ON A SIMPLIFIED METHOD OF THE ESTIMATION OF THE CORRELOGRAM FOR A STATIONARY GAUSSIAN PROCESS, II

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§1. Summary.

For the estimation of the correlogram of a real valued weakly stationary process $x(t)$, we usually use the estimate using the term $x(t)x(t+h)$. We try to replace the term $x(t)x(t+h)$ by the term $x(t)\ \mathrm{sgn}(x(t+h))$. In the previous paper [2], we showed that, when the variance is known, we can get an unbiased estimate by this replacement for a Gaussian process, and also showed its variance for a simple markov Gaussian process. In this paper, we shall evaluate its variance for a general Gaussian process, and show that this estimate is a consistent estimate under a some condition. And especially, we compare, numerically, its variance with that of usual estimate, for the second-order process.

§2. The estimate and its variance.

Let $x(t)$ be a real valued weakly stationary process with continuous time parameter t, such that $Ex(t)=0$, $Ex(t)^2=\sigma^2$, $Ex(t)x(t+h)=\sigma^2\rho_h$. We assume the variance σ^2 to be known. And, given observations $\{x(t),\ t=1, 2, \cdots, N, \cdots, N+h\}$, we consider to estimate the correlogram ρ_h, where N and h are positive integers. We shall try to replace the term $x(t)x(t+h)$ of the usual estimate

$$\tilde{\gamma}_h=\frac{1}{\sigma^2}\frac{1}{N}\sum_{t=1}^{N}x(t)x(t+h)$$

by the term $x(t)\ \mathrm{sgn}(x(t+h))$, where $\mathrm{sgn}(y)$ means 1, 0 and -1, correspondingly as $y>0$, $y=0$ and $y<0$.

For a Gaussian process, the estimate

$$\gamma_h=\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}\frac{1}{N}\sum_{t=1}^{N}x(t)\ \mathrm{sgn}(x(t+h))$$

is an unbiased estimate [2]. We shall determine the variance of this estimate. Now,

$$\mathrm{Var.}(\gamma_h)=E\gamma_h^2-\rho_h^2,$$

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$$E\gamma_h^2 = E\left(\sqrt{\frac{\pi}{2}}\ \frac{1}{\sigma}\ \frac{1}{N}\sum_{t=1}^{N} x(t)\ \mathrm{sgn}(x(t+h))\right)^2$$

$$= \frac{\pi}{2}\ \frac{1}{\sigma^2}\cdot\frac{1}{N^2}E\left(\sum_{t=1}^{N}\sum_{s=1}^{N} x(t)\ \mathrm{sgn}(x(t+h))x(s)\ \mathrm{sgn}(x(s+h))\right)$$

and, we shall evaluate the value of

$$E(x(t)\ \mathrm{sgn}(x(t+h))x(s)\ \mathrm{sgn}(x(s+h))).$$

i) When t, s, $t+h$ and $s+h$ are all distinct, we put

$$x(t) = Ax(s) + Bx(t+h) + Cx(s+h) + \varepsilon(t),$$

where A, B and C are constants and $\varepsilon(t)$ is a stochastic process such that

a) $E\varepsilon(t) = 0$,

b) $\varepsilon(t)$ has no correlation with $x(s)$, $x(t+h)$ and $x(s+h)$.

So, A, B and C are all determined by the relation

$$\begin{aligned} &E(x(t) - Ax(s) - Bx(t+h) - Cx(s+h))x(s) = 0,\\ &E(x(t) - Ax(s) - Bx(t+h) - Cx(s+h))x(t+h) = 0,\\ &E(x(t) - Ax(s) - Bx(t+h) - Cx(s+h))x(s+h) = 0.\end{aligned} \tag{1}$$

As $x(t)$ is a real-valued process, we have the equivalence

$$\rho_l = \rho_{-l}.$$

Using this, we can rewrite the relation (1) as follows:

$$\begin{aligned} &A + B\rho_{s-t-h} + C\rho_h = \rho_{s-t},\\ &A\rho_{s-t-h} + B + C\rho_{s-t} = \rho_h,\\ &A\rho_h + B\rho_{s-t} + C = \rho_{s-t-h}.\end{aligned} \tag{2}$$

