ON CERTAIN COEFFICIENT INEQUALITIES OF UNIVALENT FUNCTIONS

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1. Introduction. In the present paper we shall establish some coefficient inequalities of univalent functions by Schiffer's variational method [2].

Let S be a family of normalized functions $f(z)$ regular and univalent in $|z|<1$, that is, let $f(z)$ have the Taylor expansion

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n$$

in $|z|<1$.

THEOREM 1. *In S there holds an inequality*

$$|a_4-3a_2a_3+2a_2^3|\leqq 2.$$

Equality can occur only for the Koebe function $z/(1-e^{i\theta}z)^2$.

THEOREM 2. *In S there holds an inequality*

$$\mathrm{Re}\left\{a_5-2a_2a_4-\frac{3}{2}a_3^2+4a_2^2a_3-\frac{79}{54}a_2^4\right\}\leqq\frac{1}{2}.$$

Equality can occur only for every function in S satisfying an equation

$$\frac{1}{w^2}\sqrt{\left(1+\frac{4}{3}a_2w\right)^3}=\frac{1}{z^2}-\left(2a_3-\frac{4}{3}a_2^2\right)-z^2.$$

2. Proof of theorem 1. By Schiffer's variational method for a problem

$$\max_{S}\mathrm{Re}(a_4-3a_2a_3+2a_2^3),$$

we have a differential equation

$$\left(\frac{dw}{dt}\right)^2\frac{1}{w^5}[(a_2^2-a_3)w^2+1]+1=0$$

satisfied by the image curve of $|z|=1$ for every extremal function $w(z)$. This extremal function $w(z)$ satisfies a differential equation

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$$z^2\frac{w'^2}{w^5}[(a_2{}^2-a_3)w^2+1]$$

$$=\frac{1}{z^3}-\frac{a_2}{z^2}+(a_4-3a_2a_3+2a_2{}^3)-\bar{a}_2z^2+z^3.$$

The image curve of $|z|=1$ by $w(z)$ has at least one finite end point, which corresponds to a zero point of $w'(z)$ on $|z|=1$. Therefore the right hand side term $g(z)$ can be factorized in the following form

$$g(z)=\frac{1}{z^3}(z-E)^2(z^4+Dz^3+Cz^2+Bz+A).$$

Further there holds a functional equation

$$g(z)=\overline{g\left(\frac{1}{\bar{z}}\right)}.$$

These lead to the following relations:

$$AE^2=1,\quad |E|=1,\quad |A|=1,\quad BE^2-2AE=-a_2,\quad C-2ED+E^2=0,$$
$$B-2EC+E^2D=3(a_4-3a_2a_3+2a_2{}^3),\quad D=\bar{B}A,\quad C=\bar{C}A.$$

Let E be $e^{i\theta}$ and if $C\neq 0$, then $A=e^{-2i\theta}$, $C=re^{-i\theta}e^{p\pi i}$ $(r\neq 0)$, $D=(re^{-2i\theta}e^{p\pi i}+e^{i\theta})/2$, $B=(re^{p\pi i}+e^{-3i\theta})/2$, where p is an integer.

Case I. $e^{p\pi i}=1$. Then we have

$$3(a_4-3a_2a_3+2a_2{}^3)=\cos 3\theta-r.$$

On the other hand for Koebe's function $3(a_4-3a_2a_3+2a_2{}^3)=6$. Therefore we have an inequality $6\leqq\cos 3\theta-r$. This leads to an absurdity relation $r\leqq -5$.

Case II. $e^{p\pi i}=-1$. Then we have

$$a_2=\frac{3}{2}e^{-i\theta}+\frac{r}{2}e^{2i\theta},\qquad 3(a_4-3a_2a_3+2a_2{}^3)=\cos 3\theta+r.$$

