu'/g_\nu-g_0'/g_0||^2_{\Omega_\nu}\to 0$, from the weak convergence of g_ν'/g_ν and the convergence of the norms. By Lemma 5 the property 1 is obvious and the circle with radius Q consists of the boundary points of the image domain from the facts that $|g_0|<Q$ and $||\rho_0||^2=2\pi/\log(Q/q)$.

Let Γ be the family of curves defined in 3. Then $\mathrm{mod}\,\Gamma\leqq 2\pi/\log(Q/q)$ from the monotonity. In order to estimate $\mathrm{mod}\,\Gamma$ from below, we make an exhaustion $\{\varDelta_n\}$ of the image domain $\varDelta$ in the direction of $g_0(C)$ in which the circle with radius q is contained in $\varDelta_1$, and put $V_n=\varDelta_n-\varDelta_{n-1}$ $(n\geqq 2)$, $V_1=\varDelta_1-\{|z|\leqq q\}$. Let $\{V_{nm}\}$ be an exhaustion of V_n in the sense of Remark after Theorem 3. Since the set of slits in V_n is a minimal set of radial slits we can make by Lemma 3 $(1+\varepsilon)$-quasiconformal mapping functions F_n defined in $V_{nm(\varepsilon)}$ for sufficiently large m, which maps the domain onto a domain with the same relative boundary as V_n and a finite number of radial slits, where ε is arbitrarily close to zero. We set $\varDelta^\varepsilon=\cup_n V_{nm(\varepsilon)}$ and $G^\varepsilon=F_n$ in $V_{nm(\varepsilon)}$. G^ε maps the subdomain $\varDelta^\varepsilon$ onto a domain with $g_0(C)$, the circle with radius q and a countable number of radial slits as its boundary. Let $l(\theta)$ be the logarithmic length of the line segment in the direction of the argument θ joining the circle with radius q and $g_0(C)$. Then we have by Schwarz' inequality

$$\mathrm{mod}\,\Gamma^\varepsilon\geqq\int_0^{2\pi}\frac{d\theta}{l(\theta)} \tag{11}$$

which is a inequality of Strebel [28] for a special case where Γ^ε is the image of Γ under G^ε. Because of the fact that $l(\theta)\leqq\log(Q/q)$, we have from (11)

$$\mathrm{mod}\,\Gamma\geqq 2\pi\Big/(1+\varepsilon)\log\frac{Q}{q}$$

which implies that $\mathrm{mod}\,\Gamma=2\pi/\log(Q/q)$. We can easily see the property 2, since otherwise the module would exceed $2\pi/\log(Q/q)$ by the inequality (11).

Since the sequence $\{\bar{\rho}_n\}(\{\underline{\rho}_n\})$ is a minimal sequence for the module problem in $\varDelta\cap A(q,Q)$, it becomes a Cauchy sequence from the arguments in No. 3 and converges to a unique generalized extremal metric ρ_0, which implies that g_n originally tends to g_0, denoted by g_{aC}.

Furthermore we show

COROLLARY 3. *If a domain $\varDelta$ satisfies the following conditions: $\varDelta$ is contained in a disk with radius Q and the properties* 1, 3 *hold using the outer boundary component instead of $g_{aC}(C)$, then the module of the family of locally rectifiable curves joining two circles with radii q, Q $(q<Q)$ is equal to* $2\pi/\log(Q/q)$.

The result was stated by Strebel [27], but his proof based on incomplete discussion, because the continuity of extremal distances is not guaranteed for the exhaustion in his proof and we give another one.

Proof. We take such q that a closed disk $|w|\leqq q$ is contained in $\varDelta$. Let G^ε, $\varDelta^\varepsilon$ be the ones defined in Theorem 14. Then by the inequality (11) and the proterty 3, we see that the outer boundary component of $\varDelta$ is a circle with radius Q having possible incisions of angular measure zero. Let Λ, Λ^ε be the curve family defined in the collorary and the similar one in $G^\varepsilon(\varDelta^\varepsilon)$ respectively. Since the radial slits and incisions of $G^\varepsilon(\varDelta^\varepsilon)$ are of angular measure zero, we have $\mathrm{mod}\,\Lambda^\varepsilon=2\pi/\log(Q/q)$. The inverse image of any member of Λ^ε is contained by Λ and hence $\operatorname{mod}\Lambda=2\pi/\log(Q/q)$ because of the admissibility of the metric $(|w|\log(Q/q))^{-1}$ for the family Λ. For general $q<Q$, the same arguments as in the proof of Theorem 9 imply the assertion.

We call the image domain Δ under g_{aC} a *quasi-minimal* radial slit disk, but the function has no extremal property as in Theorem 5 in general [5, 26]. Professor Oikawa [16] obtained another interesting extremal property of g_{aC} which characterizes it. The following characterization is also due to him[3].

THEOREM 15. *Let Ω be a domain contained in a disk $|z|<Q$ and containing a closed disk $|z|\leqq q$. Then Ω is a quasi-minimal radial slit disk if and only if $2\pi/\log(Q/q)=\operatorname{mod}\Lambda=\operatorname{mod}\Gamma$, where Γ, Λ are defined in Theorem 14 and its corollary.*

Proof. The necessity is obvious from Theorem 14 and its corollary. Let C be the outer boundary of Ω. The boundary components of Ω other than C are quasi-minimal from the first half of the condition and C is a circle with radius Q having possible radial incisions of angular measure zero from the inequality (11). Note that the condition is valid for all $q'<q$, since (11) is effective for the circles $|z|=q'$ and $|z|=Q$. Then we have $R(0,C)=Q$ from the expression (10). Put $\rho=|g'_{aC}/(g_{aC}\log(Q/M_q(g_{0C})))|$ in the inverse image of $A(M_q(g_{0C}),Q)$ and $\rho_0=(|z|\log(Q/q))^{-1}$. ρ belongs to the generalized admissible class P^* for the family Γ and as is shown by Strebel [27], ρ_0 is its generalized extremal metric, since ρ_0 is extremal for the family Λ, $\Gamma\supset\Lambda$ and their modules are equal. Then by (5) we have

$$\frac{1}{2}\|\rho-\rho_0\|^2_{A(q,Q)}\leqq 2\pi\Big/\log\frac{Q}{M_q(g_{0C})}-2\pi\Big/\log\frac{Q}{q}.$$

Multiplying it by $\log(Q/q)\log(Q/M_q(g_{0C}))$ and tending q to zero, we have $g'_{0C}/g_{0C}=1/z$.

From the above therem and Corollary 3 we obtain the following characterization stated by Strebel [27, 28] for the annulus case.

COROLLARY 4. *The conditions in Corollary 3 characterizes g_{aC}.*

If we take off the condition $Q=R(a,C)<\infty$ in Theorem 14, the sequence $\{g_n\}$ contains convergent subsequences with limits g_0. By Lemma 5, (11) and the property ii) in No. 15 we can deduce the following.

REMARK (Strebel [28]). The image of the boundary components other than C under g_0 is a quasi-minimal set of radial slits and that of C is the point at infinity with possible radial incisions of angular measure zero emanating from it, if $R(a,C)=\infty$. Moreover $g_0(\Omega)\cup\{|w|>Q\}$ is a minimal radial slit plane. The uniqueness of g_0 is not known.

3) Oral suggestion.

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