From the equation (2), we have

$$A = \frac{1}{\Delta}\begin{vmatrix} \rho_{s-t} & \rho_{s-t-h} & \rho_h \\ \rho_h & 1 & \rho_{s-t} \\ \rho_{s-t+h} & \rho_{s-t} & 1 \end{vmatrix},\quad B = \frac{1}{\Delta}\begin{vmatrix} 1 & \rho_{s-t} & \rho_h \\ \rho_{s-t-h} & \rho_h & \rho_{s-t} \\ \rho_h & \rho_{s-t+h} & 1 \end{vmatrix},$$

and $$C=\frac{1}{\Delta}\begin{vmatrix} 1 & \rho_{s-t-h} & \rho_{s-t} \\ \rho_{s-t-h} & 1 & \rho_h \\ \rho_h & \rho_{s-t} & \rho_{s-t+h} \end{vmatrix},$$

where

$$\Delta=\begin{vmatrix} 1 & \rho_{s-t-h} & \rho_h \\ \rho_{s-t-h} & 1 & \rho_{s-t} \\ \rho_h & \rho_{s-t} & 1 \end{vmatrix}.$$

Therefore, we have

$$\begin{aligned} & E(x(t)/x(s),\ x(t+h),\ x(s+h)) \\ &= E(Ax(s)+Bx(t+h)+Cx(s+h)+\varepsilon(t)/x(s),\ x(t+h),\ x(s+h)) \\ &= Ax(s)+Bx(t+h)+Cx(s+h). \end{aligned}$$

And, so it holds

$$\begin{aligned} & E(x(t)x(s)/x(t+h),\ x(s+h)) \\ &= E[x(s)(E(x(t)/x(s),\ x(t+h),\ x(s+h)))/x(t+h),\ x(s+h)] \\ &= E[x(s)(Ax(s)+Bx(t+h)+Cx(s+h))/x(t+h),\ x(s+h)] \\ &= E[Ax(s)^2+Bx(s)x(t+h)+Cx(s)x(s+h)/x(t+h),\ x(s+h)]. \end{aligned}$$

In the next place, let us put

$$x(s)=Fx(t+h)+Gx(s+h)+\eta(s),$$

where $\eta(s)$ is a stochastic process such as

a′) $E\eta(s)=0$,

b′) $\eta(s)$ has no correlation with $x(t+h)$ and $x(s+h)$.

From this condition, we can express as

$$\begin{aligned} & E(x(s)-Fx(t+h)-Gx(s+h))x(t+h)=0, \\ & E(x(s)-Fx(t+h)-Gx(s+h))x(s+h)=0. \end{aligned} \tag{3}$$

Writing (3) in the correlogram's terms, we have

$$\begin{aligned} F \quad + G\rho_{s-t} &= \rho_{s-t-h}, \\ F\rho_{s-t} + G \quad &= \rho_h. \end{aligned} \tag{4}$$

By solving the equation (4), we have

$$F = \frac{1}{D}\begin{vmatrix} \rho_{s-t-h} & \rho_{s-t} \\ \rho_h & 1 \end{vmatrix} \text{ and } G = \frac{1}{D}\begin{vmatrix} 1 & \rho_{s-t-h} \\ \rho_{s-t} & \rho_h \end{vmatrix},$$

where

$$D = \begin{vmatrix} 1 & \rho_{s-t} \\ \rho_{s-t} & 1 \end{vmatrix}.$$

Substituting the above expression, we get

$$\begin{aligned} &E(x(t)x(s)/x(t+h),\ x(s+h)) \\ &= E[A(Fx(t+h)+Gx(s+h)+\eta(s))^2 + B(Fx(t+h)+Gx(s+h)+\eta(s))x(t+h) \\ &\quad + C(Fx(t+h)+Gx(s+h)+\eta(s))x(s+h)/x(s+h),\ x(t+h)] \\ &= (AF^2+BF)x(t+h)^2 + (2AFG+BG+CF)x(t+h)x(s+h) \\ &\quad + (AG^2+CG)x(s+h)^2 + AE(\eta(s)^2/x(t+h),\ x(s+h)). \end{aligned}$$

And, as $\eta(s)$ is independent of $x(t+h)$ and $x(s+h)$, we have

$$\begin{aligned} &E(\eta(s)^2/x(t+h),\ x(s+h)) \\ &= E[(x(s)-Fx(t+h)-Gx(s+h))^2/x(t+h),\ x(s+h)] \\ &= E[(x(s)-Fx(t+h)-Gx(s+h))^2/x(t+h)=0,\ x(s+h)=0] \\ &= E[x(s)^2/x(t+h)=0,\ x(s+h)=0] = \frac{\sigma^2\Delta}{D}. \end{aligned}$$

Consequently, it follows

$$\begin{aligned} &E(x(t)\ \mathrm{sgn}(x(t+h))x(s)\ \mathrm{sgn}(x(s+h))) \\ &= E[\mathrm{sgn}(x(t+h))\ \mathrm{sgn}(x(s+h))(E(x(t)x(s)/x(t+h),\ x(s+h)))] \\ &= (AF^2+BF)E(x(t+h)^2\ \mathrm{sgn}(x(t+h))\ \mathrm{sgn}(x(s+h))) \\ &\quad + (2AFG+BG+CF)E(|x(t+h)||x(s+h)|) \\ &\quad + (AG^2+CG)E(x(s+h)^2\ \mathrm{sgn}(x(t+h))\ \mathrm{sgn}(x(s+h))) \\ &\quad + A\frac{\sigma^2\Delta}{D}E(\mathrm{sgn}(x(t+h))\mathrm{sgn}(x(s+h))). \end{aligned}$$