Since $|a_2|\leqq 2$, we have

$$r^2+6r\cos 3\theta-7\leqq 0.$$

By this inequality we can say

$$0<r\leqq -3x+\sqrt{7+9x^2},\qquad x=\cos 3\theta.$$

This leads to an inequality $0<r\leqq 7$. On the other hand by Koebe's function

$$6\leqq\cos 3\theta+r,$$

thus $5\leqq r$. Let y be $x+r$, then

$$5r^2-6yr+7\geqq 0,$$

that is,

$$6y \leqq \frac{5r^2+7}{r}.$$

Since the right hand side term is monotone increasing in $5 \leqq r \leqq 7$, we have $y \leqq 6$. If the equality occurs, then $r=7$ and hence $x=-1$, that is, $3\theta=2p\pi+\pi$ for an integer p. Then

$$a_2=2 \quad \text{or} \quad 2e^{2\pi i/3} \quad \text{or} \quad 2e^{4\pi i/3},$$

which lead to three Koebe's functions

$$\frac{z}{(1-z)^2}, \qquad \frac{z}{(1-e^{2\pi i/3}z)^2}, \qquad \frac{z}{(1-e^{4\pi i/3}z)^2},$$

respectively.

If $r=0$, then $D=E/2$ and $B=\bar{D}A$. Thus putting $E=e^{i\theta}$, then $A=e^{-2i\theta}$, $D=e^{i\theta/2}$, $B=e^{-3i\theta/2}$. Hence

$$a_2=-\frac{e^{-3i\theta}}{2}e^{2i\theta}-2e^{-2i\theta}e^{i\theta}=-\left(2+\frac{1}{2}\right)e^{-i\theta}.$$

Thus $|a_2|=2+1/2>2$. This is a contradiction.

N. Suita gave another proof of this theorem. His method is quite similar in Charzynski-Schiffer's paper [1] in which they gave a quite elementary proof of the Bieberbach conjecture for the fourth coefficient.

3. Proof of theorem 2. We shall consider an extremal problem

$$\max_S \operatorname{Re} F, \quad F=a_5-2a_2a_4-\frac{3}{2}a_3^2+4a_2^2a_3-\frac{79}{54}a_2^4.$$

We shall denote the maximum value by F_0. Then any extremal function satisfies a differential equation

$$z^2\frac{w'^2}{w^6}\left(1+\frac{a_2}{3}w\right)^2\left(1+\frac{4}{3}a_2w\right)$$

$$=\frac{1}{z^4}[1+Rz^3+Tz^4+\bar{R}z^5+z^8] \ (\equiv g(z)),$$

$$R=2a_4-4a_2a_3+\frac{58}{27}a_2^3, \qquad T=4F.$$

Case I. $a_2\neq 0$. If $w(z_0)=-3/a_2$ for $|z_0|<1$, then z_0 and $1/\bar{z}_0$ are two double zeros of $g(z)$. A point on $|z|=1$, which corresponds to a finite end point of the image curve of $|z|=1$ by the mapping $w(z)$, is also a double zero of $g(z)$. If $w(z_0)=-3/a_2$ for some z_0 on $|z|=1$, then the image $w(|z|=1)$ locally forks to four analytic curves whose directions are equally distributed and one of which runs into $w=\infty$. Thus there are at least three finite end points in general which correspond to three double zeros of $g(z)$. If the number of finite end points is less than 3, then $-3/a_2$ also corresponds to a double zero or two double zeros of $g(z)$ according as the number of finite end points is two or one. Therefore in any cases $g(z)$ is of the following form

$$g(z)=\frac{1}{z^4}(W+Vz+Uz^2+z^3)^2(Y+Xz+z^2).$$

Further $g(z)$ satisfies a functional equation $g(z)=\overline{g(1/\bar{z})}$. Thus there hold the following relations:

$$W^2Y=1,\quad |W|=|Y|=1,\quad \bar{U}=\bar{W}V,\quad \bar{X}=\bar{Y}X,$$

$$2U+X=0.$$

$$2V+U^2+2UX+Y=0,$$

$$W^2+2VWX+YV^2+2UWY=0,$$

$$T=V^2+2UW+2XW+2XUV+2VY+U^2Y,$$

$$R=2VW+2UWX+V^2X+2WY+2UVY.$$

If $X=0$, then $U=V=0$ and hence $Y=0$, which contradicts $|Y|=1$. Let U and V be $re^{i\beta}$ and $re^{i\alpha}$, respectively, since $|U|=|V|$ by $|W|=1$. Then $Y=e^{2i\beta}$, $X=-2re^{i\beta}$, $W=e^{i(\beta+\alpha)}$. By two relations