Now, we shall put, for simplicity, $x(t+h)=x$ and $x(s+h)=y$ and further put

$$f(x, y) = \frac{1}{2\pi\sigma^2\sqrt{D}} e^{-(x^2-2\rho_{s-t}xy+y^2)/2\sigma^2 D}$$

Then we have

$$E(x(t+h)^2 \operatorname{sgn}(x(t+h)) \operatorname{sgn}(x(s+h)))$$

$$=\int_{y=0}^{\infty}\int_{x=0}^{\infty} x^2 f(x, y)dxdy - \int_{y=0}^{\infty}\int_{x=-\infty}^{0} x^2 f(x, y)dxdy - \int_{y=-\infty}^{0}\int_{x=0}^{\infty} x^2 f(x, y)dxdy$$

$$+\int_{y=-\infty}^{0}\int_{x=-\infty}^{0} x^2 f(x, y)dxdy.$$

Being

$$\int_{y=0}^{\infty}\int_{x=-\infty}^{0} x^2 f(x, y)dxdy = \int_{y=0}^{\infty}\int_{x=0}^{\infty} x^2 f(-x, y)dxdy = \int_{y=0}^{\infty}\int_{x=0}^{\infty} x^2 f(x, -y)dxdy$$

$$=\int_{y=-\infty}^{0}\int_{x=0}^{\infty} x^2 f(x, y)dxdy$$

and

$$\int_{y=-\infty}^{0}\int_{x=-\infty}^{0} x^2 f(x, y)dxdy = \int_{y=0}^{\infty}\int_{x=0}^{\infty} x^2 f(-x, -y)dxdy = \int_{y=0}^{\infty}\int_{x=0}^{\infty} x^2 f(x, y)dxdy,$$

so it holds

$$E(x(t+h)^2 \operatorname{sgn}(x(t+h)) \operatorname{sgn}(x(s+h)))$$

$$=2\left(\int_{y=0}^{\infty}\int_{x=0}^{\infty} x^2 f(x, y)dxdy - \int_{y=0}^{\infty}\int_{x=0}^{\infty} x^2 f(x, -y)dxdy\right).$$

Let us use the expansion (Rice [4], section 3. 5)

$$\int_0^{\infty}\int_0^{\infty} u^l v^m \exp(-u^2-v^2-2auv)dudv = \frac{1}{4}\sum_{k=0}^{\infty}\frac{(-2a)^k}{k!}\Gamma\left(\frac{l+k+1}{2}\right)\Gamma\left(\frac{m+k+1}{2}\right)$$

and put

$$I(-2a, l, m) = \sum_{k=0}^{\infty}\frac{(-2a)^k}{k!}\Gamma\left(\frac{l+k+1}{2}\right)\Gamma\left(\frac{m+k+1}{2}\right).$$

Consequently, we get

$$E(x(t+h)^2 \operatorname{sgn}(x(t+h)) \operatorname{sgn}(x(s+h)))$$

$$=2\times\frac{\sigma^2 D^{3/2}}{2\pi}(I(2\rho_{s-t}, 2, 0) - I(-2\rho_{s-t}, 2, 0)) = \frac{\sigma^2 D^{3/2}}{\pi} S_1(\rho_{s-t})$$

where

$$S_1(\rho_{s-t})=2\left(\sum_{m=0}^{\infty}\frac{(2\rho_{s-t})^{2m+1}}{(2m+1)\,!}\Gamma(m+2)\Gamma(m+1)\right).$$

Similarly,

$$E(|x(t+h)||x(s+h)|)$$

$$=2\times\frac{\sigma^2 D^{3/2}}{2\pi}(I(2\rho_{s-t},1,1)+I(-2\rho_{s-t},1,1))=\frac{\sigma^2 D^{3/2}}{\pi}S_2(\rho_{s-t}),$$

$$E(x(s+h)^2\operatorname{sgn}(x(t+h))\operatorname{sgn}(x(s+h)))$$

$$=2\times\frac{\sigma^2 D^{3/2}}{2\pi}(I(2\rho_{s-t},0,2)-I(-2\rho_{s-t},0,2))=\frac{\sigma^2 D^{3/2}}{\pi}S_1(\rho_{s-t})$$

and

$$E(\operatorname{sgn}(x(t+h))\operatorname{sgn}(x(s+h)))$$

$$=2\times\frac{\sqrt{D}}{4\pi}(I(2\rho_{s-t},0,0)-I(-2\rho_{s-t},0,0))=\frac{\sqrt{D}}{2\pi}S_3(\rho_{s-t}),$$

where

$$S_2(\rho_{s-t})=2\left(\sum_{m=0}^{\infty}\frac{(2\rho_{s-t})^{2m}}{(2m)\,!}\Gamma(m+1)^2\right)$$

and

$$S_3(\rho_{s-t})=2\left(\sum_{m=0}^{\infty}\frac{(2\rho_{s-t})^{2m+1}}{(2m+1)\,!}\Gamma(m+1)^2\right).$$

As the result, we obtain

$$E[x(t)\operatorname{sgn}(x(t+h))x(s)\operatorname{sgn}(x(s+h))]$$

$$=(AF^2+BF)\frac{\sigma^2 D^{3/2}}{\pi}S_1(\rho_{s-t})+(2AFG+BG+CF)\frac{\sigma^2 D^{3/2}}{\pi}S_2(\rho_{s-t})$$

$$+(AG^2+CG)\frac{\sigma^2 D^{3/2}}{\pi}S_1(\rho_{s-t})+A\frac{\sigma^2\Delta}{2\pi\sqrt{D}}S_3(\rho_{s-t}).$$

ii) When $s=t+h$, $s+h=t+2h$. The situation is the same when $t=s+h$. In this case, we have

$$E(x(t)x(s)\operatorname{sgn}(x(s+h))\operatorname{sgn}(x(t+h)))$$

$$=E(x(t)x(t+h)\operatorname{sgn}(x(t+h))\operatorname{sgn}(x(t+2h)))$$

$$=E[x(t+h)\operatorname{sgn}(x(t+h))\operatorname{sgn}(x(t+2h))E(x(t)/x(t+h),x(t+2h))].$$

As the above, we shall put

$$x(t)=Hx(t+h)+Kx(t+2h)+\delta(t)$$

where H and K are constants and $\delta(t)$ is a stochastic process such as

a$''$) $E\delta(t)=0$,

b$''$) $\delta(t)$ has no correlation with $x(t+h)$ and $x(t+2h)$.

H and K are determined by the conditions

$$\begin{aligned} &E(x(t)-Hx(t+h)-Kx(t+2h))x(t+h)=0, \\ &E(x(t)-Hx(t+h)-Kx(t+2h))x(t+2h)=0. \end{aligned} \tag{5}$$

This conditions are equivalent to

$$\begin{aligned} &H+K\rho_h=\rho_h, \\ &H\rho_h+K=\rho_{2h}. \end{aligned} \tag{6}$$

By solving (6), we get

$$H=\frac{1}{D_h}\begin{vmatrix} \rho_h & \rho_h \\ \rho_{2h} & 1 \end{vmatrix} \quad \text{and} \quad K=\frac{1}{D_h}\begin{vmatrix} 1 & \rho_h \\ \rho_h & \rho_{2h} \end{vmatrix},$$

where

$$D_h=\begin{vmatrix} 1 & \rho_h \\ \rho_h & 1 \end{vmatrix}.$$

We have

$$E(x(t)/x(t+h),\ x(t+2h))=Hx(t+h)+Kx(t+2h)$$

and

$$\begin{aligned} &E[x(t)x(t+h)\operatorname{sgn}(x(t+h))\operatorname{sgn}(x(t+2h))] \\ &=HE(x(t+h)^2\operatorname{sgn}(x(t+h))\operatorname{sgn}(x(t+2h)))+KE(|x(t+h)||x(t+2h)|). \end{aligned}$$

Using the same method as in i), we get

$$E(x(t)x(t+h)\operatorname{sgn}(x(t+h))\operatorname{sgn}(x(t+2h)))=H\frac{\sigma^2 D_h{}^{3/2}}{\pi}S_1(\rho_h)+K\frac{\sigma^2 D_h{}^{3/2}}{\pi}S_2(\rho_h).$$

iii) When $t=s$, it holds that

$$E(x(t)^2\operatorname{sgn}^2(x(t+h)))=Ex(t)^2=\sigma^2.$$

Therefore, using the above results, we obtain, by putting $s-t=k$,

$$\mathrm{Var.}(\gamma_h)=E^2\gamma_h-\rho_h^2$$

$$=\frac{1}{N^2}\left\{\sum_{k=1}^{h-1}+\sum_{k=h+1}^{N-1}\right\}(N-k)\Big[(AF^2+BF)D^{3/2}S_1(\rho_k)+(2AFG+BG+CF)D^{3/2}S_2(\rho_k)$$

$$+(AG^2+CG)D^{3/2}S_1(\rho_k)+\frac{A\Delta}{2D^{1/2}}S_3(\rho_k)\Big]$$

$$+\frac{1}{N^2}(N-h)[HD_h^{3/2}S_1(\rho_h)+KD_h^{3/2}S_2(\rho_h)]+\frac{\pi}{2}\cdot\frac{1}{N}-\rho_h^2.$$