$$2V+U^2+2UX+Y=0,$$

$$W^2+2VWX+YV^2+2UWY=0$$

we have

$$2V\bar{U}-3\bar{U}U^2+U=0 \quad\text{and}\quad V^2-3U\bar{U}V^2+2VU^2=0.$$

Thus we have

$$2re^{i(\alpha-2\beta)}-3r^2+1=0,$$

$$2re^{i(2\beta-\alpha)}-3r^2+1=0.$$

Hence we have

$$e^{i(4\beta-2\alpha)}=1.$$

If $e^{i(2\beta-\alpha)}=1$, then $3r^2-2r-1=0$. Therefore $r=1$. Since $W^2Y=1$, we have $1=e^{i(4\beta+2\alpha)}=e^{8\beta i}$, that is, $8\beta=2p\pi$ for some integer p. By the expression of T by V, U, W, X, Y, we have $T=-2e^{4\beta i}=\pm 2$. Further then $R=0$. If $e^{i(2\beta-\alpha)}=-1$, then $3r^2+2r-1=0$ and hence $r=1/3$. Further $e^{8\beta i}=1$. Hence $T=10e^{4\beta i}/27=\pm 10/27$.

Case II. $a_2=0$. Then the extremal function $w(z)$ satisfies a differential equation

$$z^2 \frac{w'^2}{w^6} = \frac{1}{z^4}[1+2a_4z^3+(4a_5-6a_3^2)z^4+2\bar{a}_4z^5+z^8].$$

In this case at the point at infinity $w=\infty$ the trajectory forks to four analytic curves whose successive two curves make the angle $\pi/2$ there. Thus the right hand side term is of perfectly square form

$$\frac{1}{z^4}(1+Az+Bz^2+Cz^3+Dz^4)^2.$$

Then $A=B=C=0$ and $D^2=1$. Hence $a_4=a_2=0$ and $4a_5-6a_3^2=\pm 2$. Hence in any case we have

$$F_0 = \max_S \operatorname{Re} F = \frac{1}{2}.$$

Now we shall examine when equality occurs. In case I, the extremal function satisfies the differential equation

$$z^2 \frac{w'^2}{w^6}\left(1+\frac{a_2}{3}w\right)^2\left(1+\frac{4}{3}a_2w\right) = \frac{1}{z^4}+2+z^4.$$

Integrating this differential equation, we have

$$\frac{1}{w^2}\sqrt{\left(1+\frac{4}{3}a_2w\right)^3} = \frac{1-2Cz^2-z^4}{z^2}, \qquad 2C=2a_3-\frac{4}{3}a_2^2.$$

In Case II, the extremal function satisfies the differential equation

$$z^2 \frac{w'^2}{w^6} = \frac{(1+z^4)^2}{z^4}.$$

Integrating this, we have

$$\frac{1}{w^2} = \frac{1-2a_3z^2-z^4}{z^2}.$$

This is contained in the earlier case with $a_2=0$.

4. By Schiffer's variational method certain extermal problem leads us to a differential equation satisfied by every extremal function

$$z^2 \frac{w'^2}{w^{m+1}} P(w) = Q(z),$$

where $P(w)$ and $Q(z)$ are a polynomial of w and a rational function of z, respectively. If $P(w)$ is of perfect square form, then the exact estimation for the original problem can be done relatively simple. This has very close relation to the one obtained by the Faber polynomial. Such an example is the following

$$\left|a_4-2a_2a_3+\frac{13}{12}a_2^3+2\alpha\left(a_3-\frac{3}{4}a_2^2\right)+\alpha^2a_2\right|\leqq\frac{2}{3}+2|\alpha|^2,$$

which was most effectively used by Charzynski and Schiffer [1].

Our two results are not of the above form and are few examples showing that $P(w)$ is not perfectly square but exact estimation is possible only in the elementary manner. However there is no general theory exhausting all the combinations of coefficients for which exact estimations are possible by only the algebraic calculations from the corresponding Schiffer's differential equations.

REFERENCES

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