3. Comparison of γ_h with $\tilde{\gamma}_h$.

Now we shall compare the estimate γ_h with the estimate $\tilde{\gamma}_h$, which is usually used. The estimate γ_h and the estimate $\tilde{\gamma}_h$ are both unbiased estimates. Here the comparison is made on the point of variance.

It holds, for a stationary Gaussian process with mean 0, that

$$E(x(t)x(t+h)x(s)x(s+h))$$

$$=(Ex(t)x(t+h))(Ex(s)x(s+h))+(Ex(t)x(s))(Ex(t+h)x(s+h))$$

$$+(Ex(t)x(s+h))(Ex(s)x(t+h)),$$

when $t<s$ and $t+h\neq s$. Using the above relation, we obtain

$$\mathrm{Var.}(\tilde{\gamma}_h)=E(\tilde{\gamma}_h^2)-\rho_h^2$$

$$=E\left(\frac{1}{\sigma^2}\frac{1}{N}\sum_{t=1}^{N}x(t)x(t+h)\right)^2-\rho_h^2$$

$$=\frac{1}{\sigma^4}\frac{1}{N^2}\sum_{t=1}^{N}\sum_{s=1}^{N}Ex(t)x(t+h)x(s)x(s+h)-\rho_h^2$$

$$=\frac{2}{N^2}\sum_{k=1}^{N-1}(N-k)(\rho_k^2+\rho_h^2+\rho_{h+k}\rho_{h-k})+\frac{1}{N}(1+2\rho_h^2)-\rho_h^2.$$

Let us compare the variance of γ_h with that of $\tilde{\gamma}_h$, numerically. For this, we shall consider the second-order process in the sense of Bartlett [1]. That is, $x(t)$ is subjected to the equation

$$d\dot{x}(t)+\alpha\dot{x}(t)dt+\beta x(t)dt=dy(t), \tag{7}$$

where $\dot{x}(t)$ is a mean square differential coefficient of $x(t)$, $d\dot{x}(t)$ is the change in $\dot{x}(t)$ in dt and $y(t)$ is the orthogonal process of the accumulated impulse effects.

Then we find that correlogram ρ_τ satisfies the equation

$$\rho_\tau''+\alpha\rho_\tau'+\beta\rho_\tau=0 \qquad (\tau>0),$$

where $\rho_\tau' = d\rho_\tau/d\tau$, etc., whence we have

$$\rho_\tau = Ae^{\lambda_1\tau} + Be^{\lambda_2\tau} \qquad (\tau > 0),$$

where λ_1 and λ_2 are the roots of $\lambda^2 + \alpha\lambda + \beta = 0$. Furthermore ρ_τ must satisfy the condition

$$\rho_0 = 1 \qquad \text{and} \qquad \rho_0' = 0.$$

This leads finally to

$$\rho_\tau = \frac{\lambda_1}{\lambda_1 - \lambda_2} e_2^{\lambda_2\tau} - \frac{\lambda_2}{\lambda_1 - \lambda_2} e^{\lambda_1\tau} \qquad (\tau > 0).$$

Table 1

h	ρ_h	$N=50$		$N=250$	
		Var. (γ_h)	Var. $(\tilde{\gamma}_h)$	Var. (γ_h)	Var. $(\tilde{\gamma}_h)$
1	0.9572	0.0720	0.2487	0.0149	0.0516
2	0.8536	0.0772	0.2269	0.0159	0.0471
3	0.7192	0.0850	0.1985	0.0173	0.0411
4	0.5759	0.0954	0.1706	0.0194	0.0353
5	0.4384	0.1074	0.1480	0.0218	0.0306
6	0.3154	0.1199	0.1327	0.0244	0.0273
7	0.2116	0.1315	0.1243	0.0268	0.0256
8	0.1282	0.1412	0.1212	0.0289	0.0250
9	0.0647	0.1487	0.1214	0.0305	0.0250
10	0.0189	0.1539	0.1230	0.0317	0.0254
11	−0.0119	0.1573	0.1251	0.0324	0.0259
12	−0.0307	0.1592	0.1268	0.0329	0.0263
13	−0.0403	0.1602	0.1280	0.0331	0.0265
14	−0.0432	0.1606	0.1286	0.0332	0.0267
15	−0.0416	0.1608	0.1289	0.0332	0.0267
16	−0.0373	0.1608	0.1289	0.0332	0.0268
17	−0.0316	0.1607	0.1288	0.0332	0.0267
18	−0.0254	0.1607	0.1287	0.0332	0.0267
19	−0.0194	0.1607	0.1286	0.0332	0.0267
20	−0.0140	0.1607	0.1285	0.0332	0.0267
21	−0.0095	0.1608	0.1285	0.0332	0.0267
22	−0.0058	0.1608	0.1285	0.0332	0.0266
23	−0.0030	0.1608	0.1285	0.0332	0.0266
24	−0.0009	0.1608	0.1285	0.0332	0.0266
25	0.0004	0.1608	0.1285	0.0332	0.0266
30	0.0016	0.1608	0.1285	0.0332	0.0267

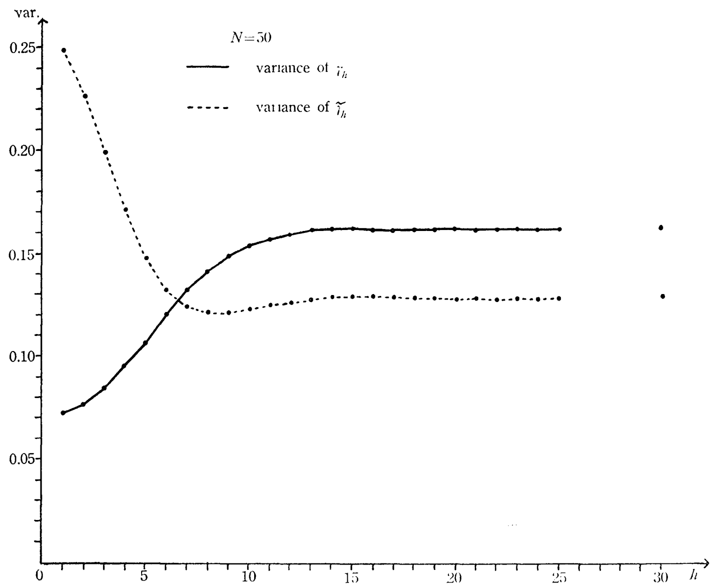

Fig. 1

For numerical computation, we shall take

$$\alpha=-2\log 0.8 \qquad \text{and} \qquad \beta=2\,(\log 0.8)^2.$$

In this case, the correlogram is

$$\rho_\tau=\sqrt{2}\,(0.8)^{|\tau|}\cos\,(|\tau|\log 0.8+\pi/4).$$

By taking $N=50$ and 250, the numerical results are shown in Table 1 and Figure 1.
This results are like as in the case of a simple Markov process [2]. For a small value of lag h, the variances of the estimate γ_h are smaller than that of the estimate $\tilde{\gamma}_h$.

Originally, the model (7) is discussed in Takahasi and Husimi [5]. So we have taken this model in this time and discussed from the statistical point of view. As we have shown, γ_h is a fairly good estimate for a Stationary Gaussian process.

4. Consistency of the estimate γ_h.

Let us further assume that the correlogram ρ_τ has the following property:

for any positive number ε, *there exists a number* τ_ε *such that* $|\rho_\tau|<\varepsilon$ *is statisfied for any number* τ *such as* $\tau>\tau_\varepsilon$, *i.e.* $\lim_{\tau\to\infty}\rho_\tau=0$.

In this case, we can prove that γ_h is a consistent estimate. The proof is as follows.

In the expression of the variance of γ_h, we shall put, for simplicity,

$$U_1(k)=AF^2+BF, \qquad U_2(k)=2AFG+BG+CF, \qquad U_3(k)=AG^2+CG.$$

Then, it holds

$$\frac{1}{N^2}\sum_{k=1}^{h-1}(N-k)\Big[U_1(k)D^{3/2}S_1(\rho_k)+U_2(k)D^{3/2}S_2(\rho_k)$$
$$+U_3(k)D^{3/2}S_1(\rho_k)+\frac{A\varDelta}{2D^{1/2}}S_3(\rho_k)\Big]=O\Big(\frac{1}{N}\Big)$$

and

$$\frac{1}{N^2}(N-h)[HD_h{}^{3/2}S_1(\rho_h)+KD_h{}^{3/2}S_2(\rho_h)]=O\Big(\frac{1}{N}\Big).$$

Now we shall evaluate the value of

$$\frac{1}{N^2}\sum_{k=h+1}^{N-1}(N-k)\Big[U_1(k)D^{3/2}S_1(\rho_k)+U_2(k)D^{3/2}S_2(\rho_k)+U_3(k)D^{3/2}S_1(\rho_k)+\frac{A\varDelta}{2D^{1/2}}S_3(\rho_k)\Big].$$

For any positive number ε, there exist a positive number $K=K(\varepsilon)$ such that

$$|\rho_k|<\varepsilon \quad \text{and} \quad |\rho_{k-h}|<\varepsilon$$

are satisfied for any k being larger than K.

It holds

$$|A|=\frac{1}{\varDelta}\begin{vmatrix}\rho_k & \rho_{k-h} & \rho_h \\ \rho_h & 1 & \rho_k \\ \rho_{k+h} & \rho_k & 1\end{vmatrix}=\left|\frac{\rho_k+\rho_k\rho_{k+h}\rho_{k-h}+\rho_h^2\rho_k-\rho_h\rho_{k+h}-\rho_h\rho_{k-h}-\rho_k^3}{1+2\rho_k\rho_h\rho_{k-h}-\rho_h^2-\rho_{k-h}^2-\rho_k^2}\right|$$

and, for any $k>K$,

$$|\rho_k+\rho_k\rho_{k+h}\rho_{k-h}+\rho_h^2\rho_k-\rho_h\rho_{k+h}-\rho_h\rho_{k-h}-\rho_k^3|$$
$$\leqq|\rho_k|+|\rho_k\rho_{k+h}\rho_{k-h}|+|\rho_h^2\rho_k|+|\rho_h\rho_{k+h}|+|\rho_h\rho_{k-h}|+|\rho_k^3|$$
$$<6\varepsilon,$$
$$|1+2\rho_k\rho_h\rho_{k-h}-\rho_h^2-\rho_{k-h}^2-\rho_k^2|$$
$$\geqq1-\rho_h^2-2|\rho_k\rho_h\rho_{k-h}|-\rho_{k-h}^2-\rho_k^2$$
$$\geqq1-\rho_h^2-4\varepsilon^2=(1-\rho_h^2)\Big(1-\frac{4\varepsilon^2}{1-\rho_h^2}\Big).$$

Now we can say

$$1-\frac{4\varepsilon^2}{1-\rho_h{}^2}\geqq\frac{1}{2}.$$

So we have

$$|A|\leqq\frac{12}{1-\rho_h{}^2}\,\varepsilon=a\varepsilon,\qquad a=\frac{12}{1-\rho_h{}^2}.$$

In the next place,

$$|B|=\left|\frac{\rho_h+\rho_h\rho_{k+h}\rho_{k-h}+\rho_h\rho_k{}^2-\rho_k\rho_{k-h}-\rho_k\rho_{k+h}-\rho_h{}^3}{1+2\rho_k\rho_h\rho_{k-h}-\rho_h{}^2-\rho_{k-h}{}^2-\rho_k{}^2}\right|$$

$$\leqq\frac{|\rho_h-\rho_h{}^3|+4\varepsilon^2}{(1-\rho_h{}^2)\left(1-\dfrac{4\varepsilon^2}{1-\rho_h{}^2}\right)}$$

$$=|\rho_h|+\frac{4\varepsilon^2|\rho_h|+4\varepsilon^2}{(1-\rho_h{}^2)\left(1-\dfrac{4\varepsilon^2}{1-\rho_h{}^2}\right)}$$

$$\leqq|\rho_h|+b\varepsilon^2.$$

Similary we obtain

$$|C|\leqq\varepsilon c,\ |F|\leqq\varepsilon f\ \text{ and }\ |G|\leqq|\rho_h|+\varepsilon^2 g.$$

In the above expression, a, b, c, f and g are constants which are independent of k and N.

Let us evaluate the value of $S_1(\rho_k)$, $S_2(\rho_k)$ and $S_3(\rho_k)$.

$$|S_1(\rho_k)|=2\left|\sum_{m=0}^{\infty}\frac{(2\rho_k)^{2m+1}}{(2m+1)!}\Gamma(m+2)\Gamma(m+1)\right|$$

$$\leqq 4|\rho_k|\sum_{m=0}^{\infty}(2\rho_k)^{2m}\leqq 4\varepsilon\sum_{m=0}^{\infty}(2\varepsilon)^{2m}=\frac{4\varepsilon}{1-(2\varepsilon)^2}\leqq l_1\varepsilon,$$

$$|S_2(\rho_k)|=2\left|\sum_{m=0}^{\infty}\frac{(2\rho_k)^{2m}}{(2m)!}\Gamma(m+1)^2\right|$$

$$=2\left(1+(2\rho_k)^2\left(\sum_{m=1}^{\infty}\frac{(2\rho_k)^{2(m-1)}}{(2m)!}\Gamma(m+1)^2\right)\right)$$

$$\leqq 2\left(1+(2\varepsilon)^2\,\frac{1}{1-(2\varepsilon)^2}\right)\leqq 2(1+l_2\varepsilon^2)$$

and

$$|S_3(\rho_k)|=2\left|\sum_{m=0}^{\infty}\frac{(2\rho_k)^{2m+1}}{(2m+1)!}\Gamma(m+1)^2\right|$$

$$\leqq 4|\rho_k|\left(\sum_{m=1}^{\infty}(2\rho_k)^{2m}\right)\leqq \frac{4\varepsilon}{1-(2\varepsilon)^2}\leqq l_3\varepsilon,$$

where l_1, l_2 and l_3 are constants which are independent of k and N. From the above results, it holds

$$|U_1(k)D^{3/2}S_1(\rho_k)|\leqq(af^2\varepsilon^3+f|\rho_h|\varepsilon+bf\varepsilon^3)\cdot 1\cdot l_1\varepsilon\leqq\varepsilon^2 d_1,$$

$$|U_2(k)D^{3/2}S_2(\rho_k)|\leqq(2af|\rho_h|\varepsilon^2+2afg\varepsilon^4+|\rho_h|^2+b|\rho_h|\varepsilon^2+g|\rho_h|\varepsilon^2$$
$$+bg\varepsilon^4+fc\varepsilon^2)\cdot 1\cdot 2(1+l_2\varepsilon^2)\leqq 2(\rho_h{}^2+\varepsilon^2 d_2),$$

$$|U_3(k)D^{3/2}S_1(\rho_k)|\leqq(a(|\rho_h|+\varepsilon^2 g)^2\varepsilon+c(|\rho_h|+\varepsilon^2 g)\varepsilon)\cdot 1\cdot l_1\varepsilon\leqq d_3\varepsilon^2$$

and

$$\left|\frac{A\varDelta}{2D^{1/2}}S_3(\rho_k)\right|\leqq|A||\varDelta||S_3(\rho_k)|\leqq(a\varepsilon)(1-\rho_h{}^2+4\varepsilon^2)(l_3\varepsilon)\leqq d_4\varepsilon^2,$$

where d_1, d_2, d_3 and d_4 are constants which are independent of k and N. Accordingly, we get

$$\frac{1}{N^2}\sum_{k=K+1}^{N-1}(N-k)\left[U_1(k)D^{3/2}S_1(\rho_k)+U_2(k)D^{3/2}S_2(\rho_k)+U_3(k)D^{3/2}S_1(\rho_k)+\frac{A\varDelta}{2D^{1/2}}S_3(\rho_k)\right]$$

$$\leqq\frac{1}{N^2}\sum_{k=K+1}^{N-1}(N-k)[\varepsilon^2 d_1+2\rho_h{}^2+2\varepsilon^2 d_2+\varepsilon^2 d_3+\varepsilon^2 d_4]$$

$$=(\varepsilon^2 d_1+2\rho_h{}^2+2\varepsilon^2 d_2+\varepsilon^2 d_3+\varepsilon^2 d_4)\left(\frac{1}{N^2}\sum_{k=K+1}^{N-1}(N-k)\right)$$

$$=(\varepsilon^2 d_1+2\rho_h{}^2+2\varepsilon^2 d_2+\varepsilon^2 d_3+\varepsilon^2 d_4)\frac{1}{2}\left(1-\frac{K+1}{N}\right)\left(1-\frac{K}{N}\right)$$

$$=\rho_h{}^2-\frac{(2K+1)}{N}\rho_h{}^2+\frac{K(K+1)}{N^2}\rho_h{}^2+\frac{\varepsilon^2}{2}(d_1+2d_2+d_3+d_4)\left(1-\frac{K+1}{N}\right)\left(1-\frac{K}{N}\right)$$

$$=\rho_h{}^2+O\left(\frac{1}{N}\right)+O(\varepsilon^2).$$

And it holds that

$$\frac{1}{N^2}\sum_{k=h+1}^{K}(N-k)\left[U_1(k)D^{3/2}S_1(\rho_k)+U_2(k)D^{3/2}S_2(\rho_k)+U_3(k)D^{3/2}S_1(\rho_k)+\frac{A\varDelta}{2D^{1/2}}S_3(\rho_k)\right]$$

$$=O\left(\frac{1}{N}\right).$$

Finally, we obtain

$$\text{Var.}(\gamma_h)=O\left(\frac{1}{N}\right)+O(\varepsilon^2),$$

so

$$P(|\gamma_h-\rho_h|>\theta)\leqq\frac{\text{Var.}(\gamma_h)}{\theta^2}=\frac{1}{\theta^2}\left(O\left(\frac{1}{N}\right)+O(\varepsilon^2)\right).$$

This shows that the estimate γ_h is a consistent estimate.

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